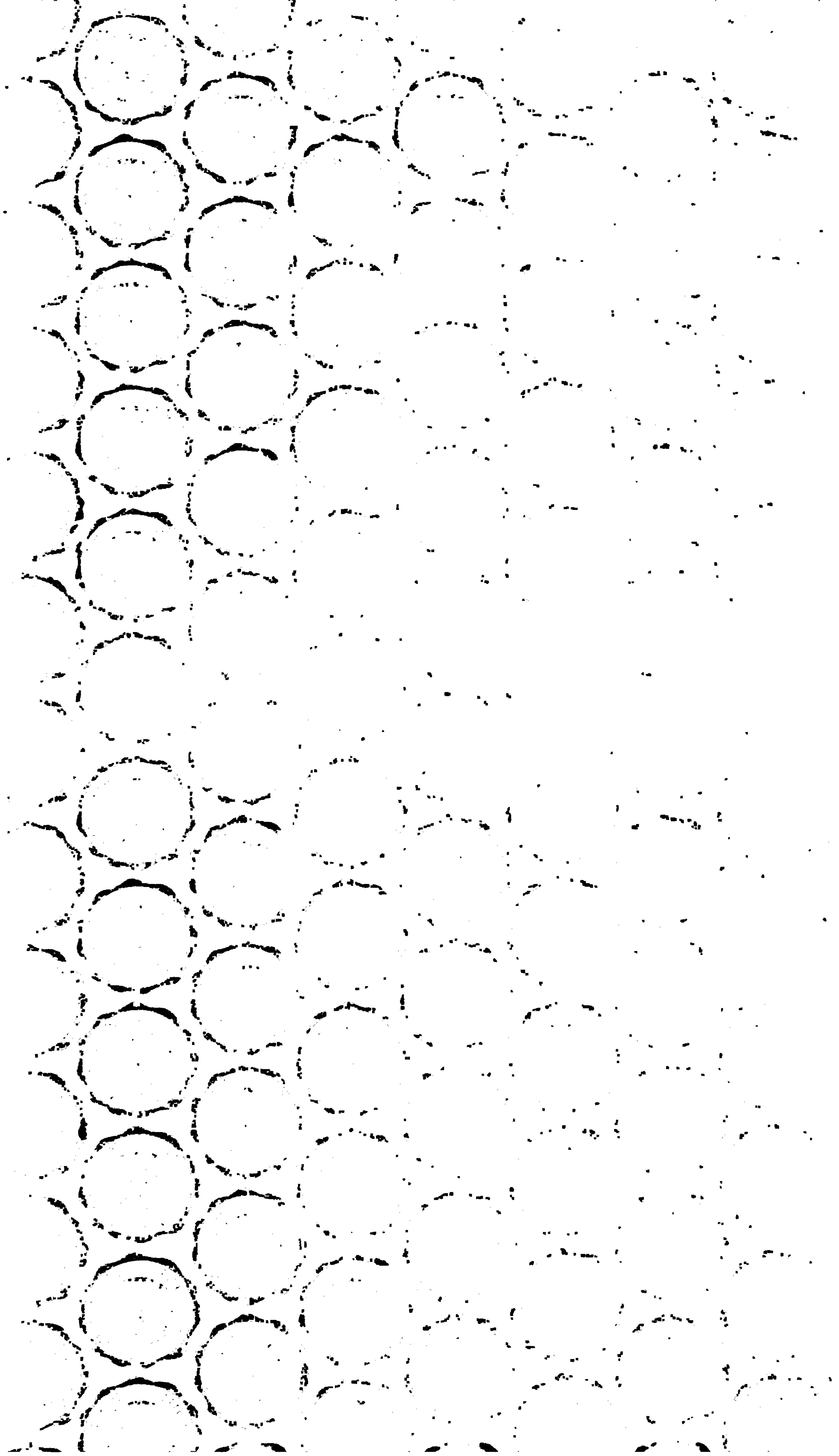

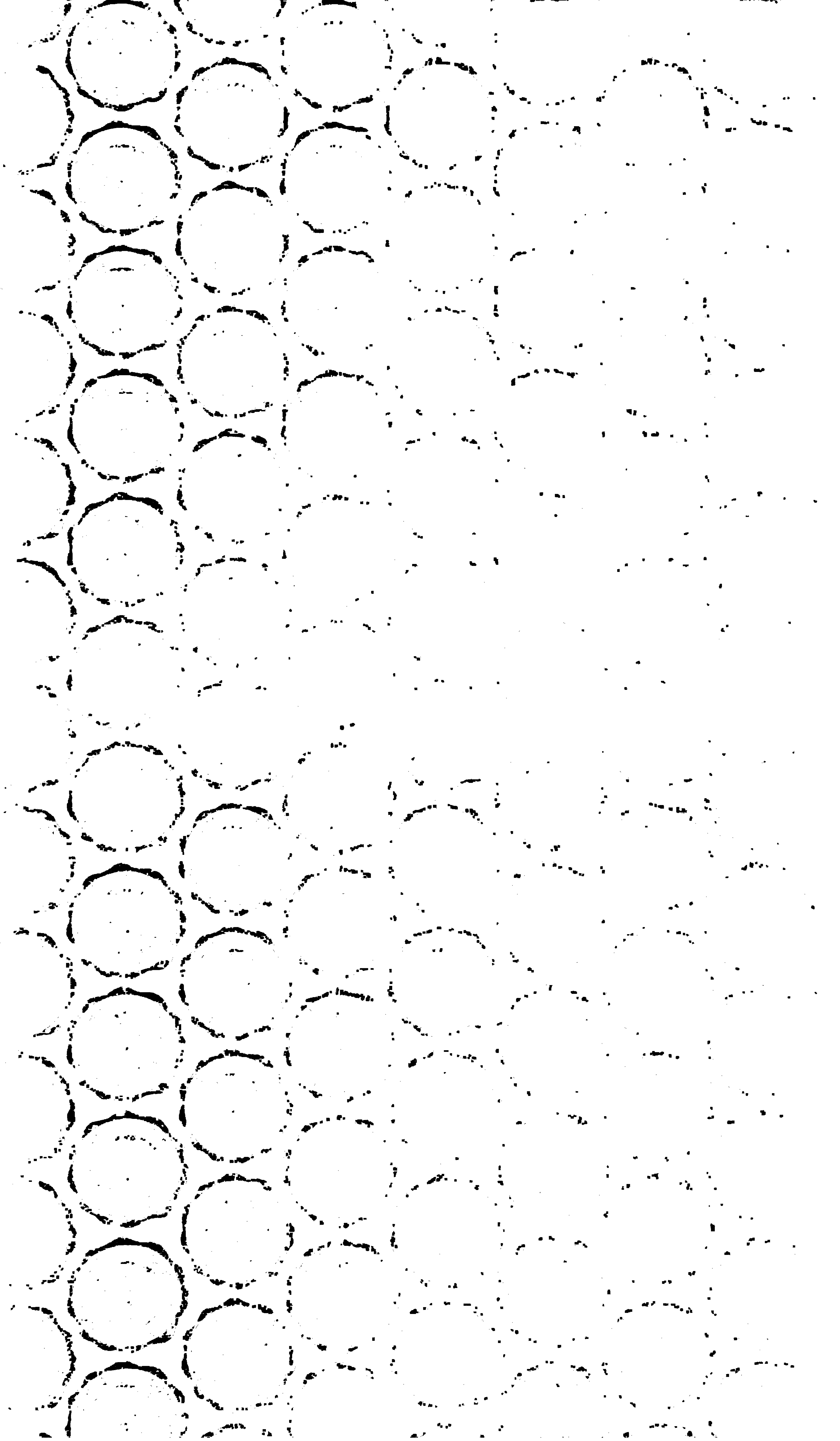

A
COMPLETE COURSE
OF
PURE MATHEMATICS.

BY

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TRANSLATED FROM THE FRENCH

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TABLE OF CONTENTS

OF THE

SECOND VOLUME.

[N. B. The figures refer to the pages.]

BOOK V.—ALGEBRA

BOOK VI.—ANALYSIS OF THREE DIMENSIONS.

BOOK VII.—DIFFERENTIAL AND INTEGRAL CALCULUS.

A COMPLETE COURSE

OF

PURE MATHEMATICS.

BOOK V.

THE HIGHER BRANCHES OF ALGEBRA.

I.—COMBINATIONS AND POWERS.

PERMUTATIONS AND COMBINATIONS.

475. WHEN a number of terms are composed of similar or different letters, placed in various orders, we shall give to these assemblages the name of *Arrangements* or *Permutations*; but if one at least of the letters be different in each term, and no regard be paid to the order in which they are arranged, the terms will then be called *Combinations*.*

Thus *abc*, *bac*, *cba*, *bca* are 4 permutations, and *abc*, *abd*, *bcd*, *acd* 4 combinations of 3 letters each.

To denote the number of permutations that can be formed with m letters, taking them p and p, we shall write $[mPp]$; the number of combinations will be indicated by $[mCp]$.

* The combinations have also been called *different Products*; but this expression we reject as defective; for *ab* and *cd*, though they are different combinations of two letters, may yet form equal products, as $3 \times 8 = 6 \times 4 = 2 \times 12$.

Permutations and Arrangements have also been distinguished from each other, in the first of these terms being confined to the arrangements of p letters among themselves, or taken p and p; but this distinction is of no material advantage, and we shall not make use of it, any more than of various other denominations.

Let it be proposed *to find the number y of all the permutations of m letters taken p and p*, $y = [mPp]$. In the first place, let those arrangements only be considered, which commence each with some one letter, as a, whilst they differ, either in respect to some other letter on the right of a, or as to the order in which the remaining letters are placed. If, now, this initial a be suppressed, we shall have an equal number of assemblages of $p-1$ letters each; and these will evidently be all the possible arrangements of the other $m-1$ letters $b, c, d \ldots$ taken $p-1$ together, and their number will be expressed by $\varphi = [(m-1)\ P\ (p-1)]$. Conversely, therefore, if we take these $m-1$ letters $b, c, d \ldots$, form with them all the permutations of $p-1$ letters each, and then place a at the head of each term, we shall have all the permutations, p and p, which have a for their initial letter; for no one of these last can be omitted, or be repeated more than once, without the same error presenting itself in the assemblages that result on the suppression of the initial a; *i. e.* without some permutation of the letters $b, c, d \ldots$, taken $p-1$ together, having also been so omitted or repeated, which is contrary to the supposition.

Thus, there are precisely as many arrangements of $m-1$ letters, taken $p-1$ together, as there are arrangements of m letters, taken p and p, in which a stands first; and this number is φ.

Reasoning for b in the same manner that we have done for a, we shall similarly find that there are φ permutations which commence with b; there will also be φ having c at their head...; and since each letter must stand first in its turn, the number y required will be composed of φ repeated as many times as there are letters; whence

$$y = m\varphi, \text{ or } [mPp] = m\,[(m-1)\ P\ (p-1)].$$

From this it follows, that

1°. To determine the number y'' of the arrangements of m letters, taken 2 and 2, φ being in this case the number of arrangements of $m-1$ letters taken 1 and 1, or $\varphi = m-1$, we have $y'' = m\,(m-1)$.

2°. If the number y''' be required of the arrangements of m letters, taken 3 and 3, p becomes 3, and φ denotes the number of arrangements of $m-1$ letters taken 2 and 2; *i. e.* φ is equivalent to y'' when m is changed into $m-1$; whence $\varphi = (m-1)\ (m-2)$, and $y''' = m\,(m-1)\ (m-2)$.

3°. We find, in like manner, for the number of arrangements, taking the letters 4 and 4,

$$y^{IV} = m\,(m-1)\ (m-2)\ (m-3);$$

and it is evident that, generally, to pass from one of these equations to the succeeding one, we must change m into $m-1$, and then multiply by m; an operation which comes to the same thing with annexing to

the factors m, $m-1$... the integer immediately inferior to the last of these factors. For p letters, therefore, this last multiplier will be $m-(p-1)$; whence

$$y=[mPp]=m(m-1)\ (m-2)\ \ldots\ldots\ (m-p+1)\ \ldots\ldots\ (1),$$

the number of factors being p. Thus 9 things, taken 4 and 4, can be changed among themselves in as many different ways as is expressed by the product of the four factors $9.8.7.6=[9P4]=3024$; which is the number of different ways in which 9 persons can occupy 4 places. The arrangements of m things 1 and 1, and 2 and 2, are together equal in number to $m+m(m-1)=m^2$.

Making $m=p$, we obtain the number z of the arrangements of p letters, taken p and p; *i. e.* each term containing all the letters:

$$z=[pPp]=p(p-1)\ (p-2)\ \ldots\ldots\ 3.2.1=1.2.3\ldots p\ \ldots\ (2).$$

The number of permutations of 7 letters among themselves is $1.2.3\ldots 7=5040$.

The number of arrangements of the 7 notes of the musical scale is $1.2.3\ldots 7=5040$; taking into account the semi-tones, we have 479001600.

476. Let us now investigate *the number x of the different combinations of m letters taken p and p*, $x=[mCp]$. Suppose these x combinations to be already effected, and written successively in an horizontal line; under the first of them write down all the permutations of the p letters which enter into it, and we shall have a vertical column consisting of z terms (2). The second term of the horizontal line will in like manner give a vertical column of z terms, constituting the several permutations of the p letters contained in that term, and in which one letter at least will be different from those which enter into the first combination. The third combination will also give z results different from the others, &c.; and thus a table will be formed consisting of x columns, each containing z terms; making in all xz results, which will evidently constitute all the arrangements possible of our m letters, taken p and p, without any one of them being omitted or repeated more than once. But the number of these being y (1), we have $xz=y$, whence $x=\dfrac{y}{z}=\dfrac{mPp}{pPp}$, or

$$x=[mCp]\times\frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\ldots\times\frac{m-p+1}{p}\ \ldots\ldots\ (3).$$

The equations (1) and (2) being each of them composed of p factors, the equation (3) will also have p; and these factors will be fractions, the terms of which are integral, and follow the natural order, decreasing from m for the numerator, and increasing from 1 up to p

for the denominator. Since x must, from the nature of the case, be an integral number, *the formula* (1) *must be exactly divisible by* (2); as might also be proved directly.

477. We have

$$[mCq] = x' = \frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}\ldots\ldots\ldots\frac{m-q+1}{q}.$$

Suppose $p > q$; then the factors of this equation will all enter into equation (3), which may consequently be written

$$x = x'.\frac{m-q}{q+1}.\frac{m-q-1}{q+2}\ldots\ldots\ldots\ldots\frac{m-p+1}{p}.$$

I. Let us see if it be possible for x and x' to be equal. It is evident that, in this case, the product of the several factors following x' must reduce itself to 1; or the numerators must form the same product as the denominators; whence, taking the latter in inverse order, we have, for the equation of condition,

$$(m-q)\ (m-q-1)\ \ldots\ldots = p\ (p-1)\ \ldots\ldots\ldots\ (q+1).$$

But, the two sides of this containing the same number of successively decreasing factors, the equality will be impossible, unless each factor on the one side be equal to the one of the same rank on the other; since, otherwise, the common factors being suppressed, there would remain on the one side factors all greater than those on the other, and the same in number. Thus, that x and x' may be equal, our equation requires that $m - q = p$; whence we have this theorem:

$$[mCp] = [mCq], \text{ when } m = p + q.$$

For instance, 100 letters, taken 88 and 88, and taken 12 and 12, must give an equal number of combinations; and, in fact, 100 C 88 has for its numerator 100.99... 90.89.88... 13, and 1.2.3... 12.13... 88 for its denominator; whence, suppressing the common factors 13.14... 88, there remains $\dfrac{100.\,99\ldots\,88}{1.\,2.\,3\ldots\,12} = 100\ C\,12$. This theorem will serve to facilitate the calculations of the formula (3), when $p > \frac{1}{2}m$; thus we should sooner find 100 C 4 than 100 C 96 $= 3921225$.

"From this we shall conclude, that if the numbers of combinations of m letters taken 1 and 1, 2 and 2, 3 and 3... be written successively, *the same values will recur in inverse order beyond the middle term.* Thus, for 8 letters, these numbers are 8, 28, 56, 70, 56, 28, 8.

II. Suppose $q = p - 1$; then x has only one factor more than x', and we have

$$x = x'\frac{m-p+1}{p}, \text{ or } [mCp] = [mC(p-1)].\frac{m-p+1}{p}\ \ldots\ (4).$$

1°. Hence we get this rule for deducing successively, one from the other, the numbers of combinations of m letters taken 1 and 1, 2 and 2, 3 and 3...

Write the two series $m, m-1, m-2...,$ and $1, 2, 3...$; with these respective numbers form the fractions $\frac{m}{1}, \frac{m-1}{2}, \frac{m-2}{3}...$; and, lastly, multiply each of these fractions by the product of all the preceding ones. For instance, if 8 letters are to be combined, we write $\frac{8}{1}, \frac{7}{2}, \frac{6}{3}...$, and we have $8, 8 \times \frac{7}{2} = 28, 28 \times \frac{6}{3} = 56...$ Thus it is we find that 8 numbers of the lottery form 8 *extraits*, 28 *ambes*, 56 *ternes*, 70 *quaternes*, and 56 *quines*. The whole 90 numbers give 90 extraits, 4005 ambes, 117480 ternes, 2555190 quaternes, 43949268 quines.*

2°. In our successive factors $\frac{m}{1}, \frac{m-1}{2}, \frac{m-2}{3}...,$ the numerators go on decreasing, the denominators increasing, and the first are > 1. The products increase continually, so long as the order i is such that

$$\frac{m-i+1}{i} = \text{or} > 1, \; i.\,e.\; i = \text{or} < \frac{m+1}{2}.$$

Beyond this they decrease, and we have seen that the products recur in inverse order. Let us investigate the greatest term.

---

* The above terms will be explained by the following scheme of the French lottery. The lottery is composed of 90 numbers, and any quantity of numbers, from one to ninety, may be chosen, the person depositing on each number whatever sum he thinks proper. Of the 90 numbers, only 5 are drawn at one time, producing the following prizes:

Five prizes of *simple Extraits*,
Ten of *simple Ambes*,
Ten of *Ternes*,
Five of *Quaternes*,
Five of *determined Extraits*, and
Ten of *determined Ambes*.

The simple *Extrait* is formed by the coming up of one, two, three, four, or five numbers from the wheel, on any one of which *separately* a certain sum had been deposited; and for each number that thus comes up, the purchaser is entitled to 15 times the sum deposited thereon.

The simple *Ambe*, *Terne*, or *Quaterne* is similarly formed by the coming up of any two, three, or four of the numbers taken in one ticket, the order in which the numbers turn up *not* having been named by the purchaser; and the value of the prize is respectively 270, 5500, or 75000 times the sum deposited.

The *determined* chances consist in the purchaser's fixing the order in which the numbers shall come up. For the determined Extrait 70 times the amount deposited is paid, and for the determined Ambe, 5100 that amount.

1st Case, m odd. Since $m+1$ will be even, we may assume $i=\frac{m+1}{2}$; and the last fractional factor then becoming $=1$, it will again give the preceding term, which is of the rank $i=\frac{1}{2}(m-1)$; and has for its last factor $\frac{\frac{1}{2}(m+3)}{\frac{1}{2}(m-1)}=\frac{m+3}{m-1}$; thus *the terms go on increasing up to the one in the middle, which is repeated, and has for its value* $\left[mC\frac{m+1}{2}\right]$.

2nd Case, m even. The product increases no farther than the rank $i=\frac{1}{2}m$; for if we assume a greater value than this, as $i=\frac{1}{2}(m+2)$, the condition specified will not be satisfied. Thus *the middle term, and the one which is the greatest of all, is not repeated;* it has for its value $\left[mC\,\frac{m}{2}\right]$, the last factor of which is $\frac{\frac{1}{2}m+1}{\frac{1}{2}m}$.

3°. The equation (4) also gives

$$x+x'=x'\times\frac{m+1}{p};$$

and, since $q=p-1$, there results, from the equation of N°. 477,

$$x+x'=\frac{m+1}{1}\cdot\frac{m}{2}\cdots\frac{m-p+2}{p},$$

the second side of which, compared with equation (3), gives

$$[(m+1)\,Cp]=[mCp]+[mC(p-1)].$$

From this relation we are enabled, by means of a simple addition, to deduce the combinations of $m+1$ letters from those of m letters; and it is on this principle that, in the following Table, which is called the *arithmetical Triangle of Pascal, each number is the sum of the two corresponding terms of the line preceding.* Thus for the 7th line we have 1, 7, 21, 35, 35, 21, 7, 1;
and to form the 8th line we shall put $1+7=8$, $7+21=28$, $21+35=56$, $35+35=70$, &c.

This law explains the recurrence of the same terms in inverse order; since, supposing only that this is the case for one line, it must necessarily be so for the line that follows. As to the formation of the terms, those of any line may either be deduced step by step from each other (1°.), or by means of the terms of the line preceding (3°.); or, lastly, directly from the equation (3), which is the *general term.*

COEFFICIENTS OF THE BINOMIAL; OR, NUMBERS OF COMBINATIONS.

1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1
1	11	55	165	330	462	462	330	165	55	11
1	12	66	220	495	792	924	792	495	220	66
1	13	78	286	715	1287	1716	1716	1287	715	286
1	14	91	364	1001	2002	3003	3432	3003	2002	1001
1	15	105	455	1365	3003	5005	6435	6435	5005	3003
1	16	120	560	1820	4368	8008	11440	12870	11440	8008
1	17	136	680	2380	6188	12376	19448	24310	24310	19448
1	18	153	816	3060	8568	18564	31824	43758	48620	43758
1	19	171	969	3876	11628	27132	50388	75582	92378	92378
1	20	190	1140	4845	15504	38760	77520	125970	167960	184756
0	1 and 1	2 and 2	3 and 3	4 and 4	5 and 5	6 and 6	7 and 7	8 and 8	9 and 9	10 and 10

III. Let $1, m, a, b, c \ldots b, a, m, 1$, be the numbers of any line; those of the line following (3°.) are $1, 1 + m, m + a, a + b \ldots m + 1, 1$; and the sum of the terms of the even orders is $1 + m + a + b \ldots + m + 1$, which is the same with the sum of the terms of the odd orders, and also the sum of the terms of the line preceding. If, therefore, we add together all the terms of the line $m + 1$, we shall have the double of the sum of the line m. But the second line of the table is $1 + 2 + 1 = 4 = 2^2$; so that the succeeding lines have for their sum $2^3, 2^4, 2^5 \ldots 2^m$. Thus, *the sum of all the combinations of m letters is 2^m; that of either the even or the odd orders is 2^{m-1}, the same with what we find for the sum of all the combinations of $m - 1$ letters.*

478. Let the m letters $a, b, c, d \ldots$ be divided into two sets, the number of letters in the one being m', in the other m'' $(m = m' + m'')$; and let us then investigate the several combinations p and p that can be formed with p' of the first set of letters joined to p'' of the others $(p = p' + p'')$. For this purpose form all the combinations of the first set of letters taken p' and p', and those of the second taken p'' and p''; they will be in number $m'Cp'$ and $m''Cp''$; and each of the first results being now coupled with each of the second, p' factors on the one hand, united with p'' on the other, will form p factors;

and it is evident that these systems will make up all those required. Their number therefore is

$$X = [m'Cp'] \times [m''Cp'']\ldots\ldots (5).$$

I. Into how many of the combinations does the letter a enter? Here $m' = p' = 1$, and $X = (m-1)\,C\,(p-1)$.

II. How many of the combinations contain a without b, and b without a? In this case, $m' = 2$, $p' = 1$; whence $X = 2 \times [(m-2)\,C\,(p-1)]$.

III. How many contain both a and b? $m' = p' = 2$, $X = (m-2)\,C\,(p-2)$.

IV. How many contain neither a nor b? $m' = 2$, $p' = 0$, and $X = (m-2)\,Cp$.

V. Among the combinations of m letters, taken p and p, how many are there which contain two of the three letters a, b, c? $m = 3$, $p' = 2$, $X = 3 \times [(m-3)\,C\,(p-2)]$.

VI. The combinations of 10 letters taken 4 and 4 are 210 in number: supposing three letters a, b, c to be specified, it may be asked, how many of these combinations there are which do not contain one of these letters, how many contain only one, how many two, and how many all the three; we find

1°. Not one of the three letters	$3C0 \times 7C4 = 1 \times 35 =$	35
2°. Only one	$3C1 \times 7C3 = 3 \times 35 =$	105
3°. Two	$3C2 \times 7C2 = 3 \times 21 =$	63
4°. Three	$3C3 \times 7C1 = 1 \times 7 =$	7
Total number of combinations		210

As to the *permutations of m letters p and p, which contain p' letters taken from among m' that are specified*, their number $Y = X \times 1.2.3\ldots p$. For we have only to take each of the X combinations, and form the permutations p and p of the p letters that enter into it.

But if it be also proposed that the m' letters should each of them occupy throughout a particular and previously assigned place, the permutations in the different terms of X will then have to be taken only in respect to the p'' letters not selected; whence

$$Y = X \times 1.2.3\ldots p'' = [m'Cp'] \times [m''Pp''].$$

This, however, is on the assumption that the m' selected letters, which have their p' places fixed, may occupy any of them indifferently; for, otherwise, they may be changed among themselves in the places

assigned, and the preceding product must then be multiplied by $1.2.3\ldots p'$, or

$$Y = [m'Pp'] \times [m''Pp'']$$

479. *To form all the possible permutations p and p of the m letters* $a, b, c\ldots v$. Set off $p-1$ of the letters, as $i, k\ldots v$; and place one of them, as i, by the side of each of the other $m-p+1$ letters $a, b, c\ldots h$; whence ia, ib, $ic\ldots$ Let i be now changed successively into $a, b, c\ldots h$, and we shall have all the arrangements 2 and 2 of the $m-p+2$ letters $a, b\ldots h, i$. At the head of each of these results place the next suppressed letter k; then change k successively into $a, b\ldots h, i$, and we shall have all the permutations 3 and 3 of the $m-p+3$ letters $a, b\ldots i, k$; and so on.

For instance, to form the permutations 3 and 3 of the five letters a, b, c, d, e, we set off d and e; and first of all, placing d by the side of a, b, c, we have da, db, dc; then changing d into a, b, and c, there result all the arrangements 2 and 2 of the 4 letters a, b, c, d:

$$da,\ db,\ dc,\ ad,\ ab,\ ac,\ ba,\ bd,\ bc,\ ca,\ cb,\ cd.$$

It remains to place e at the head of each of these terms; (eda, edb, $edc\ldots$); then to change e into a, into b, into c, and into d; and we shall have the 60 arrangements required.*

480. Let it be proposed to *form the several combinations* p and p. In the first place, to obtain those of 2 letters each, take a and prefix it to $b, c\ldots$, (ab, ac, $ad\ldots$); these will be the combinations 2 and 2 into which a enters. In like manner, place b by the side of $c, d\ldots$; then c by the side of the letters $d, e\ldots$ on its *right*, &c., and we shall have all the combinations 2 and 2.

To obtain the combinations 3 and 3, combine all the letters but a, 2 and 2, in the manner just mentioned; then annex a to each term, b to each of those in which b is not found already, c to each of those which contain neither b nor c, &c., and we shall have the combinations 3 and 3.

Generally, to form the combinations p and p, set off $p-2$ letters $i, k\ldots v$, and combine the others $a, b, c\ldots h$ 2 and 2; annex to each result one of the suppressed letters i, then a to the terms without a,

---

* This theory will enable us to discover the *enigma* or *anagram* that may be formed from the letters of a particular word. These laborious trifles are sometimes very happy in their results. In *Frère Jacques Clément*, the assassin of Henry III., we find, letter for letter, *C'est l'enfer qui m'a créé.* Jablonski formed the anagrams of *Domus Lescinia*, in favour of Stanislaus, of the house of the Leczinski, and discovered the following: *Ades incolumis, omnis es lucida, mane sidus loci, sis columna Dei, I scande solium.* The last was prophetic: Stanislaus became king of Poland.

b to those containing neither a nor b, &c., and you will have the combinations 3 and 3 of the letters $a, b, c \dots h, i$; again, introduce l into each term, a into each of those which do not contain a, &c., and you will have the combinations 4 and 4 of $a, b \dots i. l$; and this must be continued till all the $p - 2$ letters have been introduced.

DEVELOPMENT OF THE POWER OF A POLYNOMIAL.

481. When we make $a = b = c\dots$, the product of m factors $(x + a)(x + b)(x + c)\dots$ becomes $(x + a)^m$; and consequently, the development of the m^{th} power of a binomial reduces itself to the effecting this product, and then making the 2nd terms $a, b, c\dots$ equal; a process which allows of our recognizing the law observed by the different terms of the product, before they undergo the requisite reduction. But it has been seen [N°. 97, 4°.] that this product is of the form

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} \dots + abcd\dots,$$

A being the sum $a + b + c\dots$ of the 2nd terms of the binomial factors, B the sum $ab + ac + bc\dots$ of their products 2 and 2, C that of the products 3 and 3, $abc, abd\dots$, &c.; whence, making $a = b = c\dots$, the several terms of A become $= a$, those of B are each $= a^2$, those of $C = a^3\dots$; those of $N = a^n$.

Hence, A becomes a repeated m times, or ma.

For B, a^2 must be repeated as many times as there are products 2 and 2, or $B = a^2 [mC2] = \frac{1}{2}m(m-1)a^2$.

For C, a^3 is to be taken as many times as there are combinations given by m letters 3 and 3; or $C = \frac{1}{6}m(m-1)(m-2)a^3$; and so on.

For a term Na^{m-n} of any rank n, we have $N = [mCn]a^n$; and, finally, the last term is a^m.

Hence we have this formula, discovered by Newton:

$$(x+a)^m = x^m + max^{m-1} + m.\frac{m-1}{2}a^2x^{m-2} + m.\frac{m-1}{2}.\frac{m-2}{3}a^3x^{m-3} \dots + a^m \dots (6).$$

$$T = [mCn]a^nx^{m-n} = m.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m-n+1}{n}a^nx^{m-n} \dots (7),$$

T being the term which has n terms before it, or the general term from which all those of the development of $(x + a)^m$ may be deduced by taking $n = 1, 2, 3\dots$

To obtain the development of $(x - a)^m$, a must be changed in the above formula into $-a$; *i. e.* the terms in which a appears with an odd exponent must be taken with a contrary sign.

482. The formula (6) *is composed of $m + 1$ terms, and the coefficients are all integral*; those of the first 20 powers have been already given

[page 6]. The exponents of a go on increasing by unity each term; and those of x decrease by the same quantity, the sum of these two powers of a and x being m for each term; so that [page 4, 1°.], *if any term be multiplied by $\frac{a}{x}$, and by the exponent of x, and then be divided by the number of the term in the series, we shall have the term following.* For example, we find

$$(x+a)^9 = x^9 + 9ax^8 + 36a^2x^7 + 84a^3x^6 + 126a^4x^5 + 126a^5x^4 + \ldots$$

To obtain $(2b^3 - 5c^3)^9$, we shall, in this equation, assume $x = 2b^3$, and $a = -5c^3$, when there will result $2^9b^{27} - 9.5c^3.2^8b^{24} + 36.5^2.c^6.2^7b^{21}\ldots$, or

$$(2b^3 - 5c^3)^9 = 512b^{27} - 45.256c^3b^{24} + 36.25.128c^6.b^{21}\ldots$$

Moreover, we know that, in formula (6),

1°. Beyond the middle term, the coefficients recur in inverse order, those equally distant from the two extremes being equal; these coefficients go on increasing up to the middle term, the value of which has been given [page 5, 2°.]

2°. Any one of the coefficients of the m^{th} power, being added to the one that follows it, gives the coefficient of the $(m+1)^{th}$ power, which has the same rank as the latter of the coefficients [see page 6].

3°. The sum of all the coefficients of the m^{th} power is $= 2^m =$ the sum of all those of the even, or of the odd, orders in the $(m+1)^{th}$ power, as appears from page 7. And in fact, making $x = a = 1$, the equation (6) reduces itself to $2^m =$ the sum of all the coefficients.

4°. When $x = 1$, and $a = z$, the equation (6) becomes

$$(1+z)^m = 1 + mz + m.\frac{m-1}{2}z^2 + m.\frac{m-1}{2}.\frac{m-2}{3}z^3\ldots + z^m \ldots\ldots (8).$$

This expression being much simpler than that of formula (6), we shall bring any proposed power whatever under the same form. Thus, for $(A+B)^m$, we shall divide the binomial by A, in order to reduce the first term to unity; multiply by A^m, in order to restore the quantity to its proper value $= A^m\left(1+\frac{B}{A}\right)^m$; and then, making the fraction $\frac{B}{A} = z$, we shall have a case of equation (8). Having therefore *formed the consecutive products of the factors* m, $\frac{1}{2}(m-1)$, $\frac{1}{3}(m-2)$, $\frac{1}{4}(m-3)\ldots$ in the manner stated [page 4], we shall have the coefficients of the development, and these must then be multiplied by the successive ascending powers of z. For instance, to develop $(2a+3b)^8$, we shall take $(2a)^8\left(1+\frac{3b}{2a}\right)^8$, and make $\frac{3b}{2a} = z$. We

then form the fractions $\frac{8}{1}, \frac{7}{2}, \frac{6}{3}, \frac{5}{4}\ldots$, and, by successive multiplication, obtain the coefficients 8, 28, 56, 70; the last of which belonging to the middle term, the following coefficients are 56, 28, 8. Lastly, annexing to these coefficients, taken in order, the ascending powers $z, z^2, z^3\ldots$, multiplying the whole by $256a^8$, and then putting for z the fraction which it represents, we obtain

$$(2a + 3b)^8 = 256a^8 + 3072a^7b + 16128a^6b^2 + 48384a^5b^3 + 90720a^4b^4 \\ + 108864a^3b^5\ldots\ldots$$

483. To develop $(a + b + c + d\ldots + i)^m$, we shall reduce it to a binomial form by assuming $b + c\ldots + i = z$: then

$(a + z)^m$ has for its general term............... $[mC\alpha]\ a^\alpha z^p$,

α and p being any numbers whatever, provided only that $\alpha + p = m$. If now we assume $c + d\ldots + i = y$, we have $z = b + y$, and the general term of $z^p = (b + y)^p$ is .... $[pC\beta]\, b^\beta y^q$,

with the condition $\beta + q = p$; *i. e.* $\alpha + \beta + q = m$.

In like manner, assuming $d + e\ldots + i = x$, the general term of

$y^q = (c + x)^q$ is .... $[qC\gamma]\, c^\gamma x^r$,

where $\gamma + r = q$, or .... $\alpha + \beta + \gamma + r = m$.

Ascending upwards through these successive substitutions, it is evident that the general term of the development required is

$$N = [mC\alpha].[pC\beta].[qC\gamma]\ldots a^\alpha b^\beta c^\gamma\ldots i^u.$$

The sum $\alpha + \beta + \gamma + \ldots + u$ must be $= m$; but otherwise $\alpha, \beta, \gamma\ldots$ are arbitrary numbers, denoting the ranks of the several general terms in their respective series. The denominator of the coefficient of N is $1.2.3\ldots\alpha \times 1.2.3\ldots\beta\ldots$, taking as many series of factors as there are exponents, with the exception of the last u. For the sake of analogy, let the product $1.2.3\ldots u$ be introduced into the denominator, and consequently into the numerator also, which will then take the form

$$m(m-1)\ldots(m-\alpha+1) \times p(p-1)\ldots(p-\beta+1) \times q\ldots(q-\gamma+1)\ldots u(u-1)\ldots 2.1.$$

But $p = m - \alpha$, and therefore the factors $p, p - 1\ldots$ continue the series $m(m-1)\ldots$ down to $(p - \beta + 1)$, when it is still continued by $q = p - \beta$, and so on, to $u(u-1)\ldots 2.1$; so that the numerator is the series of decreasing factors $m(m-1)\ldots$ down to 2.1, and may consequently be written $1.2.3\ldots(m-1)\,m$. The general term required is therefore

$$N = \frac{1.2.3\ldots m \times a^\alpha.b^\beta.c^\gamma\ldots i^u}{1.2.3\ldots\alpha \times 1.2.3\ldots\beta \times 1.2.3\ldots\gamma \times 1.2\ldots u}\ldots\ldots\ldots(9).$$

The exponents $\alpha, \beta, \gamma\ldots$ are all the positive and integral numbers whatever from 0, with the condition only that their sum $= m$; and the development must admit as many terms of this form as there can be

found values that fulfil the above condition, among all the combinations possible. The denominator is composed of as many series of factors $1.2.3\ldots\alpha$, $1.2.3\ldots\beta\ldots$, as there are exponents.

Thus, for $(a+b+c)^{10}$, one of the terms is

$$\frac{1.2.3\ldots 10\, a^5b^3c^2}{1.2.3.4.5 \times 1.2.3 \times 1.2} = 2520\, a^5b^3c^2,$$

and the same coefficient will affect the terms $a^3b^2c^5$ and $a^2b^3c^5\ldots$

484. In all this the exponent m has been supposed to be a *positive integer;* if it be not so, the development of $(1+z)^m$ is still unknown, and it remains to prove that, under all circumstances, it will retain the same form (8). The investigation in regard to $(1+z)^m$ will comprehend the proposition for the developing any polynomial whatever; for, multiplying the equation (8) by x^m, we have the series for $(x+xz)^m$, or, making $xz=a$, for $(x+a)^m$. This transformation, therefore, brings us to the equation (6), which being thus demonstrated for any exponent m whatever, we may then apply the principle of N°. 483.

Hence, m and n denoting any numbers whatever, assume

$$x = 1 + mz + \tfrac{1}{2}m(m-1)z^2 + \&c.,$$
$$y = 1 + nz + \tfrac{1}{2}n(n-1)z^2 + \&c.;$$

making $p=m+n$, we deduce

$$xy = 1 + pz + \tfrac{1}{2}p(p-1)z^2 + \&c.$$

In fact, without troubling ourselves with the multiplication of the polynomials x and y, which would only give us the first terms of an indefinite series, without informing us of the law which it follows, we may observe that, if m and n are integral and positive, it has been already proved that $x=(1+z)^m$, $y=(1+z)^n$, whence $xy = (1+z)^{m+n} = (1+z)^p$; and in this case the product xy will be such as we have assumed it to be. And though m or n be not integral and positive, the result must be the same, since the rules for the multiplication of two polynomials do not depend on the values that may be assigned to the letters of the factors. For instance, the term involving z^2 in xy, must be the product of certain terms of x and y, terms which will be the same whatever be the values of m and n; and since this product is $\tfrac{1}{2}p(p-1)z^2$ in one case, it must be so in all others.

Hence,

1°. If m be *integral* and *negative*, since n is arbitrary, assume $n=-m$; n then will be integral and positive, in which case we know that $y=(1+z)^n$; also $p=0$ reduces the third equation to $xy=1$, whence $x=y^{-1}=(1+z)^{-n}=(1+z)^m$.

2°. When m is a *fraction, positive or negative,* assume $n=m$; whence $p=2m$, $xy=x^2$, and consequently $x^2=1+pz+\ldots$

Let this last equation be again multiplied by x; then we shall have
$$x^3 = 1 + qz + \tfrac{1}{2}q(q-1)z^2 + \ldots,\ q \text{ being} = n + p = 3m;$$
similarly,
$$x^4 = 1 + rz + \ldots, \text{ where } r = 4m, \text{ and lastly,}$$
$$x^k = 1 + lz + \tfrac{1}{2}l(l-1)z^2 + \ldots, \text{ where } l = km.$$

Let k be taken = the denominator of the fraction m; then km or l will be integral, in which case it has been already proved that the development is that of $(1+z)^l$; hence $x^k = (1+z)^l$, and consequently, since $l = km$, we have $x = (1+z)^m$.

3°. m being *irrational or transcendental* [see note, N°. 516]. Let n and h be two numbers between which m is comprised; since then each term of $x = 1 + mz\ldots$ lies between the corresponding terms in the series $(1+z)^n$ and $(1+z)^h$, it is obvious that x itself lies between these two series, the difference between which may be diminished *ad libitum*. Hence $(1+z)^n$ approximates indefinitely to x, as n approximates to m; let α be the difference, or $(1+z)^n = x + \alpha$.

If, in like manner, β be the difference between $(1+z)^n$ and $(1+z)^m$, we have $1+z)^n = (1+z)^m + \beta$, whence $x + \alpha = (1+z)^m + \beta$, where α and β allow of continued diminution; and consequently (N°. 113) $x = (1+z)^m$.

4°. And lastly, *the exponent being imaginary*: it is only by convention that expressions of this sort can be treated according to the same rules with those that are real, since no just idea can be formed of a calculation, the elements of which are symbols that represent no real magnitudes; thus, in the present case, there is no room for any demonstration.

485. We shall now apply the formula (6) to some examples.

I. To develop $\dfrac{a}{\alpha + \beta x} = \dfrac{a}{\alpha}\cdot\dfrac{1}{1+kx}$, k being $= \dfrac{\beta}{\alpha}$, we must form the series for $(1+kx)^{-1}$ [N°. 482, 4°]. The coefficients will in this case have for their factors -1, $\tfrac{1}{2}(-1-1)$, $\tfrac{1}{3}(-1-2)\ldots$, which are all equal to -1, and their products therefore are alternately $+1$ and -1; whence we get the progression by quotient,
$$1 - kx + k^2x^2 - k^3x^3 + \ldots, \text{ the ratio of which is } -kx.$$
Consequently,
$$\frac{a}{\alpha+\beta x} = \frac{a}{\alpha}\left(1 - \frac{\beta x}{\alpha} + \frac{\beta^2x^2}{\alpha^2} - \frac{\beta^3x^3}{\alpha^3}\ldots \pm \frac{\beta^n x^n}{\alpha^n}\ldots\right).$$

II. For $\sqrt{(a^2 \pm x^2)}$ we shall write $a\sqrt{\left(1 \pm \dfrac{x^2}{a^2}\right)} = a\sqrt{(1 \pm y^2)}$,

making $x = ay$. In order now to develop the power $\frac{1}{2}$ of $1 \pm y^2$, we must combine the factors $\frac{1}{2}$, $\frac{1}{2}(\frac{1}{2}-1)$, $\frac{1}{2}(\frac{1}{2}-2)$... or $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{6}$, $-\frac{5}{8}$..., and we shall find the coefficients to be fractions, the numerators of which are formed of the odd factors 1. 3. 5. 7..., and the denominators of the even factors 2. 4. 6. 8. Hence there results

$$\sqrt{(1 \pm y^2)} = 1 \pm \frac{y^2}{2} - \frac{1.y^4}{2.4} \pm \frac{1.3.y^6}{2.4.6} - \frac{1.3.5.y^8}{2.4.6.8} \pm \dots\dots,$$

$$\sqrt{(a^2 \pm x^2)} = a\left(1 \pm \frac{x^2}{2a^2} - \frac{1.x^4}{2.4a^4} \pm \frac{1.3.x^6}{2.4.6.a^6} - \frac{1.3.5.x^8}{2.4.6.8\,a^8} \pm \dots\right).$$

III. We shall in like manner obtain

$$(1 \pm y^2)^{-\frac{1}{2}} = 1 \mp \frac{1.y^2}{2} + \frac{1.3.y^4}{2.4} \mp \frac{1.3.5.y^6}{2.4.6} + \frac{1.3.5.7.y^8}{2.4.6.8} \mp \dots,$$

$$(a^2 \pm x^2)^{-\frac{1}{2}} = \frac{1}{a}\left(1 \mp \frac{x^2}{2a^2} + \frac{1.3.x^4}{2.4.a^4} \mp \frac{1.3.5.x^6}{2.4.6.a^6} + \frac{1.3.5.7.x^8}{2.4.6.8.a^8} \mp \dots\right),$$

$$\sqrt[3]{(a+x)} = \sqrt[3]{a}\left(1 + \frac{x}{3a} - \frac{x^2}{9a^2} + \frac{5x^3}{81a^3} - \frac{10x^4}{243a^4} + \frac{22x^5}{729a^5}\dots\right),$$

$$\sqrt[3]{(1-y^3)} = \left(1 - \frac{y^3}{3} - \frac{y^6}{9} - \frac{5y^9}{81} - \frac{10y^{12}}{243} + \frac{22y^{15}}{729} - \dots\right),$$

$$(1-a)^{-2} = 1 + 2a + 3a^2 + 4a^3 \dots + (n+1)\,a^n \dots$$

486. When m is a prime number, the coefficients in the development of $(x+a)^m$ are all multiples of m, with the exception of those of the terms x^m and a^m; for the equation (3), page 3 gives

$$1.2.3\dots p \times [mCp] = m\,(m-1)\,(m-2)\,\dots\dots\,(m-p+1);$$

and, since the second side is a multiple of m, the first must be so also; but m is supposed to be prime and $> p$, and therefore mCp must be divisible by m. It may be proved in the same manner that all the terms of $(a+b+c\dots)^m$ are multiples of m, excepting $a^m + b^m + c^m$; and consequently, K denoting some integral quantity, we have

$$(a+b+c\dots)^m = a^m + b^m + c^m \dots + mK.$$

If we assume $1 = a = b = c\dots$, and h be the number of terms in the polynomial, we find $h^m = h + mK$; whence $h^m - h$ is a multiple of m, or $\dfrac{h\,(h^{m-1}-1)}{m}$ = an integer. Hence, if the prime number m do not divide h, it must divide $h^{m-1} - 1$; which constitutes the *theorem of Fermat*, enunciated thus: *If the integer h is not a multiple of the prime number m, the remainder from the division of h^{m-1} by m is unity.*

This theorem may also be enunciated in the following manner: since $m-1$ is some even number, as $2q$, we may assume $h^{m-1}-1 = (h^q+1)(h^q-1)$; and m therefore must divide one or other of these two factors, *i. e.* the remainder from the division of h^q by m is ± 1, when m is a prime number > 2, and $q = \frac{1}{2}(m-1)$.

EXTRACTION OF FOURTH, FIFTH...... ROOTS.

487. The rules that we have given (N^{os}. 62 and 67) for the extraction of square and cube roots may now be extended to those of any degree. For instance, to obtain the 4th root of 548464, let the highest 4th power contained in this number be represented by A, the tens of the root by a, and the units by b. Since, then, $A = (a+b)^4 = a^4 + 4a^3b...$, the first term a^4 is the 4th power of the figure of the tens, to the right of which four cyphers should be annexed. Separating, therefore, the four figures 8464, we see that 54 contains the fourth power of the tens' figure, considered as simple units; and since 16 is the highest fourth power comprised in 54, it follows that 2, the fourth root of 16, is the tens' figure. Subtracting 16 from 54, and reinstating the figures that were set apart, the remainder 388464 contains the four other parts of $(a+b)^4$, or $4a^3b + ...$ But $4a^3b$ is terminated by three cyphers arising from a^3; whence, marking off the three figures 464, the remainder 388 contains four times the product of the units b by the cube of the figure 2 of the tens, considered as simple units, or $4 \times 8b = 32b$; the same remainder contains also the thousands arising from $6a^2b^2 + ...$ The quotient 10, of 388 divided by 32, will therefore be b or $> b$; and b must in fact be reduced to 7, or the root to 27, as may be proved, in the same manner as for the cube root [see vol. i. page 75], by forming, in the manner given below, the quantity $b(4a^3 + 6a^2b + 4ab^2 + b^3)$. We find for the remainder 17023. To carry the approximation farther, we must add four cyphers, mark off three of them, and divide 170230 by $4a'^3$, making $a' = 27$. Since $4a'^3 = 4a^3 + 12a^2b + 12ab^2 + 4b^3$, it appears that, to form this divisor $4a'^3$, $6a^2b + 8ab^2 + 3b^3$ must be added to the part within the brackets above, &c.

54·8464	27·2 *root*			
16				
	32 *1st divis.*	$4a^3$	....	53063
388·464	168............	$6a^2b$	....	168
371441	392	$4ab^2$	...*doubled*...	784
	343.........	b^3	...*trebled*...	1029
170230				
	53063 × 7 = 371441 *2nd divis.*			78732 = 4 × 27^3.

It is easily seen that this course of calculation, so convenient for

determining each partial divisor, is general, whatever be the degree of the root to be extracted.

488. The logarithmic tables render these extractions very easy; but they prove insufficient when we wish to approximate to the root within closer limits than those to which the tables extend. In this case we make use of the following methods.

I. The series [II. page 14] enable us to extract square roots with a very great degree of accuracy. To obtain $\sqrt{N}$, divide N into two parts a^2 and $\pm x^2$, so that the first shall be an exact square, and very large in comparison with the second; $\sqrt{N} = \sqrt{(a^2 \pm x^2)}$ will be given by a highly convergent series. Let $\sqrt{2}$, for example, be required: we have $\sqrt{8} = 2\sqrt{2}$; and since $8 = 9 - 1$, we shall take $a = 3$, $x^2 = 1$; whence $\sqrt{8} = 3\,(1 - \frac{1}{18} - \frac{1}{648}\ldots)$. To render the series more rapidly convergent, take the three first terms, which amount to 2·829, and compare the square of this fraction with 8; it will be seen that $8 = 2{\cdot}829^2 - 0{\cdot}003241$; whence

$$\sqrt{8} = 2{\cdot}829 \times \sqrt{\left(1 - \frac{3{\cdot}241}{2829}\right)} = 2{\cdot}8284271247784;$$

and, finally, taking the half, you have $\sqrt{2} = 1{\cdot}4142135623892$. The logarithmic tables give the first approximation, which is then extended by the above process.

We must be careful to take into account all the terms of the series, which, when reduced to decimals, have significant figures in the order of those which it is intended to retain in the result; the first term neglected should commence with 0·000000... to one rank farther than the degree of approximation required.

That we may be at liberty to consider the first part of a series as forming an approximate value of its whole sum, it is necessary that the series be *convergent* [see the ex. of N°. 99;] and for this it is not enough that the initial terms decrease, since it is possible that, farther on, they may proceed to increase. But take the *general term I*, or the n^{th} term of the series; in its expression change n into $n - 1$, and divide by the result; the quotient will be the factor which, being multiplied into the $(n-1)^{th}$ term, produces the n^{th}; and accordingly as this factor is $>$ or $<$ 1, the series will, at this point, go on increasing or decreasing. In order therefore *that the convergence may extend to infinity, this factor must be $<$ 1, however great we suppose n to be.*

Thus for $(x + a)^m$, it follows from equation (4) that the quotient of which we speak is in this case $\frac{m-n+1}{n}\cdot\frac{a}{x}$; and this is the factor which changes the $(n-1)^{th}$ term into the n^{th} [N°. 482]. Assuming this fraction

< 1, we shall have $n > \frac{(m+1)a}{x+a}$; whence we may feel assured that, having arrived at a certain rank, the series will subsequently be convergent on to infinity. It is so from the very commencement, when $a < x$ and $m < 1$, which is the case in the example given above.

For the series $x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} \ldots \frac{x^{2n-1}}{2.3\ldots 2n-1}\ldots$, which is that for sin. x [see N°. 587], changing n into $n-1$, dividing, &c. we find, for the factor which leads from one term to the next, $\frac{x^2}{(2n-2)(2n-1)}$; whence the condition for convergence is $(2n-2)(2n-1) > x^2$. Assuming $2n.2n > x^2$, or $n > \frac{1}{2}x$, we see that the condition is fulfilled so long as $n > \frac{1}{2}x$.

II. Suppose that we are already acquainted with an approximate value a of the m^{th} root of a given number N, which has been divided into a^m and b, the correction x due to a being very small. We have then

$$N = a^m \pm b, \text{ and } \sqrt[m]{N} = a \pm x, \text{ whence } a^m \pm b = (a \pm x)^m,$$

where b and x are supposed to be very small relatively to a. Developing, we have $b = x\,(ma^{m-1} \pm A'xa^{m-2} + A''x^2a^{m-3}\ldots)$, $m, A', A''\ldots$being the coefficients of the equation (6), page 10. For a first approximation, neglect the small terms in $x^2, x^3\ldots$, *i. e.* assume $b = mxa^{m-1}$; whence the correction x will be derived, within a very little. This value of x being then substituted in the term $A'xa^{m-2}$, and the following terms neglected, we obtain a new equation, which leads to this still more approximate value

$$x = \frac{2ab}{2ma^m \pm (m-1)\,b};$$

and this quantity, substituted in $\sqrt[m]{N} = a \pm x$, gives the approximate root of N. For instance, for $m = 2$ and 3 we find

$$\sqrt{N} = \sqrt{(a^2 \pm b)} = a \pm \frac{2ab}{4a^2 \pm b} = a \pm \frac{2ab}{3a^2 + N}$$

$$\sqrt[3]{N} = \sqrt[3]{(a^3 \pm b)} = a \pm \frac{ab}{3a^3 \pm b} = a \pm \frac{ab}{2a^3 + N}.$$

The approximation may be carried on with very great rapidity by making use of these formulæ several times successively, as we did in order to obtain $\sqrt{8}$. Assume $a = 2.8$; whence $a^2 = 7{\cdot}84$, $b = +\,0{\cdot}16$, and $\sqrt{8} = 2{\cdot}8 + \frac{0{\cdot}16}{...}$ $= 2{\cdot}82842$. Assuming now $a = 2{\cdot}82842$, whence a^2 and b, we arrive in the end at the same value of $\sqrt{8}$, that has been obtained previously.

FIGURATE NUMBERS.

489. This name is given to the numbers following:

1st order	1.	1.	1.	1.	1.	1.	1.	1.	1.	1....
2nd......	1.	2.	3.	4.	5.	6.	7.	8.	9.	10....
3rd......	1.	3.	6.	10.	15.	21.	28.	36.	45.	55....
4th......	1.	4.	10.	20.	35.	56.	84.	120.	165.	220....
5th......	1.	5.	15.	35.	70.	126.	210.	330.	495.	715....
6th......	1.	6.	21.	56.	126.	252.	462.	792.	1287.	2002....
7th......	1.	7.	28.	84.	210.	462.	924.	1716.	3003.	&c.

The law which these numbers follow is this: *Each term is the sum of the one on its left, added to that which is immediately above it;* $2002 = 1287 + 715$. From this generation, compared with that of the table [page 6], we shall conclude that the numbers are the same, but ranged in a different order. A line of the former table, as 1, 7, 21, 35... is in the present one an hypothenuse; and we therefore have $T = [mC(p-1)]$ for the value of any term of the order p, or that is taken in the p^{th} line, and on the m^{th} hypothenuse.

Taking two consecutive lines:

$$(p-1)^{th} \text{ order} \ldots\ldots\ldots 1.a\ldots\ldots\ldots q.\ r.\ s.\ t.\ v\ldots,$$
$$p^{th}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1.A\ldots\ldots\ Q.\ R.\ S.\ T.\ V\ldots,$$

we have $\qquad A = 1 + a,\ldots,\ R = Q + r,\ S = R + s,\ T = S + t\ldots$

1°. These equations being all added together, the result is $T = 1 + a\ldots r + s + t$; so that any term T is the sum of all the terms of the preceding order, up to the one t which is in the same vertical column; or, in other words, *the general term of the order* p *is the term of summation of the order* $p-1$.

2°. It will in like manner be seen that *any term is the sum of the preceding column limited to the same order.* This results also from the fact that the p^{th} column *is formed of the same numbers as the order* p; for these terms, taken two and two, are those which recur on the same hypothenuse, as being equally distant from the extremes [see page 4, 1].

3°. On any hypothenuse, the terms range themselves one place ahead of each other in the consecutive lines. Thus, for T and v, if T is the n^{th} term of the order p, or in the n^{th} column and the p^{th} line, v is the $(n+1)^{th}$ term of the order $p-1$; the term of the preceding line is the $(n+2)^{th}$ of the order $p-2\ldots$; and in order therefore to ascend to the 2nd order $1.2.3\ldots m$, we must to the rank n add $p-2$, the difference of the two orders; *i.e.* the term m, the N° of the hypothenuse, will occupy the rank $n+p-2$, or $\qquad m = n + p - 2$.

Consequently the equation $T = mC(p-1)$ reduces itself [page 4] to

$$T = [(n+p-2)\,C\,(p-1), \text{ or } (n-1)]\ldots (10);$$

whence, developing by the equation (3), page 3, and taking the factors of the numerators in inverse order,

$$T = \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdots \frac{n+p-2}{p-1} = \frac{p}{1} \cdot \frac{p+1}{2} \cdots \frac{p+n-2}{n-1}.$$

The first or the second of these expressions for *the general term* T is used in preference, accordingly as p is $<$ or $>$ n. This also serves to prove that the n^{th} term of the order p is the same as the p^{th} term of the order n.

Assuming $p = 3, 4, 5 \ldots$, we have

3rd order $1.3.\ 6.10 \ldots T = \frac{1}{2} n(n+1) = (n+1)\, C2$;

4th ...... $1.4.10.20 \ldots T = \frac{1}{6} n(n+1)(n+2) = (n+2)\, C3$;

5th .... $1.5.15.35 \ldots T = \frac{1}{24} n(n+1)(n+2)(n+3)$; &c.

In the development of $(x+a)^{-h}$, we have [N°. 482, 4°.] for coefficients

$$1, -h, \tfrac{1}{2}h(h+1), -\tfrac{1}{6}h(h+1)(h+2), \ldots$$

If now h be integral, these factors enter into the equation (10), when p is replaced in it by h; so that the successive coefficients of the power $-h$ of a binomial are the p^{th} column, or the p^{th} line of our table, with alternate signs: $\pm T = (h+n-2)\, C(h-1)$ or $(n-1)$.

For example,

$(x \pm a)^{-1}$ coeff.	1	∓ 1	$+1$	∓ 1	$+1$	...
$(x \pm a)^{-2}$	1	∓ 2	3	∓ 4	5	...
$(x \pm a)^{-3}$	1	∓ 3	6	∓ 10	15	...
$(x \pm a)^{-4}$	1	∓ 4	10	∓ 20	35	...

To obtain the *term of summation* Σ or the sum of the first n terms of the order p, in the table of N°. 489, it will be sufficient (1°.) *to investigate the n^{th} term of the order $p+1$*; *i. e.* in (10) to change p into $p+1$.

Comparing the terms T, t and S, we have

$T = mC(p-1)$, $t = (m-1)\, C(p-2)$, $S = (m-1)\, C(p-1)$;

whence, developing and reducing [equation 3, N°. 476], we find

$$T = \frac{n+p-2}{n-1} \times S = \frac{n+p-2}{p-1} \times t \ldots \quad (11).$$

These formulæ serve for deducing one from the other and step by step the terms which compose either the p^{th} line or the n^{th} column. Thus $p = 6$ gives $T = \frac{n+4}{n-1} . S$; and making $n = 2, 3, 4 \ldots$, we find $\frac{6}{1}, \frac{7}{2}, \frac{8}{3}, \frac{9}{4} \ldots$ for the multipliers of each term S of the 6^{th} order, giving by their product the following term T. For $n = 7$, $T = \frac{p+5}{p-1}$. t gives $\frac{7}{1}, \frac{8}{2}, \frac{9}{3} \ldots$ as the factors which lead from a term t of the 7^{th} column to the next term T.

This equation (11), when p is changed into $p+1$, T into Σ, and t into T, becomes $\Sigma = \frac{p+n-1}{p} \times T$, an equation which expresses the sum Σ of the series of the order p continued to the n^{th} term T. Thus, for the 7^{th} order, $\Sigma = \frac{1}{7}(n+6) . T$; and the 7^{th} series, terminated at the 9^{th} term 3003, has for its sum $\frac{1}{7} \times 15 \times 3003 = 6435$.

490. We have taken for the origin of our table the series 1. 1. 1. 1...; if we take $1. \delta. \delta. \delta...$, and follow the same course of generation, the 2nd order will be the equidifference $1. 1+\delta. 1+2\delta. 1+3\delta...$, and so on for the following orders, as we see in this table, of which the preceding one is but a particular case:

1st order	1.	$\delta.$	$\delta.$	$\delta.$	$\delta...$
2nd......	1.	$1+\delta.$	$1+2\delta.$	$1+3\delta.$	$1+4\delta...$
3rd	1.	$2+\delta.$	$3+3\delta.$	$4+6\delta.$	$5+10\delta...$
4th	1.	$3+\delta.$	$6+4\delta.$	$10+10\delta.$	$15+20\delta...$
5th	1.	$4+\delta.$	$10+5\delta.$	$20+15\delta.$	$35+35\delta...$
6th	1.	$5+\delta.$	$15+6\delta.$	$35+21\delta.$	$70+56\delta...$

It is evident that the terms have all the form $T = A + B\delta$; and on comparing the numbers with those of the first table, it will be found that A is the term of the same rank n in the preceding order $p-1$, whilst the factor B is the term of the same order p in the preceding rank $n-1$: thus,

$$T = n^{th} \text{ term of the order } (p-1) + [(n-1)^{th} \text{ term of the order } p]\delta$$

$$= [(n+p-3)\, C\,(n-2) \text{ or } (p-1)]. \left(\frac{p-1}{n-1} + \delta\right)$$

$$= (n-1)\frac{n}{2}.\frac{n+1}{3}\cdots\frac{n+p-3}{p-1}.\left(\frac{p-1}{n-1}+\delta\right);$$

which is the general term of this latter table. *The term of summation Σ of the order p is the general term of the order $p+1$*, as before. Thus, $p=3$ gives, for the third order,

$$T = n + \tfrac{1}{2}n\delta(n-1),\ \Sigma = \tfrac{1}{2}n(n+1)[1+\tfrac{1}{3}\delta(n-1)].$$

In the first of the subjoined examples we make $\delta = 2$, and the squares 1. 4. 9. 16... are derived from the odd progression 1. 3. 5. 7...; in the second series $\delta = 3$, &c.

1. 2. 2. 2. 2...	1. 3. 3. 3. 3...	1. 4. 4. 4...
1. 3. 5. 7. 9...	1. 4. 7. 10. 13...	1. 5. 9. 13...
1. 4. 9. 16. 25...	1. 5. 12. 22. 35...	1. 6. 15. 28...
$T = n^2$	$T = n.\frac{3n-1}{2}$	$T = n(2n-1)$
$\Sigma = n.\frac{n+1}{2}.\frac{2n+1}{3}$	$\Sigma = n^2.\frac{n+1}{2}$	$\Sigma = n.\frac{n+1}{2}.\frac{4n-1}{3}.$

491. If the side al [*fig.* 1] of the triangle alm be cut into $n-1$ equal parts, in the points $b, d, f\ldots$, and $bc, de, fg\ldots$ be drawn parallel to the base lm, these lines will increase in length in the same proportion as the numbers 1. 2. 3. 4... If now one point be placed in a, 2 on the line bc (one in b and c each), 3 on de, 4 on fg..., the sum of these points, commencing from a, is successively 1. 3. 6. 10...; and the triangle alm will contain as many of these points as is marked by the n^{th} of these numbers of the 3^{rd} order, which have, for this reason, been styled *triangular*. When the triangle is equilateral, these points are equidistant.

In like manner, in a polygon of m sides, draw diagonals from one of the angles a, and divide these lines and the sides of the angle a into $n-1$ equal parts: the corresponding points being then joined by straight lines, $n-1$ polygons will be formed having the angle a common, and $m-2$ sides parallel; and the perimeters of these sides will increase in the same proportion as 1. 2. 3. 4... Now place a point at each angle, one in the middle of the parallel sides of the 2nd polygon, 2 on each of the sides of the 3rd, &c.; these sides then will each contain 1, 2, 3... points more, and the $m-2$ parallel sides of any polygon will on the whole contain $m-2$ points more than in the preceding polygon. Make δ therefore equal to $m-2$, and the area of our polygon will contain the number of points (equidistant, if the figure is regular) expressed by the n^{th} term of the series of the 3rd order, derived from $1.\delta.\delta.\delta\ldots$ It is on this account that the names of *Square, Pentagonal, Hexagonal*... have been assigned to the numbers of these series, of which we have given the general terms and those of summation for $\delta = 2, 3, 4$, or $m = 4, 5, 6$; and generally, the numbers of the 3rd order are all called *polygonal*, because they may be contained, equidistantly from each other, in a polygonal figure.

Reasoning in the same manner for a trihedral angle, we shall see that the series 1. 4. 10. 20... represents the number of points that can be placed on parallel planes, whence these numbers have received the name of *Pyramidal*. The *polyhedral* numbers constitute the series of the 4th order, of which the general terms and those of summation are determined by making $p = 4$ and 5. Analogy has led us to generalize these ideas, and the appellation of *figurate numbers* has been given to all those which come under the law of N°. 489, and are comprised in the preceding table, though in fact these numbers cannot all be really represented by the figures of Geometry, beyond the 4th order.

ON THE PERMUTATIONS AND COMBINATIONS, WHEN THE LETTERS ARE NOT ALL DISSIMILAR.

492. Form the product of the polynomial $a+b+c\ldots$, taken several times as a factor, observing, in each term, to write the letter which is the multiplier in the 1st rank, and to leave each letter of the multiplicand in its place.

$$
\begin{array}{ll}
A\ldots & a+b+c\ldots \\
 & \underline{a+b+c\ldots} \\
B\ldots & aa+ab+ac\ldots \\
 & ba+bb+bc\ldots \\
 & ca+cb+cc\ldots \\
 & \ldots\ldots\ldots\ldots \\
 & \underline{a+b+c}
\end{array}
$$

$$
\begin{array}{ll}
C. & aaa+aab+aac\ldots+aba+abb+abc\ldots+aca+acb+acc\ldots \\
 & baa+bab+bac\ldots+bba+bbb+bbc\ldots+bca+bcb+bcc\ldots \\
 & caa+cab+cac\ldots+cba+cbb+cbc\ldots+cca+ccb+ccc\ldots
\end{array}
$$

The product B is formed of the permutations 2 and 2 of the letters $a, b, c\ldots$; C, of the permutations 3 and 3, &c., admitting that any letter may enter 1, 2, 3... times into each term; and so on for the others. For, that two arrangements 3 and 3 in which a is the initial letter may be repeated twice, or that one of them may be omitted in C, the system of the two letters on the right of a, must be an arrangement of 2 letters itself so repeated or omitted in B.

The product B has m lines and m terms in each line, m being the number of letters $a, b, c\ldots$; so that there are m^2 arrangements 2 and 2. The product C has also m lines, each consisting of m^2 terms, which makes m^3 arrangements 3 and 3...; and, lastly, *m^n is the number of permutations n and n of m letters, when each letter may enter* 1, 2, 3... *n times in the results;* n may also be $> m$. For instance, 9 digits, taken 4 and 4, give 9^4, or 6561 different numbers.

The sum of the arrangements of m letters, taken 1 and 1, 2 and 2, 3 and 3... n and n, is $m+m^2+m^3\ldots+m^n = m.\dfrac{m^n-1}{m-1}$. With 5 figures, taken singly, or 2, or 3 together, $\frac{5}{4}(5^3-1)$ or 155 different numbers may be written.

Suppose there are n dice $A, B, C\ldots$, each having f faces marked with the letters $a, b, c\ldots$; a throw of these dice will produce a system such as $abacc\ldots$ If now we take the first die A, and turn up its different faces successively, without making any change in the rest, the above system will produce f; so that our n dice give f times more results than the $(n-1)$ other dice $B, C\ldots$ Two dice therefore give f^2 throws, 3 give f^3, 4 give $f^4\ldots$, and *n dice with f faces produce f^n different throws.* We here consider identical results, as different, when they are given by different dice.

If the 1st die has f faces, the 2nd f', the 3rd f''..., the number of throws is $f \times f' \times f''$...

493. Suppose there are m vacant places A, B, C..., which are to be occupied by m letters, viz. α places by a, β places by b, &c.: let us see in how many different ways this distribution can be effected. It is manifest that to place the α letters a, nothing more is requisite than to take α of the letters A, B, C..., and put them equal to a; and this can be done in as many different ways as it is possible to equate α of the letters A, B, C... to a; $[mC\alpha]$ therefore marks in how many ways α places can be occupied, among m which are vacant.

There remain, in each term, $m - \alpha$ vacant places, of which β may be filled by the letter b, in as many ways as $(m - \alpha)\ C\beta$ denotes; and the product $mC\alpha \times (m - \alpha)\ C\beta$ indicates in how many ways α letters a and β letters b can be distributed, in m vacant places.

In the $m - \alpha - \beta$ places which remain unoccupied, γ letters c must be placed, and each term will produce a number $(m - \alpha - \beta)\ C\gamma$...; and so on, till there are no longer any places vacant, which will be the case when we have $\theta C\theta = 1$. Consequently, *if we wish to distribute the m factors $a^\alpha b^\beta c^\gamma$... in all the different ways possible, or to form all the arrangements that the factors will admit of, the results will be equal in number to n,* formula (9) p. 12, which is the coefficient of the general term of a polynomial.

For example, the 10 factors $a^4b^3c^2d$ form permutations the number of which is $N = \dfrac{1.2.3\ldots 10}{2.3.4 \times 2.3 \times 2} = 12600$. The 7 letters of the word *Etienne* may be arranged in 420 different ways.

This coefficient N expresses also *how many throws there are which, with m dice having each f faces, can produce a given result.* For if these dice have on their faces the f letters a, b, c..., and it be proposed that α of the dice should present the face a, this will be the same as though α letters a ought to take their place in ranks the number of which is m; which gives $mC\alpha$ throws for turning up the α letters a. That β of our $m - \alpha$ other dice may now present the face b, β places must in like manner be filled up among the $m - \alpha$ vacant ones; whence each of our preceding results will produce $(m - \alpha)\ C\beta$; and so on.

494. Let us next investigate *the number of combinations of the letters* $a.b.c...$, *allowing that each factor may appear several times in the different terms* (as in N°. 492, except that the order of the factors is here indifferent.) Multiply the polynomial $a + b + c...$ several times by itself, taking as factors of any term $a, b, c...$ those terms only of the multiplicand which are in the same column with it, or on its left. It is evident then that we shall have for our successive results the required combinations 2 and 2, 3 and 3...

$$\begin{array}{rrrr}
a + & b + & c + & d... \\
a + & b + & c + & d... \\
\hline
aa + & bb + & cc + & dd... \\
 & + ab + & bc + & cd... \\
 & & + ac + & bd... \\
 & & & + ad... \\
\hline
aaa + & bbb + & ccc + & ddd... \\
 & + abb + & bcc + & cdd... \\
 & + aab + & acc + & bdd... \\
 & & + bbc + & add... \\
 & & + abc + & ccd... \\
 & & + aac + & \&c...
\end{array}$$

As to the number of the combinations, each column of a product contains as many terms as there are in the column immediately above, *plus* as many as there are in the columns on the left. If, therefore, $1, \alpha, \beta, \gamma...$ be the numbers of terms in the columns of a product, those of the following product are $1, 1 + \alpha, 1 + \alpha + \beta, 1 + \alpha + \beta + \gamma...$, a series which is deducible from $1.\alpha.\beta...$ according to the law of the figurate numbers [N°. 489]. Hence, for the combinations 2 and 2, the successive columns contain 1. 2. 3. 4... terms; for those 3 and 3, the numbers are 1. 3. 6. 10...; and for the combinations p and p, we have the series of the p^{th} order. The total number of the combinations, or that of the terms of a product, is the sum of the series, extended to 2, 3, 4... columns, accordingly as 1, 2, 3... letters are to be combined; for n letters, the n first terms of the order p must be added together, *i. e.* the n^{th} term of the order $p + 1$ must be taken. Thus, *the whole number of combinations of n letters taken p and p, allowing that each letter may appear 1, 2, 3... times, is the n^{th} term of the order $p + 1$.* We must therefore change p into $p + 1$ in the equ. (10) p. 19., and we shall have
$$T = [(n + p - 1)\, Cp \text{ or } (n - 1)]... \quad (12)$$

$$= n.\frac{n+1}{2}.\frac{n+2}{3}...\frac{n+p-1}{p} = (p+1)\frac{p+2}{2}...\frac{p+n-1}{n-1};$$

where n may be $>, =$ or $<p$. For example, 10 letters 4 and 4 give 715 results; 4 letters 10 and 10 give 286. It also appears that n letters, taken p and p, and $p + 1$ letters, taken $n - 1$ and $n - 1$, give the same number of combinations, since n may be replaced by $p + 1$ and p by $n - 1$, without any alteration in respect to T.

The development of $(a + b + c...)^p$ is composed [N°. 483] of as many terms of the form $Na^\alpha b^\beta c^\gamma...$ as it is possible to assume different numbers for the exponents $\alpha, \beta, \gamma...$, their sum only being always $= p$. The whole number of the terms is therefore equal to that of the combinations p and p, that can be formed with the n letters $a, b, c...$,

assigning to them as values all the exponents from zero to p. It is evident that T *is the number of terms of the power* p *of the polynomial* $(a + b + c...)$.

If the sum of the combinations of n letters taken 1 and 1, 2 and 2,... p and p be required, we must add the n^{th} number of the successive orders 1. 2. 3... $p + 1$ in the table of N°. 489, or the n^{th} column, which we know to have for its sum the $(n + 1)^{th}$ number of the same order $p + 1$. Changing, therefore, n into $n + 1$ in the equ. (12), we shall have, for the sum required, $S = [(n + p)\ Cp, \text{ or } n] - 1$.

This subtractive unit corresponds to the combinations 0 and 0, which must here be omitted. For example, 5 letters combined from 1 and 1, to 4 and 4, or 4 letters from 1 and 1 to 5 and 5, give this number of results

$$\frac{6.\,7.\,8.\,9}{1.\,2.\,3.\,4} - 1 = 125.$$

To obtain the combinations from p and p to p' and p', the formula must be applied twice (to the numbers p and p' successively) and the results substracted: 5 letters taken from 4 and 4 to 6 and 6 form 461 — 125, or 336 combinations.

495. Let it be proposed to find *all the combinations of the letters of the monomial* $a^{\alpha} b^{\beta} c^{\gamma}...$, *taken* 1 and 1, 2 and 2, 3 and 3, *up to the dimension* $\alpha + \beta + \gamma...$ The letter a may be affected with the exponents $1, 2, 3... \alpha$; b in like manner with $1, 2, 3... \beta$, &c.; and the question evidently reduces itself to the finding all the divisors of $a^{\alpha} b^{\beta} c^{\gamma}...$, which are the terms of the product [note p. 26. 1st vol.]

$$(1 + a + a^2...a^{\alpha})\ (1 + b + b^2... b^{\beta})\ (1 + c... c^{\gamma})...$$

The number of terms, or that of the combinations required is $(1 + \alpha)(1 + \beta)(1 + \gamma)...$ Thus, $a^5b^4c^3d^2$ has 360 divisors (6. 5. 4. 3), including unity; and there are therefore 359 ways of combining the factors 1 and 1, 2 and 2, 3 and 3, &c.

And if, *among these divisors, those only be required which contain* a, since the others must divide $b^{\beta}c^{\gamma}...$, and these are in number $(1 + \beta)(1 + \gamma)...$, subtracting them, there remains $\alpha(1 + \beta)(1 + \gamma)...$ for the number of divisors which admit a; as though, to the several combinations without a, we had annexed the factors $a, a^2, a^3...$

To determine how many, among the divisors of $a^{\alpha} b^{\beta} c^{\gamma}...$, there are which contain $a^m b^n$, we shall take all those of $c^{\gamma} d^{\delta}...$, the number of which is $(1 + \gamma)(1 + \delta)...$, and annex $a^m b^n$ to each of them; and the number of the results will be that of the divisors in question.

CHANCES.

496. In speculating on an event of chance, the skill and prudence consist in ascertaining the greatest number of favorable chances; the event becomes *probable* in proportion to the value and the number of these chances. Events are *equally possible,* when there are equal reasons for expecting that each will happen, so that we should feel alike undecided in our suppositions as to which really will happen, and players, who divided these chances evenly among themselves, should each have equal grounds of hope, and the same right to look for success. We judge of the degree of *probability* of an event, by comparing the number of chances which produce it with the total number of all the chances equally possible.

The degree of probability is measured by a fraction, the denominator of which is the number of all the events equally possible, and the numerator the number of chances that are favorable. Suppose I wish to throw 5 and 2 with two dice, the faces of which are marked 1, 2, 3, 4, 5, 6: there are but two cases, among the 36 equally possible, in which 5 and 2 can be turned up; and the probability therefore is $\frac{2}{36}$ or $\frac{1}{18}$. If I propose to throw 7 for the sum of the points, I reckon three double chances, that are favorable, 5 and 2, 6 and 1, 4 and 3; and I therefore have $\frac{6}{36}$ or $\frac{1}{6}$ for the probability. The odds are 1 to 5 that I shall succeed.

We must therefore *ascertain the number of all the chances equally possible, then of those that are favorable, and form a fraction with these numbers.* When the probability is $>\frac{1}{2}$, there is *likelihood; incertitude,* if this fraction is $\frac{1}{2}$, *i. e. we may bet indifferently for or against the event.* The chances for and against an event being united together, the result must always be unity.

We shall now give some applications of these principles:

Of 32 cards, 12 are court-cards, and 20 plain; one card only being drawn, the probability of its being a court-card is $\frac{12}{32}=\frac{3}{8}$. The odds therefore are 3 to 5 on drawing a court-card, and 5 to 3 on drawing a plain one.

Among m cards, there are p of a specified sort; what is the probability of drawing m' that shall be all of that sort? The number of possible cases is mCm'; that of the favorable cases is pCm'; and the probability required is $\frac{pCm'}{mCm'}$. In a pack of 52 cards, for instance, there are 13 hearts; three cards being drawn indiscriminately, the probability that all three are hearts is $13C3 : 52C3$, or $\frac{286}{22100}=\frac{1}{77}$ nearly.

Among m cards, there are a hearts and a' spades; and $m'+m''$ cards

are drawn; what is the probability that m' of them are hearts and m'' spades? $mC\ (m' + m'')$ is the number of all the chances possible. The a hearts, combined m' and m', form aCm' systems; the a' spades, $a'Cm''$; coupling these chances together [N°. 478], the number of favorable chances is $[aCm'].[a'Cm'']$; and this is the numerator required. It would be $[aCm'].[a'Cm''].[a''Cm''']$, if there were also a'' diamonds of which m'' were to be drawn, &c.

The wheel of a lottery contains m numbers of which p are drawn; a player having taken m' of these numbers, what is the probability that *precisely* p' of them will be drawn? The total number of chances is mCp, the denominator required; and the number of favorable chances has been found [N°. 478], where it appears that the numerator is

$$X = [(m - m')\ C\ (p - p')].[m'Cp'].$$

In the French lottery, $m = 90$, $p = 5$, and the denominator is $90C5 = 43949268$. Supposing that a player have taken 20 numbers, or $m' = 20$, if he wish that there should turn up *precisely*

$1 = p'$, the numerator is $20\ [70C4]$, the probability... 0.4172
$2 = p'$ .... $20.\frac{19}{2}\ [70C3]$, ... 0.2367
$3 = p'$ .... $20.\frac{19}{2}.\frac{18}{3}\ [70C2]$, ... 0.0626
$4 = p'$ ....$70\ [20C4]$ 0.0077
$5 = p'$ .... $[20C5]$ 0.0003

If it be proposed only that one at least of the numbers should be drawn, *i. e.* that 1, 2, 3, 4 or 5 should turn up, we must take the sum 0.7245. For the condition that two at least be drawn, add the above results, except the first; and you have the probability 0.3073, &c. If you wish that no one of the numbers should be drawn, make $p' = 0$, or take the complement of 0.7245 to 1; and you will have 0.2755 for the probability.

These problems may be expressed thus: among m cards, there are m' of a specified sort; p cards are drawn, and it is proposed that there should be, *either precisely, or at least,* p' of them taken from among those specified: to find the probability. For example, a piquet-player having received 12 cards, concludes from his hand that, among the 20 other cards, there are 7 hearts; what is the probability that, if he take 5 cards more, there will be precisely 3 hearts among them? We have $m = 20$, $m' = 7$, $p = 5$, $p' = 3$; whence results $\frac{[13C2].[7C3]}{20C5} = \frac{2730}{15504}$, about $\frac{3}{17}$. Reasoning as before, we shall have for the probability that there will be at the least 3 hearts, $\frac{3206}{15504}$, or about $\frac{6}{29}$.

A purse contains 12 counters, of which 4 are white; 7 being drawn, what is the probability that there will be precisely 3 white ones among them? $m = 12$, $m' = 4$, $p = 7$, $p' = 3$; whence we deduce $\frac{280}{792}$, very

nearly $\frac{8}{17}$. The probability of drawing at least 3 white counters among the 7 is $\frac{14}{33}$.

497. Two events A, A' are such that p, p' causes are favorable to their taking place, whilst q, q' causes militate against them; it is presumed that the events can happen together or separately, and that they are independent of each other: required what are the probabilities in all the different cases. Imagine two dice, one of them with $p + q$ faces, p of them white, and q black; the other with $p' + q'$ faces, p' red, and q' blue: it is evident that a throw of each of these dice separately will lead to results corresponding to our two events. The event A will be realized, if we turn up one of the p white faces, and will fail, if we turn up one of the q black faces, &c. The total number of chances is [p. 24] $(p + q)(p' + q')$, the common denominator of all our probabilities.

If we wish for a black and a red face to come together, the q black and the p' red faces give qp' combinations; and these being the favorable cases, the probability is

$$\frac{qp'}{(p + q)(p' + q')} = \frac{q}{p + q} \times \frac{p'}{p' + q'};$$

which is that of A''s happening without A. Similar reasoning will apply to the other cases.

It will be observed that we have here the product of the probabilities relative to each of the events desired; and, consequently, *if the events be independent of each other, the probability that they will happen together is the product of all the probabilities relative to each separately.* This theorem of *compound probabilities* is here demonstrated only for two events; but if there had been a third A'', or a third die with $p'' + q''$ faces, the same reasoning would have applied, and established the validity of our consequences.

With two dice with 6 faces, it is wished at one throw to turn up 4 and ace; what is the probability of success? Considering only one die, there are 6 chances, two of which (4 or ace) being favorable, the simple probability is $\frac{2}{6}$ or $\frac{1}{3}$. But, this first chance being gained, it remains for the second die to give the other point (ace or 4), another simple probability the value of which is $\frac{1}{6}$; whence the probability required is $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$; the same as though we had compared the two favorable cases with the 36 possible chances.

A pack of 32 cards is divided, according to suits, into 4 heaps, 8 hearts, 8 diamonds, &c.: required how we may venture to bet on drawing one of the 3 court-cards of the hearts. Since we are not aware which of the heaps contains the hearts, $\frac{1}{4}$ is the simple probability that

we shall fix upon the right one; and this done, we have also, out of 8 cards, to draw one of the three court-cards, another simple probability $\frac{3}{8}$; that required therefore is compounded of these two, and is $\frac{3}{32}$.

When the probabilities are compounded, they become weakened, since they result from the product of several quantities < 1. Suppose that a man, of whose veracity I am able to judge, bears evidence to me of a fact which he represents himself to have seen, and that I value at $\frac{9}{10}$ the probability that he has no wish to deceive me, and has not himself been led into error by his senses. If now, instead of having himself seen it, he has only received the fact from a witness equally trustworthy, the probability will be no more than $\frac{9}{10} \times \frac{9}{10}$ or $\frac{81}{100}$, very nearly $\frac{4}{5}$. And if there were 20 such intermediate witnesses, we should have no more than $\left(\frac{9}{10}\right)^{20}$, or scarcely $\frac{1}{8}$; and the odds would be 7 to 1 that the fact transmitted is false, though all the intermediate persons were equally deserving of credit. This diminution in the probability may be compared with the gradually lessened brightness of objects, as seen through the interposition of several pieces of glass.

498. When the simple probabilities are equal to each other, the result, or product, is a power of that quantity. Suppose that an event A is brought about by p causes, whilst q are against it; what is the probability of A coming to pass k times in n trials? It is evident that, at each trial, the simple probability is $\frac{p}{p+q}$ in favour of A, $\frac{q}{q+p}$ against it; and if the event is to be realized k times, we have the power k of the 1st fraction, and, that it may fail the other $n-k$ trials, we have the power $n-k$ of the 2nd. Hence, multiplying these two powers, there results for the compound probability

$$z = \frac{p^k . q^{n-k}}{(p+q)^n},$$

which expresses the probability that, in n trials, A will happen precisely k times, the order of succession of the events being fixed *à priori*. But if this order is arbitrary, z must be repeated as often as we can combine these results, viz. the k times that the event A happens, with the $n-k$ in which it does not, which gives the factor nCk; and consequently $z \times [nCk]$ is the probability that A will happen k times in n trials, without particularizing those in which it is so to take place.

And if the proposal be that A should happen at least k times, we must here change k into k, $k+1$... up to n, and take the sum of the results.

Hence the denominator of the probability required is $(p+q)^n$: the

numerator is obtained by developing this binomial, and stopping at the term into which p^k enters, which we must take without or with its coefficient, accordingly as regard is or is not to be paid to the k ranks in which A is to be realized in n trials. And if we wish that A should happen at least k, but at the most k' times, in the n trials, we must add all the terms in which p has the exponents $k, k+1, \dots k'$.

For example, let a die with 6 faces have 2 favorable to a player; and suppose that, in order to win, he must in 4 throws turn up one or other of these 3 times (or that, in a single throw of 4 dice, 3 faces must be favorable): required the probability of success. We have $p = 2$, $q = 4$, and $(p+q)^4 = 6^4 = 1296 =$

p^4	$=$ 16 throws which give one of the favorable faces	4 times
$4p^3q$	$=$ 128	3
$6p^2q^2$	$=$ 384	2
$4pq^3$	$=$ 512	1
q^4	$=$ 256	0

Sum $= 1296 = (p+q)^4$, the denominator of the probabilities.

Thus, the probability of one of the favorable cases occurring *precisely* 3 times is $\frac{128}{1296}$ or $\frac{8}{81}$; which must be divided by the coefficient 4, if the order in which they are to happen is fixed, and we shall have $\frac{2}{81}$. Lastly, adding the two 1st terms, we have $\frac{144}{1296}$ or $\frac{1}{9}$, for the probability that the favorable faces will present themselves at least 3 times.

What are the chances of two players M and N of equal skill; supposing that M wants 6 points and N 4 towards winning the game? The sum of these points being 10, we shall form the 9th power of $p+q$; reserve for M the 4 first terms (in which the exponent of p is at least 6), take the other 6 terms for N, and finally make $p = q = 1$. We find 130 on the one hand, 382 on the other, the sum of which is 512; and the chance of M, or the probability that he will win, is $\frac{130}{512}$, that of N is $\frac{382}{512}$. Supposing the game to be broken up without any farther trial, the stake ought to be divided between M and N in the proportion of 130 to 382, very nearly as 1 to 3; and these also are the rates at which they should sell their pretensions to the stake, if they agreed to give up their claims. When the skill of the players is not equal, but, for instance, as 3 to 2, *i. e.* when M ordinarily gains from N 3 games out of 5, or M gives to N 1 point in 3, to put them on a *par*, the calculation is the same, only assuming $p = 3$ and $q = 2$. In this case we find that the chance of M is to that of N about : : 14 : 15.

499. It frequently happens that the causes lie so concealed, or multiply themselves in so uncertain a manner, that it is impossible to bring

them clearly to light and ascertain their number; in which case the principles now laid down can no longer allow of application. Under such circumstances, recourse is had to experience, in order to ascertain whether the events are subject to a periodical return, whence we may be able to conjecture with all but certainty that the unknown cause which has frequently produced them in a regular order, again acting, will produce a recurrence of them in the same order. The number of these returns is substituted for that of the causes themselves in the calculations respecting the degrees of probability. Thus, suppose a die thrown 10 times successively has presented the face a 9 times; there must then, in its figure, its substance, or the manner of throwing it, be some secret cause which has led to this recurrence of the face a 9 times: if 100 trials have in like manner given this face a 90 times, the probability $\frac{9}{10}$ favorable to this return acquires very considerable force, which will be still farther strengthened should repeated trials continue to agree with this supposition; since if we could make an infinite number of trials, all of which presented the face a 9 times out of 10, the hypothesis would be reduced to a *certainty.*

It is thus that experience has constantly proved the following facts, though it is impossible to assign the exact causes.

1°. The number of marriages contracted in a country, for any fixed period, is to that of births and to the whole population very nearly : : 3 : 14 : 396.

2°. 16 males are born to 15 females.

3°. The population, the number of births, that of deaths and that of marriages are : : 2037615 : 71896 : 67700 : 15345; with very slight variations, the births are, annually, $\frac{1}{28}$th, the deaths $\frac{1}{30}$, and the marriages $\frac{1}{132}$ of the entire population. The difference $\frac{1}{486}$ between the births and deaths is the annual increase of the population.

4°. The average length of generations from father to son is 33 years.

5°. The number of deaths of the male sex is to that of the female : : 24 : 23; and, in any country, the number of inhabitants of the 1st sex is to that of the 2nd : : 33 : 29.

6°. The deaths among the males form $\frac{1}{28}$, among the females $\frac{1}{31}$, of the population: at Paris the total number of deaths is but $\frac{1}{32}$ of the number of inhabitants; these deaths rise annually to 22700, the mean number, and the births to 24800.

7°. The half of the entire population is upwards of 25 years of age, and one-half is renewed every 25 years.

8°. In France, $\frac{1}{17}$th of the population marry each year. The mean duration of life is $28\frac{1}{4}$ years.

On these considerations are established the Tables of population and mortality; on which subject may be consulted *L'Annuaire* du Bureau des Longitudes.

We shall add nothing farther on the doctrine of Probabilities, which is so extensive, as itself to form the subject of special Treatises. See those of MM. Laplace, Lacroix, Condorcet, Duvillard, &c.

II. RESOLUTION OF EQUATIONS.

COMPOSITION OF EQUATIONS.

500. After transposition, reduction and division by the coefficient of the highest power of x, *every equation has the form*

$$x^m + px^{m-1} + qx^{m-2} \ldots + tx + u = 0 \ldots (1),$$

which, for the sake of brevity, we shall represent by $X = 0$; $p, q \ldots u$ are known numbers, and may be positive, negative, or zero. The name of *Root* is given to every quantity a which, when substituted for x, reduces X to 0, or makes $a^m + pa^{m-1} \ldots + u = 0$.

The polynomial proposed being $kx^m + px^{m-1} + qx^{m-2} \ldots$, and a being any arbitrary quantity, let the polynomial be divided by $x - a$. According to the theorem at the end of N°. 100, let R be the remainder, and $kx^{m-1} + p'x^{m-2} + q'x^{m-3} \ldots + u'$ be the quotient from the division: this quotient, multiplied by $x - a$, and augmented by the addition of the remainder, must give back *identically* the dividend, which consequently is

$$= kx^m + \left.\begin{matrix} p' \\ -ak \end{matrix}\right| x^{m-1} + \left.\begin{matrix} q' \\ -ap' \end{matrix}\right| x^{m-2} + \left.\begin{matrix} r' \\ -aq' \end{matrix}\right| x^{m-3} \ldots\ldots + \begin{matrix} R \\ -au' \end{matrix}$$

and since we must here meet again with all the terms of the first polynomial, the factor p of x^{m-1} must be the same with $p' - ak$ which in our product also affects x^{m-1}; in like manner $q = q' - ap'$, $r = r' - aq' \ldots$ $u = R - au'$; and transposing the negative terms, there results

$$p' = p + ak,\ q' = q + ap',\ r = r' + aq' \ldots R = u + au'.$$

These equations, all of the same form, allow of our deducing the coefficients $p', q', r' \ldots$, and the remainder R, one from the other successively; for *each is composed of the coefficient of the same rank in the dividend, plus the product of the preceding coefficient in the quotient multiplied by a.*

The following are some examples of this:

Divide $4x^5 - 10x^4 + 6x^3 - 7x^2 + 9x - 11$ by $x - 2$,

Quotient...... $4x^4 - 2x^3 + 2x^2 - 3x + 3$, remainder $- 5$.

Having put down $4x^4$, the 1st term of the quotient, we form $4.2 - 10 = -2$, which is the coefficient of x^3; that of x^2 is $-2.2 + 6 = +2$; then $2.2 - 7 = -3$, &c. Had the divisor been $x + 2$, the numerical factor would have been -2 throughout, and the quotient

$$4x^4 - 18x^3 + 42x^2 - 91x + 191, \text{ remainder } - 393.$$

But we can also determine any coefficient independently of the rest. For p', q'... being successively eliminated between the foregoing equations, we find, page 115,

$$p' = ka + p,\ q' = ka^2 + pa + q, \ldots\ldots R = ka^m + pa^{m-1} \ldots + u.$$

Thus each coefficient may be formed from the proposed polynomial, by changing x into a, limiting ourselves to the first terms, as far as the one of the same order with the term required of the quotient, and suppressing the powers of a that are common to all these terms.

It appears therefore that if X, or the polynomial (1) be divided by $x - a$, and the operation be continued till x no longer enters into the dividend, we shall arrive at the remainder $a^m + pa^{m-1} \ldots + u$; an expression which is $= 0$ if a be a root, but the contrary if a be not a root. Hence X *is or is not divisible by* $x - a$, *accordingly as* a *is or is not a root of the equation* $X = 0$.

The above process is very convenient for finding the quotient of $X : (x - a)$, and may also be made use of, instead of substituting a, when we wish to ascertain whether $x = a$ is a root of the equation $X = 0$, since we should in that case find $R = 0$.

501. $X = 0$ having a for a root, let Q be the exact quotient of X divided by $x - a$, or $X = (x - a)\,Q$; Q then is a polynomial of the degree $m - 1$.

But, if b is a root of the equation $Q = 0, x - b$ also divides Q exactly, and taking Q' as the quotient, which will be of the degree $m - 2$, we have

$$Q = (x - b)\,Q', \text{ and consequently } X = (x - a)\,(x - b)\,Q'.$$

Similarly, c being a root of $Q' = 0$, and Q'' being the quotient from $Q' : (x - c)$, we have $X = (x - a)\,(x - b)\,(x - c)\,Q''$, and so on. The degree of Q, Q', Q''... is reduced by one unit for each binomial factor that is successively introduced; after $m - 2$ divisions therefore there will be $m - 2$ of these factors, and the quotient will be of the 2nd degree, and be itself decomposable into $(x - h)\,(x - l)$. Thus

admitting that every equation has one root, X is formed of the product of m binomial factors of the first degree,

$$X = (x-a)\,(x-b)\,(x-c) \ldots (x-l).$$

This equation is *Identical, i. e.* there is no difference between the two sides except in their analytical expression, a difference which ceases on the operations indicated being put in execution. Since the 2nd side becomes nothing when we assume $x = a.\ b \ldots l$, *the equation* $X = 0$ *has m roots, which are the second terms, taken with contrary signs, of its m binomial factors* [See N°. 516].

We may now prove that X *does not allow of being decomposed into any other factors* $(x-a')\ (x-b')\ (x-c') \ldots$, *the quantities* $a', b', c' \ldots$ *being, all or some of them, different from* $a, b, c \ldots$; for if, assuming $X = Q\,(x-a)$, we suppose that $x = a'$ reduces X to zero, this same value $x, = a'$ must also reduce $Q\,(x-a)$ to the same state; and since, $a' - a$ is not 0, by hypothesis, it follows that $Q = 0$ must have a' for a root. But $Q = 0$ corresponds to $Q'\,(x-b) = 0$, and it appears in the same manner that a' is a root of $Q' = 0$, then of $Q'' = 0$, &c., and lastly of $x - l = 0$, *i. e.* $a' - l = 0$, and $a' = l$ contrary to our supposition. Thus X cannot be divided by $x - a'$, any more than by $x - b'$, $x - c', \ldots$ *.

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* It must be observed that, since the nature of imaginary functions is unknown to us *à priori*, and a' must, in this reasoning, be supposed any quantity whatever, it is by no means evident that, because $x = a'$ renders $Q\,(x-a) = 0$, Q also must necessarily be nothing. It has on this account been thought proper to give the following demonstration, which is independent of this supposition. We assume at one and the same time that $X = Q\,(x-a) = M\,(x-a')$;
we divide Q by $x - a'$, making $Q = q\,(x-a') + r$, where q denotes the quotient and r the remainder independent of x, and we have the identical equation

$$M = q\,(x-a) + \frac{r\,(x-a)}{x-a'} \ldots (1).$$

We now assume

$$\phi = \frac{r\,(x-a)}{x-a'}, \text{ or } r\,(x-a) = \phi\,(x-a'),$$

where it is evident that ϕ does not contain x, since in that case the 2nd side could not be identical with the 1st; and making x = any quantity α, there results

$$r\,(\alpha - a) = \phi\,(\alpha - a'),$$

whence

$$\frac{x-a}{\alpha-a} = \frac{x-a'}{\alpha-a'},\ x\,(a'-a) - \alpha\,(a'-a) = 0.$$

But this equation must subsist whatever x be, *i. e.* $x\,(a'-a) = 0$, even for $x = 1, 2, 3, \ldots$, so that $a' = a$, contrary to the supposition. Hence the equation (1) is absurd unless at least r be 0; and $x - a'$ must divide Q or $(x-b)\,Q'$.

Hence,

1°. *Any polynomial X can be resolved only into one system of binomial factors of the* 1*st degree, and the equation* $X = 0$ *can have no more than m roots.*

2°. Any fraction $X : Y$ which, when we assume $x = a$, becomes $\frac{0}{0}$, has $x - a$ for a common factor of its two terms X and Y; and $x - a$ may also enter in any power into each of them. The value of the fraction is obtained by first suppressing the common factors $x - a$, and then making $x = a$; and *this value is nothing, infinite or finite,* accordingly as $x - a$ remains as a factor in X, or in Y, or has disappeared from both, *i. e.* according to the exponent that $x - a$ bears in X and Y.

3°. If two equations have each the same root a, they will have $x - a$ for a common factor. This is the case for the equations that follow, the method of the greatest common divisor showing that $x + 3$ is a factor of both:

$$2x^3 - 3x^2 - 17x + 30 = 0, \quad x^3 - 37x - 84 = 0.$$

Had there been no common divisor, the supposition of the coexistence of the two equations would have been absurd. When this divisor is of the 2nd degree, there are two roots common; the other roots are foreign to the problem.

4°. We may, by division, reduce the degree of an equation by as many units as there are roots known [N°. 500], the investigation of the roots and that of the factors being the same thing. The factors of the 2nd degree are $\frac{1}{2} m (m - 1)$ in number [N°. 476], since they result from the combinations 2 and 2 of those of the 1st [See N°. 520]; the number of those of the 3rd degree is $\frac{1}{6} m (m - 1)(m - 2)$, &c.

502. Since the proposed equation $x^m + px^{m-1} \ldots + tx + u = 0$ is identical with $(x - a)(x - b) \ldots$, it follows from what has been seen p. 109 of the 1st vol. that

1°. *The coefficient p of the 2nd term is the sum of the roots with their signs changed.*

It appears in like manner that $x - a'$ divides Q', then $Q'' \ldots$ and finally the last factor $x - l$, *i. e.* $x - l = (x - a')\,\theta$, θ being a number, or $x(1 - \theta) - (l - a'\theta) = 0$, whatever be the value of x; consequently $x(1 - \theta) = 0$, $\theta = 1$, and lastly $l = a'$ contrary to the hypothesis.

From this we conclude that if the product XY of two polynomials X and Y is divisible by $x - a$, X or Y is so also, and $x = a$ renders this factor $= 0$: and if XY is divisible by P, the factors of P must all be found in X and Y

2°. *The coefficient q of the 3rd term is the sum of the products of every two of the roots.*

3°. *r is the sum of the products 3 and 3, with contrary signs, &c.*

Finally, *the last term u is the product of all the roots, only with a contrary sign, if the degree m be odd.*

When an equation is devoid of the 2nd term px^{m-1}, the sum of the roots is nothing; and if there be no last term u, one of the roots is $= 0$.

TRANSFORMATION OF EQUATIONS.

508. Let the equation be $kx^m + px^{m-1}..., + tx + u = 0...$ (1).

To transform this equation is to compose another, such that its roots y shall have to those x of the one proposed a relation given by an equation between x and y. Our object, therefore, will be, to eliminate x between the last of these equations and the first. If, for example, we wish to diminish all the roots x by the quantity i, we have $x - i = y$; and we must therefore substitute $i + y$ for x in (1); whence

$$(i+y)^m + p(i+y)^{m-1} + q(i+y)^{m-2}... + t(i+y) + u = 0...(2).$$

Without detaining ourselves with the development of the powers of $i + y$, it follows from the law of the successive terms in the formula of the binomial [N°. 482], that the transformed equation (2), arranged according to the ascending powers of y, is

$$X + X'y + \tfrac{1}{2}X''y^2 + \tfrac{1}{6}X'''y^3... + ky^m = 0;$$

where X is the sum of the 1st terms, or the polynomial proposed, x only being replaced by i; *X' is deduced from X by multiplying each term by the exponent of i and diminishing that exponent by unity;* X' is called the Derivative of X, X'' is the derivative of X', X''' that of X''... Thus the successive coefficients of the transformed equation may be deduced without developing the powers of $i + y$, and we shall have *

* The process explained p. 34, is very convenient for determining, in each particular case, the values of the coefficients X, X', $\tfrac{1}{2}X''$... of the transformed equation in y. For, let X be divided by $x - i$, and let A be the quotient and a the remainder; A being now divided by $x - i$, let B be the quotient and b the remainder; let C, c be the quotient and remainder from the division of B by $x - i$, and so on: we shall have

$$X = A(x - i) + a,\ A = B(x - i) + b,\ B = C(x - i) + c...;$$

and successively eliminating A, B, C..., we find

$$X = a + b(x - i) + c(x - i)^2 + ... + k(x - i)^m;$$

whence it appears that the remainders a, b, c... from our successive divisions are the coefficients of the transformed equation, y being $= x - i$. The process of

$$\begin{aligned}
X &= ki^m + pi^{m-1} + qi^{m-2} \ldots + ti + u, \\
X' &= mki^{m-1} + (m-1)\,pi^{m-2} + (m-2)\,qi^{m-3} \ldots + t, \\
X'' &= m\,(m-1)\,ki^{m-2} + (m-1)\,(m-2)\,pi^{m-3} + \ldots \\
\&c. &= \&c.
\end{aligned}$$

Thus, to make $x = y + 2$ in $x^3 - 5x^2 + x + 7 = 0$, we have

$$i^3 - 5i^2 + i + 7,\ 3i^2 - 10i + 1,\ 6i - 10;$$

and putting 2 for i and dividing X'' by 2, we have $-3, -7, +1$; whence

$$-3 - 7y + y^2 + y^3 = 0.$$

If, on the contrary, the roots x are all to be increased by i, we must assume $x = y - i$, *i. e.* change i in our previous expressions into $-i$, or take the odd powers of i with contrary signs.

504. The arbitrary quantity i may be disposed of in such a manner as to clear the proposed equation of one of its terms. Let the transformed equation (2) be arranged according to the decreasing powers of y:

$$\left.\begin{array}{r|l|l}
ky^m + mik & y^{m-1} + \tfrac{1}{2}m\,(m-1)\,i^2k & y^{m-2} + \ldots + ki^m \\
+ \quad p & \quad + \quad (m-1)\,i\,p & \qquad\qquad + pi^{m-1} \\
 & \quad + \qquad\qquad\quad q & \qquad\qquad + qi^{m-2} \\
 & & \qquad\qquad \&c.
\end{array}\right\} = 0.$$

To get quit of the 2nd term, we must assume $mik + p = 0$; whence

$$i = -\frac{p}{mk},\ x = y - \frac{p}{mk}.$$

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p. 34, gives us each remainder and each quotient $A, B\ldots$; and the calculation presents itself as in the following example, where y is assumed $= x - 3$.

$$\begin{array}{lrrrrr}
\text{Proposed equation}\ldots & 2x^4 - & 7x^3 - & 12x^2 + & 4x + & 129 = 0 \\
 & 2 & -1 & -15 & -41 & +6 \\
\text{Factor } 3 & 2 & +5 & +0 & -41 & \\
 & 2 & +11 & +33 & & \\
 & 2 & +17 & & & \\
\text{Transformed equation}\ldots & 2y^4 + & 17y^3 + & 33y^2 - & 41y + & 6 = 0
\end{array}$$

The first line $2, -1, -15\ldots$ is composed of the coefficients of the first quotient A, the second of those of the second quotient B, the third of $C\ldots$, and each line has one term less than the preceding line. The last term of each line is the remainder from the division by $x - 3$, $a = 6$, $b = -41,\ldots$; and these therefore are the coefficients required in inverse order.

This mode of calculation is particularly convenient when $i = 1$, *i. e.* when the unknown quantity in the transformed equation is $y = x - 1$, since we have then only additions to perform, according to the law of the Table of Figurate Numbers. The following are two examples:

$$\begin{array}{llll}
x^3 - 12x^2 + 41x - 29 = 0 & & & \\
1 \quad -11 \quad +30 \quad +1 & & & \\
1 \quad -10 \quad +20 & & & \\
1 \quad -9 & & & \\
y^3 - 9y^2 + 20y + 1 = 0 & & &
\end{array}
\qquad
\begin{array}{l}
x^4 - 6x^3 + 7x^2 - 7x + 7 = 0 \\
1 \quad -5 \quad +2 \quad -5 \quad +2 \\
1 \quad -4 \quad -2 \quad -7 \\
1 \quad -3 \quad -5 \\
y^4 - 2y^3 - 5y^2 - 7y + 2 = 0
\end{array}$$

Thus, *x must be changed into y minus the coefficient p of the 2nd term, divided by the product of the coefficient k of the 1st, and by the degree m of the equation;* observing that if p and k have different signs, the subtraction becomes an addition ($y+$, instead of $y-$). The sum of the roots of the transformed equation is nothing; and the several roots have therefore been increased or diminished by a quantity i, which has rendered the positive parts equal to the negative.

The calculation will proceed more rapidly by assuming $x + \frac{p}{mk} = y$; developing the power m, and multiplying by k, we arrive at once at the value of the two 1st terms $kx^m + px^{m-1}$. Thus, for $x^3 - 6x^2 + 4x - 7 = 0$, we shall assume $x - 2 = y$, according to our theorem: the cube gives $x^3 - 6x^2 = y^3 - 12x + 8$, and the proposed equation becomes $y^3 - 8x + 1 = y^3 - 8y - 15 = 0$, which is the one required.

For $x^2 + px + q = 0$, we shall assume $x + \frac{1}{2}p = y$; whence, squaring, we get $x^2 + px = y^2 - \frac{1}{4}p^2$ and the transformed equation $y^2 = \frac{1}{4}p^2 - q$. Hence we deduce y, and subsequently the roots x of the proposed equation; so that this gives us a new method of solving equations of the 2nd degree.

If we wish *to get quit of the 3rd term,* we shall assume

$$\tfrac{1}{2}m(m-1)i^2k + (m-1)ip + q = 0,$$

an equation which, in general, will lead to irrational or imaginary values of i.

Finally, to remove the last term, we must assume $ki^m + pi^{m-1} \ldots + u = 0$, and we shall therefore have to solve the proposed equation itself; and, in fact, the transformed equation will have a root of y which will be $= 0$, and consequently $x = i$.

505. *That the roots x may become h times greater,* assume $y = hx$, or $x = \frac{y}{h}$ in the equation (1); then

$$\frac{ky^m}{h^m} + \frac{py^{m-1}}{h^{m-1}} + \frac{qy^{m-2}}{h^{m-2}} \ldots + \frac{ty}{h} + u = 0,$$

whence $$ky^m + phy^{m-1} + qh^2y^{m-2} \ldots + uh^m = 0.$$

This comes to the same thing with multiplying the successive terms of the equation (1) by $h^0, h^1, h^2 \ldots h^m$.

It will be observed that if the proposed equation have no fractional coefficients (and we can always get quit of fractions by reduction to the same denominator), assuming $y = kx$, *i. e.* making the arbitrary quantity $h = k$, the transformed equation becomes divisible throughout by k, and we have the equation $y^m + py^{m-1} + qky^{m-2} \ldots + uk^{m-1} = 0$, *disengaged from the coefficient of the* 1*st term.* Thus, *to clear an equation*

of fractional coefficients, we must first reduce it to a common denominator, and then get quit of the coefficient k of the 1*st term by assuming* $y = kx$, an operation which comes to the same thing with multiplying the coefficients, commencing with that of the 2nd term, by $k^0, k^1, k^2 \dots k^{m-1}$ successively.

Let the equation, for example, be $x^4 - \frac{2}{3}x^3 + \frac{5}{6}x^2 - \frac{3}{4}x - \frac{7}{2} = 0$: multiplying by 12, we have $12x^4 - 8x^3 + 10x^2 - 9x - 42 = 0$; and making $x = \frac{1}{12}y$, *i. e.* multiplying the factors 10, 9, and 42 by 12, 12^2, 12^3 respectively, there results

$$y^4 - 8y^3 + 120y^2 - 1296y - 72576 = 0.$$

It will be easily seen that to get quit at one and the same time of the 2nd term and the coefficient k of the 1st, we must assume $x = \frac{y-p}{mk}$.

That the roots x may become h times less, we must assume $x = hy$, *i. e.* divide the successive coefficients of the proposed equation by $h^0, h^1, h^2 \dots h^m$. The preceding process would give to the equation coefficients greater than before, whereas this diminishes them, and is employed with this view; unless, however, the division by $h^0, h^1 \dots$ can be effected exactly, the transformed equation will acquire fractional coefficients. Let the equation be $x^3 - 144x = 10368$; assuming $x = 12y$, we have $y^3 - y = 6$, an equation much simpler than the original one.

506. The following are two other useful transformations:

If we assume $x = -y$, which merely changes the signs of the alternate terms of the equation, the positive roots of x become negative for y, and the converse.

Making $x = \frac{1}{y}$, the greatest roots of x correspond to the least of y, and the converse; the transformed equation is said to be the *reciprocal of the one proposed.* Since $x, x^2 \dots x^m$ are replaced by the divisors $y, y^2 \dots y^m$, if we multiply throughout by y^m, the factors $x, x^2 \dots x^m$ will be replaced by $y^{m-1}, y^{m-2} \dots y^1, 1$; and this operation therefore reduces itself to distributing the powers of y in the inverse order of those of x:

$$\frac{k}{y^m} + \frac{p}{y^{m-1}} + \frac{q}{y^{m-2}} \dots + \frac{t}{y} + u = 0,$$

whence $$uy^m + ty^{m-1} + sy^{m-2} \dots + py + k = 0;$$

and if we wish also to get quit of the coefficient u, we must assume $y = \frac{y'}{u}$; whence $x = \frac{u}{y'}$, a transformation which will fulfil the two proposed conditions at once.

LIMITS OF ROOTS.

507. That $x = L$ may be a root of an equation $X = 0$, it is necessary that, on L being substituted for x, the polynomial X should become nothing. But if $x = L$ *give a positive result*, and if it is also known that every number $> L$ fulfils the same condition, it is evident that L is greater than any of the roots; and L in this case is called a superior limit of the roots of the equation $X = 0$.

Introducing $x - 1$ as a factor in x^n, we have

$$x^n = (x - 1)\, x^{n-1} + x^{n-1};$$

proceeding in the same manner for x^{n-1}, then for x^{n-2}, &c., there results

$$x^{n-1} = (x - \;)\, x^{n-2} + x^{n-2},\; x^{n-2} = (x - 1)\, x^{n-3} + x^{n-3}, \text{ \&c.};$$

and combining these several results, we find

$$x^n = (x - 1)\,(x^{n-1} + x^{n-2} + x^{n-3} \ldots + x + 1) + 1.$$

This theorem being applied to each of the *positive terms* of $X = kx^m + px^{m-1} + qx^{m-2}\ldots$, there results

$$\begin{array}{r|r|r|r} k(x-1)x^{m-1} + k & (x-1)x^{m-2} + k & (x-1)x^{m-3}\ldots + k & (x-1) + k \\ + p & + p & \ldots + p & + p \\ & + q & \ldots + q & + q \\ \multicolumn{4}{r}{\text{\&c.}\ldots\ldots\ldots\ldots\ldots\ldots} \end{array}$$

But X must have some negative terms; since, otherwise, no positive value of x could reduce X to nothing, and zero would be the superior limit. Let these negative terms be left in their original form, and placed in the columns in which x has the same exponent; with the exception of these coefficients, the whole will be positive, provided that we assume $x > 1$. But every column into which a negative coefficient $-s$ enters has the form $(k + p + q\ldots)(x - 1) - s$, the factor of $x - 1$ being *the sum of the positive coefficients which precede* s; and it is evident that the result will not be negative, except on the supposition that $(k + p + q\ldots)(x - 1) < s$; we shall have the sign $+$ if we assume

$$x \begin{smallmatrix}=\\>\end{smallmatrix} 1 + \frac{s}{k + p + q\ldots} \quad (M).$$

Applying this to each column in which a negative term appears, if we assume $x =$ or $>$ the greatest of the quantities M, the polynomial X will have a positive value, and this value M will fulfil the necessary condition for the limit required, since M, as also every quantity $> M$, will render all the terms of the equation positive. Hence, *divide each negative coefficient by the sum of the positive ones which precede it, take the greatest of the coefficients thus obtained, add unity to it, and you will have a superior limit of the roots.*

For the equation $4x^5 - 8x^4 + 23x^3 + 105x^2 - 80x + 11 = 0$, we divide 8 by 4, and 80 by $4 + 23 + 105$: $\frac{8}{4} > \frac{80}{132}$; consequently $1 + \frac{8}{4}$, or 3, is $> x$, and 3 is the limit.

Every number greater than the value M also possesses the property of exceeding all the roots. Now, substituting zero for each of the coefficients $p, q \ldots$, we find $x \overset{=}{>} 1 + \frac{s}{k}$, and since we can always render $k = 1$, it is usually said, that *the greatest negative coefficient of the equation, taken positively and augmented by unity, is a superior limit of the roots.* This expression is simpler than the preceding one, it is formed at a glance and without any calculation, and is preferred when our object is only to demonstrate general propositions. But when we are engaged in the investigation of the numerical values of the roots, it is of importance to select for our superior limit the lowest number we can obtain, and one approaching as nearly as possible to the greatest root; in which case it is more advantageous to employ the 1st limit (M), or the one which results from the following theorem.*

508. In X, making $x = L + y$, L being any number whatever, the transformed equation [503] will be $X + X'y + \frac{1}{2}X''y^2 \ldots + ky^m = 0$. If, now, the arbitrary quantity L be so assumed that the coefficients $X, X' X'' \ldots$ shall be all positive, no positive value of y can satisfy this equation; the real values of x therefore will correspond to negative roots of $y = x - L$, and consequently $L > x$. Hence, *any number which, substituted for x in X and its derivatives $X', X'' \ldots$, renders no one of them negative, is a superior limit of the roots of x.*

In the example cited, the derivatives are

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* It has been stated by some authors that to obtain the limit l of the roots of an equation, we must find a number l, which shall render *the 1st term greater than the sum of all the others;* whereas the expression ought to have been "*the 1st term greater than the sum of all the negative terms.*" Without staying to demonstrate the truth of this assertion, which is evident from what has gone before, we will take an example which will serve clearly to exhibit it.

The equation $x^3 - 10x + 3 = 0$ has 3 for a root, and every number > 3 is a limit. If however we assumed $x^3 > 3 - 10x$, this condition would be satisfied by $x = 0.3$ or $0.4 \ldots$, values which would not be limits.

In like manner, $x^3 - 6x + 4 = 0$ has its greatest root a little above 5, so that $x = 6$ is the limit. And yet $x = \frac{2}{3}$ renders $x^2 > 4 - 6x$.

The error alluded to, which exists only in the enunciation, arises from the circumstance that, in ascertaining whether a number is a limit, it is unnecessary to take into account the effect of the positive terms; so long as the 1st term exceeds the sum of the negative ones, the positive terms, which combine to augment the positive part, will, *à fortiori*, give a positive value to the result.

$20x^4 - 32x^3 + 69x^2 \ldots,\ 80x^3 - 96x^2 + 138x \ldots,\ 240x^2 - 192x \ldots;$
and it will be easily seen that $x = 1$ renders the equation proposed and its derivatives all positive; thus $x < 1$, a limit lower than the one previously obtained.

It is observable, that if the signs of the odd powers of y be changed, which comes to the same thing with assuming $x = L - y$, the real roots of y, which before were all negative, will become positive. *We can, therefore, by means of the superior limit of the roots of an equation $X = 0$, transform it into another which has no negative root.*

509. Change x into $-x$, *i. e.* change the signs of the terms of the even order; the positive values of x then will become negative; and if we investigate the superior limit L', $-L'$ will be below the negative roots, *i. e.* the roots of x will all be comprised between $-L'$ and L. In our example, we have $4x^5 + 8x^4 + 23x^3 - 105x^2 - 80x - 11$, for which $\frac{105}{35} + 1$ is the limit; the negative roots therefore lie between -4 and 0, and all the roots between -4 and $+1$.

510. The positive roots of the equation $X = 0$ are all comprised between zero and the limit L. Making $x = \dfrac{1}{z}$, the greatest roots of x will correspond to the least of z; so that if we investigate the superior limit l of the roots of z, or $z < l$, we shall have $x > \dfrac{1}{l}$; and thus therefore we shall obtain *an inferior limit of the positive roots of x*: that of the negative roots is arrived at by changing x into $-x$, and reasoning in the same manner.

If s be *the greatest coefficient of a sign contrary to that of the last term u in the equation* $kx^m + px^{m-1} \ldots + u = 0$, since the transformed equation is $uz^m + \ldots + pz + k = 0$, we know [N° 507] that we may assume for the limit $z < 1 + \dfrac{s}{u}$; whence $x > \dfrac{u}{u + s}$; and between this value and the superior limit L are comprised all the positive roots of x. This fraction however may be replaced by a still higher limit [507, 508], which contracts the interval within which the roots are contained. In our example, the positive roots lie between $\frac{11}{91}$ and 1.

511. Retaining only the negative terms of X and the first term kx^m, there remains $kx^m - Hx^{m-h} - Nx^{m-n} \ldots$; and the number L which, substituted for x, gives a positive result, will obviously produce the same effect on X, in which the positive part is still greater. If k have the sign $-$, we may in the same manner render $kx^m >$ the

sum of the positive terms. Thus, *we shall know values L of x which give to the result of X the same sign as that of the 1st term, and which are also so taken that every number > L will fulfil the same condition.*

For $x = \frac{1}{z}$, $k + px + qx^2\ldots$ becomes $\frac{1}{z^m}(kz^m + pz^{m-1}\ldots)$; and the number L, which gives a result of the same sign as k, corresponds to the value $x = \frac{1}{L}$, which produces the same effect on $k + px + qx^2\ldots$ Thus, *we become acquainted with values of x sufficiently small, that the sign of the quantity $k + px + qx^2\ldots$ shall be that of k, and that all less numbers shall fulfil the same condition.*

In these two cases, we may assume $L = 1 + \frac{s}{k}$, s being the greatest coefficient of a sign contrary to that of k.

ON THE EXISTENCE OF ROOTS.

512. X being a polynomial which has no negative signs, *we may assume for x a series of numbers that shall give to X values continually increasing, and succeeding each other within any limits however small that can be assigned.* For, make $x = \alpha$, and $\alpha + i$; the results P, and $P + P'i + \frac{1}{2}P''i^2\ldots$ have for their difference $i(P' + \frac{1}{2}P''i\ldots)$; and the point is, to find a value of i that shall render this quantity less than any assigned number h. Our expression is entirely positive, and i very small and < 1; let $i = 1$ within the parenthesis, and suppose that $i(P' + \frac{1}{2}P''\ldots) =$ or $< h$; the condition prescribed then will evidently be fulfilled; and it is therefore satisfied by making $i =$ or $< \frac{h}{P' + \frac{1}{2}P''\ldots}$.

Taking next $x = (\alpha + i) + i'$, and reasoning in the same manner for i', we have a 3rd result which differs from the 2nd by less than h, and so on.

This being premised, let P be the sum of the positive, and N that of the negative terms of an equation $X = 0 = P - N$; suppose that $x = \alpha$ and $= \lambda$ have given results of different signs; and in the expressions P and N, respectively positive throughout, substitute for x a series of values increasing from α to λ, and taken so closely upon each other that the results of P shall differ by less than h consecutively. Among these results then there will be, *at least*, two successive ones of different signs. For example,

$$x = \eta \text{ gives } P' - N' \text{ negative, or } P' < N',$$
$$x = \theta \ \ldots\ldots \ P'' - N'' \text{ positive, or } P'' > N''.$$

Since P and N go on increasing, it is clear that the 4 numbers P', N', N''

P'' are ranged *in order of magnitude;* and since the difference between the extremes does not amount to h, $P' - N'$ and $P'' - N''$ must also be $< h$; these binomials therefore may also be made to approach indefinitely to zero, h allowing of diminution *ad libitum;* and thus $P - N$ is reduced to nothing, within at least the quantity h. The idea attached to incommensurables allows of our concluding from this that *there is at least one root of $X = 0$, between the numbers α and λ which give to X contrary signs.* The case in which $P = N$ exactly is not excluded from this reasoning.

Supposing that

$$x = \eta \text{ gives } P' - N' \text{ positive, or } P' > N',$$
$$x = \theta \text{ ...... } P'' - N'' \text{ negative, or } P'' < N'',$$

we might dispose of the four results in the order of *decreasing magnitudes,* N'', P'', P', N'; but if N varies by intervals $< h$, whence $N'' - N' = 0$, within less than h, *à fortiori,* we have $P' - N' = 0 = N'' - P''$.

The above process may also serve to approximate *ad libitum* to a root comprised between α and λ, by contracting these limits indefinitely by substitutions of intermediate numbers. [See N°. 525].

Hence, *an equation which has no real roots cannot, by the substitution of any numbers, produce results of opposite signs;* the sign of all the results will be that of the 1st term kx^m; since, when x has attained a certain limit, these results all preserve the same sign.

513. P and N in the outset converge towards each other, so long as P is $< N$; and diverge on P becoming $> N$; but if these results could again begin to converge and then to diverge, &c., $P - N$ would thus pass several times from one sign to the other, in the interval between α and λ, which is the case now to be examined.

Suppose that, the results being still of opposite signs, the equation $X = 0$ has two roots a and b between α and λ; we may [N°. 500] assume $X = (x - a) \cdot (x - b)\ Q$. Now for $x = \alpha$, $x - a$ and $x - b$ are negative; they are positive when $x = \lambda$; and their product therefore has the sign $+$ in both cases. But X must take contrary signs by supposition; so that Q must change signs, and $Q = 0$ will also have a root between α and λ; $X = 0$ therefore has 3 roots within this extent, If we suppose that there is a 4th, we shall in like manner recognize a 5th, &c. Thus, *when two values, substituted for x in $X = 0$, give results of contrary signs, this equation has* 1, 3, 5... *or an odd number of roots intercepted between these limits.*

If α and λ give results of the same sign, as $P - N$ positive, it may be that all the intermediate numbers from α to λ leave $P > N$; and no one of them giving $P = N$, there is no root in this interval. But if

$x = \theta$ give a negative result, it is then certain that there are 1, 3, 5... roots between α and θ, and 1, 3, 5... between θ and λ; which makes altogether 2, 4, 6... roots between α and λ. Thus *two quantities which, substituted for x, give results of the same sign, intercept no root, or include an even number of roots between them.*

The converse of these theorems is true; for if, for instance, there be three roots a, b, c, between α and λ, the proposed equation is

$$(x - a)(x - b)(x - c)\,Q = 0;$$

and the three first factors being negative for $x = \alpha$, and positive for $x = \lambda$, their product has different signs; but Q must retain the same sign, since, otherwise, besides our 3 roots, there would be others between α and λ; hence, &c.

The case of $a = b = c$ supposes $X = (x - a)^3 Q$; and all that has been now said still holds good, only the root a is comprised 3 times between α and λ: thus the equality of the roots does not destroy the generality of our theorems.

514. In these reasonings α and λ are supposed to be positive; if that be not the case, assume $x = y - h$ in $kx^m + px^{m-1}...$; when we have $k(y-h)^m + p(y-h)^{m-1}...$ Make $x = \alpha$ and λ; the results $k\alpha^m + p\alpha^{m-1}...$, $k\lambda^m + p\lambda^{m-1}...$ are precisely those which would be obtained by assuming $y = h + \alpha$ and $h + \lambda$ in the transformed equation; and since h is arbitrary, these substitutions may be made positive, if we wish it, and we shall therefore judge from the similar or different signs of these results whether there are 0, 2, 4..., or 1, 3, 5... roots between $h + \alpha$ and $h + \lambda$. Each intermediate root $y = h + \theta$, will give one $x = \theta$, comprised between α and λ;* hence, &c.

The theorem of N°. 512, which was demonstrated only for the case in which X has no signs but $+$, is equally true in all cases; for since P and N, considered separately, receive all values between those given by $x = \alpha$ and λ, the difference $P - N$ passes through degrees of magnitude succeeding each other within any intervals however small that can be proposed.

515. We will now examine the two cases of the degree being even or odd in the equation

$$kx^m + px^{m-1}... + tx + u = 0 = X.$$

I. If m is *even* and the last term u positive, making $x = 0$ and $=$

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* If α and λ are negative, as -2 and -7, a negative number between 2 and 7, as -4, is said to be *intermediate*; and if α and λ have different signs, as $+4$ and -3, every positive number < 4, or negative one < 3, is intermediate, as $+2$ -2 and 0. [See N°. 116].

the superior limit l, the two results will be both positive; so that if there are any positive roots, they will be of an even number [513]. The same will be the case for the negative roots, since $x =$ the superior limit $- l'$ of these will also give the sign $+$. Hence there are also 0, or 2, or 4... imaginary roots; and though there is nothing to indicate whether in fact any substitution can make X take the sign $-$, so that no root may be real; it is however certain that *the imaginary roots, the positive and the negative ones, if there are any such, are even in number, when the last term is positive and the degree even.*

When the last term u is negative, since $x = 0$ and $= l$ give results the one negative and the other positive, there is at least one positive root, and may be 3, 5... Changing x into $- x$, the signs of kx^m and u continue the same, which proves the existence of a positive root in the transformed equation, and consequently of a negative root in the one proposed; there may be 3, 5... Hence, *every equation of an even degree, in which the last term is negative, has two real roots of contrary signs; and, generally, an odd number of each sign; the imaginary roots are even in number.*

II. If m is *odd*, and the last term u negative, $x = 0$ and $= l$ give results of different signs; and, consequently, one positive root, or, 3, or 5... These roots being disengaged from the proposed equation, by dividing it by the corresponding binomial factors, the quotient will be even in degree, and the last term also will be positive, since otherwise there would still remain some positive root: thus we are brought back to the preceding case. Hence, *the positive roots of any equation of an odd degree, and in which the last term is negative, are of an odd number; the negative and imaginary roots, if there are any such, are even in number.*

When the last term u is positive, on substituting $- x$ for x, kx^m takes the sign $-$; but the signs being then changed throughout, kx^m again takes the sign $+$, whilst the last term, which was $+ u$, becomes $- u$; which brings us back to the preceding case. Hence, *every equation of an odd degree, in which the last term is positive, has an odd number of negative roots; the positive and imaginary roots, if there be any, are even in number.*

1°. *Every equation of an odd degree has one real root of a sign contrary to that of its last term.*

2°. *The imaginary roots of an equation are always even in number.*

3°. *An equation which has no real roots is necessarily of an even degree with its last term positive.*

4° Let $a, b..., - a', - b'...$ be the real roots of an equation...

$$X = T(x-a)(x-b)\dots(x+a')(x+b')\dots;$$

where it is supposed that $T=0$ has no real roots. The last term of X being the product of that of T, which is positive [3°], by $-a, -b\dots +a', +b'\dots$, its sign depends solely on the number of the negative factors; and, consequently, *the last term of an equation is positive or negative, accordingly as the number of positive roots is even or odd, without any regard to the number of the negative or imaginary roots.*

516. Thus it is proved, without any dependence on the theorem [501], that every equation $X=0$ has at least one real root, except when the degree is even and the last term positive. If it were possible to show that, in this last case, there exists, if not a real value, at least *an algebraical symbol, a function of the coefficients,** which will reduce X to zero, it would be demonstrated that every equation has one root, and therefore, according to N°. 501, that it has precisely m roots.

Let the last term $+u$ of X be changed into a negative and arbitrary quantity $-h$; then the equation $kx^m\dots+tx-h=0$ will have at least one positive root a, and be divisible by $x-a$. Thus, after several partial divisions we shall arrive at length at a dividend of the form $Ax-h$, and the following remainder $Aa-h$ must be $=0$: a therefore is a function of h, determined by this condition, and this function must certainly exist, though we be not acquainted with it; and we shall consequently have the identical equation

$$kx^m + px^{m-1}\dots + tx - h = Q(x-a).$$

* When a formula contains different letters $p, q\dots$ connected together by signs indicating the operations that are to be performed, this formula is said to be a *function* of the quantities $p, q\dots$ In the present case, the unknown quantity x is a function of the coefficients; for, if the roots exist, and be made to vary, the equation cannot retain the same coefficients, seeing that it is the product of the binomial factors $(x-a)(x-b)\dots$ The roots and the coefficients therefore have a dependence on each other, and x must contain the latter in its value; this relation between the quantities is written thus;

$$x = f(p, q\dots.), \quad x = F(p, q\dots.).$$

A division is made of functions into different species; the *implicit*, in which the quantities are involved among themselves; $y^2-2xy+1$ is an implicit function of x and y. The *explicit*, when the unknown quantities are separated in a resolved equation; $y = x \pm \sqrt{(x^2-1)}$ is an explicit function of x. The *algebraic functions* are those which involve only the operations of Algebra, extending to the extractions of roots. The *transcendental* functions contain logarithms, unknown exponents, sines, cosines, &c. We shall readily understand the farther denominations of *logarithmic, exponential, circular, functions...* We ordinarily express, in a function, those only of the letters contained in it, to which regard is intended to be paid, in the object we have in view. [See the note, N°. 620].

We may here assume for the arbitrary quantity h any number we think proper, and the identity will still subsist, provided only that we take for a the value corresponding to that of h. Make $h = -u$; then the 1st side becomes X, and the division of it by $x - a$ will be exactly possible. Thus the equation $X = 0$ has the root a; but since the radicals, with which h may be affected, may perhaps, on $-u$ being substituted for h, cease to be real, it is possible that this root a may be imaginary.

517. The theorems of N°. 515 may also be demonstrated by means of Geometry. The curve which has for its equation $y = X$ *consists of a single branch continued indefinitely in both directions as* $Q'CMN$ (fig. 2); for each value of x corresponds always to one of y. These curves have received the name of *Parabolic*, from the circumstance of one of the variables entering into the equation only in the 1st degree and in a single term, as for the common parabola. The abscissæ AR, AQ of the points R, Q..., in which the curve cuts the axis of x, correspond to $y = 0$, and are roots of the equation $X = 0$; they are positive for the sections R, Q..., on the right of the origin A, negative for R', Q'... on the left.

This being premised,

1°. if $x = AB$ and AD give results of different signs, the corresponding ordinates BC, DE are on opposite sides of the axis; and the curve in its course from C to E meets the axis at least once in R, or 3, 5... times between B and D. In like manner, when $x = AB$ and AP give results of the same sign, the ordinates BC, PM are on the same side of the axis, and the curve proceeds from C to M without cutting the axis, or cutting it in 2, 4... points. We shall readily perceive in these circumstances the proofs of what has been said [N°°. 512 and 513].

2°. When X is of an *even degree*, with its last term positive, $x = 0$ and $=$ the superior limit AP give points of the curve such as to require that from A to P there shall be no point of section with the axis, or 2, 4...; and the same for the negative values of x. But if the last term be negative, $x = 0$ gives the negative ordinate AF (fig. 3); whilst the limits B and D of the positive and negative roots give, on the contrary, the positive ordinates BC, DE; the curve therefore proceeds from E to F, and then to C, and cuts the axis at least once between D and A, and once between A and B; it may cut in 3, 5... points, either on one side of A, or the other. All this is in conformity with N° 515, I.

3°. If the degree of X is *odd*, with the last term negative, $x = 0$ and

$=$ the limit AB of the positive roots (fig. 4) give the points F and C on the opposite sides of Ax; and the curve proceeds from F to C, cutting the axis in 1, or 3, or 5... points. And if the last term be positive, $x=0$, and $=AB$, the limit of the negative roots (fig. 5), give the points C and F, and 1, 3, 5... intersections between B and A, agreeably to N°. 515, II.

Observe that the curve touches the axis of x, when two or more points of section coincide, *i. e.* when there are equal roots [See N°. 424, 523 and 713].

COMMENSURABLE ROOTS.

518. *The equation* $x^m + px^{m-1} + qx^{m-2} ... + u = 0$, *the coefficients of which are all integral, cannot have a fractional root* $x = \frac{a}{b}$. For in this case we should have $\frac{a^m}{b^m} + p\frac{a^{m-1}}{b^{m-1}} + ... = 0$, and multiplying the whole by b^m, $a^m + b\,(pa^{m-1} + qba^{m-2} ... + ub^{m-1}) = 0$; the second part of which is a multiple of b; and consequently a^m is divisible by b, which supposes, either that $b=1$, or that a and b are not prime to each other [N°. 24, 6°.]. Hence, &c.

Thus, when we have prepared the equation according to the mode of N°. 505, making $y = kx$, so as to render all the coefficients integral, and that of the first term 1, we may rest assured that there will then be no fractional value of y: the real roots of y, being divided by k, give those of x, which will be integral only when this division succeeds exactly. Thus, the investigation of the fractional roots that an equation may have is reduced to that of the integral roots of the transformed equation; and we thus obtain the rational binomial factors of the polynomial X, divided by the coefficient k of its 1st term.

Let $X = kx^m + px^{m-1} + qx^{m-2} ... sx^2 + tx + u$; and denote by $kx^{m-1} + p'x^{m-2} ... s'x^2 + t'x + u'$ the quotient of X divided by $x - a$. We have already given [N°. 500] a process for determining the quotient and the remainder R, viz.

$$ka + p = p',\ kp' + q = q' ...,\ s'a + s = t',\ t'a + t = u';$$

whilst the remainder from the division is $R = u'a + u$: hence

$$-k = \frac{p - p'}{a} ...,\ -s' = \frac{s - t'}{a},\ -t' = \frac{t - u'}{a},\ -u' = \frac{u - R}{a}.$$

Instead of calculating the coefficients $p' ... s', t', u', R$, step by step successively, we may, when R is known, obtain them in the inverse order $u', t', s' ...$; a mode which is of service in the investigation of the integral roots. For, the condition necessary and sufficient to express that $x - a$ divides X is that R be nothing; whilst, if a be an integral

number, it follows from the very nature of the calculation that $p' \dots, s', t', u'$ are all integral; thus, 1°. *a divides u; and we must look for the integral roots of the equation $X = 0$ only amongst the divisors of its last term:* 2°. *a divides $t - u'$, then $s - t' \dots$, and, finally, $p - p'$, and this last quotient is the coefficient of the first term, with a contrary sign.*

Such are the conditions which must be fulfilled by every integral root a; and it is, on the other hand, evident that every integral number a which does satisfy them is a root; since, if we compose, by the process just explained, the quotient of X divided by $x - a$, we shall arrive at a remainder nothing, and at the coefficients $k, p' \dots s', t', u'$, the same as those we had previously obtained.

This, therefore, is the course to be pursued: we must take, as well with the sign $+$ as $-$, the several divisors of the last term u, and the quotients u' of u divided by these numbers; and submit them to the tests above prescribed: if, in the course of the operations, any one of these divisors give a fractional quotient, we must reject it as incapable of being a root; but we shall recognize as such every divisor which leads in the end to $-k$, or the coefficient of the 1st term with a contrary sign. The series of numerical quotients thus obtained forms the coefficients $u', t', s' \dots k$ of the algebraical quotient of X divided by $x - a$, but with contrary signs.

± 1 divides all the numbers; so that the trials of these divisors of u giving quotients always integral, it would not be until the close of the calculation that we could ascertain, from not finding $-k$ for the quotient, that ± 1 is not a root; and we consequently prefer substituting ± 1 in the equation. We pass over likewise in our trials those divisors which exceed the limits of the roots [N°. 507].

For example, let $x^4 - x^3 - 16x^2 + 55x - 75 = 0$; since $75 = 3 . 5^2$, we easily find [N°. 25] that the divisors of 75 are $\pm$ (1, 3, 5, 15, 25 and 75). We shall exclude ± 1; for the coefficients, summed alternately, give $-90 + 54$; and whether we take ± 54, the whole sum cannot be zero. We must likewise exclude the divisors which exceed $+17$ and -37, the limits of the roots. The following is the most convenient mode of arranging the calculation, where * points out the divisors to be rejected:

$a =$	15,	5,	3,	− 3,	− 5,	−15,	−25
$-u' =$	− 5,	−15,	−25,	25,	15,	5,	3
$55 - u' =$	50,	40,	30,	80,	70,	60,	58
$-t' =$	*,	8,	10,	*,	−14,	− 4,	*
$-16 - t' =$	...,	− 8,	− 6,	...,	−30,	−20	
$-s' =$	..,	*,	− 2,	...,	+ 6,	*	
$-1 - s' =$	.........,		− 3,	....	+ 5,		
$-k =$	....		− 1,	....	− 1		

Thus the proposed equation has only two integral roots, 3 and -5; the quotient by $x+5$ is $x^3-6x^2+14x-15$; and this again being divided by $x-3$, we have

$$(x+5)(x-3)(x^2-3x+5)=x^4-x^3-16x^2 \text{ \&c.}$$

We subjoin two other examples:

	$x^3+3x^2-8x+10=0$			
$a=$	2,	-2,	-5,	-10
$-u'=$	5,	-5,	-2,	-1
$t-u'=$	-3,	-13,	-10,	-9
$-t'=$	*,	*,	$+2$,	*
$s-t'=$	.....	.....	5	....
$-k=$	.....	.....	-1	....

$8x^3-7x^2-63x+36$							
9,	6,	4,	3,	2,	-2,	-3,	-4
4,	6,	9,	12,	18,	-18,	-12,	-9
-59,	-57,	-54,	-51,	-45,	-81,	-75,	-72
*	*	*	-17,	*,	*,	$+25$,	$+18$
.....	.....	.....	-24,	.....	.....	$+18$,	$+11$
.....	.....	.....	-8,	.....	.....	*,	*

In the 1st, the factor $x+5$ gives the quotient $x^2-2x+2=0$. In the 2nd, among the divisors of 36, we make trial of those only which lie between the limits -5 and $+10$; whence we get the factor $x-3$ and the quotient $8x^2+17x-12$.

The following are some problems which may be solved by this process:

I. To find a number N, consisting of three figures x, y, z, such that, 1°. their product shall be 54; 2°. the figure in the middle be the 6th of the sum of the two others; 3°. that, subtracting 594 from the number proposed, the remainder shall be expressed by the same figures in the reverse order. These conditions give

$$xyz=54,\ 6y=x+z,\ 100z+10y+x=N-594;$$

and since $N=100x+10y+z$, this reduces the last of the preceding equations to $x-z=6$; eliminating y, we have $x^2z+xz^2=324$; and, lastly, substituting $z+6$ for x, we have $z^3+9z^2+18z=162$. But x, y, z are necessarily integral, and our method gives $z=3$; whence $x=9$, $y=2$ and $N=923$.

II. What is the base x of the system of numeration in which the number 538 is expressed by the characters (4123)?

The object here is to find the integral and positive root of the equation $4x^3+1x^2+2x+3=538$; which is 5. [See the note, p. 6. of vol. I].

Generally, if A is the number expressed by the n figures $a, b, c \ldots i$, the base x of the system is given by

$$ax^{n-1}+bx^{n-2}\ldots=A-i;$$

an equation which has only one positive root [N°. 530]. Moreover, x is integral and $>a, b, c \ldots i$.

III. Let the equation proposed be $8.(\frac{2}{5})^{x^3-5x^2+3x+3}=125$; observing that $\frac{125}{8}=(\frac{5}{2})^3$, the rules of N°. 147, 3°. give

$$(x^3-5x^2+3x+3)\log\tfrac{2}{5}=3\log\tfrac{5}{2},\text{ or } x^3-5x^2+3x+3=-3;$$

whence we deduce $x=2$, and $=\frac{1}{2}(3\pm\sqrt{21})$.

IV. For $6x^4 - 19x^3 + 28x^2 - 18x + 4 = 0$, we assume $x = \frac{1}{6}y$; whence $y^4 - 19y^3 + 168y^2 - 648y + 864 = 0$. There are no negative roots, and the positive ones are < 20. But $864 = 2^5.3^3$, which leads us to make trial of the divisors 2, 3, 4, 6... 18. We thus find $y = 3$ and 4; whence $y^2 - 12y + 72 = 0$; and, lastly, $x = \frac{1}{2}, \frac{2}{3}, 1 \pm \sqrt{-1}$.

We find in like manner that
$6x^5 + 15x^4 + 10x^3 - x = x(x+1)(2x+1)(3x^2+3x-1)$.

519. These calculations may in some cases become very long and tedious; but the following method will serve to abridge them.

Make $x = \theta$ in the proposed equation $X = 0$, and let U be the result obtained; if now we make $x = z + \theta$, the transformed equation $z^m + Pz^{m-1}...$ will have precisely U for its last term. θ may here be any integer whatever; and the integral roots of x will correspond to those of z, comprised among the divisors of U. Let these latter be denoted by $\pm d'$, $\pm d''$...; it is clear then that the integral roots of x are all comprised under the form $x = \theta \pm d$, a number which must divide the last term of X.

Thus put for x any integer θ, and take all the divisors $\pm d$ of the result U (it will be well to select θ so that U may have but a small number of divisors); it will be unnecessary then to submit to the process prescribed by the general method any other of the divisors of the last term u, than those which are comprised in the form $\theta \pm d$. We may take several values of θ, which will give as many systems of exclusion.

In the problem IV, making $y = 1$, we have $U = 366$, the divisors of which are 1, 2, 3, 6, 61; the integral roots of y therefore are of the form $1 \pm$ these divisors; and consequently y is among the numbers 2, 3, 4, 7, confining ourselves within the limits 0 and 20. After this, we need only try 2, 3, and 4, for 7 does not divide 864.

520. Let us now investigate the *commensurable factors of the 2nd degree of the equation* $X = 0$. One of these factors being $x^2 + ax + b$, the other will be of the form $x^{m-2} + p'x^{m-3} + ...$, and we shall have the identical equation,

$$X = (x^2 + ax + b)(x^{m-2} + p'x^{m-3} + q'x^{m-4}...);$$

where there are m unknown coefficients. Effecting the multiplication, and comparing the respective terms of the two sides, we shall have m equations; eliminating p', q'... between these, there will remain but two equations between a and b; and, lastly, one containing b alone, and which will be of the degree $\frac{1}{2}m(m-1)$, that being the number of the combinations 2 and 2 of the factors of the 1st degree. This last equation will give for b at least one commensurable value, or otherwise

X would have no rational factor of the 2nd degree; and this value of b, introduced into the equation between a and b, will give a, and consequently x^2+ax+b.

For example, let

$$x^4-3x^2-12x+5=(x^2+ax+b)(x^2+p'x+q');$$

then

$$a+p'=0,\ b+ap'+q'=-3,\ p'b+aq'=-12,\ q'b=5.$$

The two first give p' and q'; substituting in the two others, we have

$$2ab+3a-a^3=12,\ b^2+b(3-a^2)+5=0,$$

and eliminating b, we find $a^6-6a^4-11a^2=144$; whence $a=3$ and -3, $b=5$ and 1; and the factors consequently are

$$(x^2+3x+5)(x^2-3x+1).$$

ELIMINATION.

521. $A, a, B, b\ldots$ being functions of y, let us investigate *the several pairs of values* which, substituted for x and y in the polynomials Z and T, reduce them to zero:

$$Z=Ax^m+Bx^{m-1}\ldots,\ T=ax^n+bx^{n-1}\ldots.$$

Supposing $m=$ or $>n$, divide Z by T; Z being first multiplied, if it be necessary for the purpose of avoiding fractions, by such a factor M, as will render A a multiple of a: M will be a number or a function of y(*). Let the quotient be represented by Q, and the remainder, which will be a function of y, by R; we have then

$$MZ=QT+R\ldots(1).$$

This equation is *identical*, clear of fractions and irrational quantities, and is true whatever x and y be; we may therefore substitute for x and y one of the pairs of values in question, when Z and T will be nothing; and R therefore will be so also,

$$i.\ e.\ R=0 \text{ and } T=0.$$

Conversely, if any values of x and y render R and T each nothing, we have $MZ=0$; and consequently $M=0$, or $Z=0$:

Thus, when instead of the equations............ $Z=0,\ T=0$

we take these .... $R=0,\ T=0,$

---

* The factor M is obtained in the same manner as for the common divisor [p. 118, vol. I.]. When the degree n is $=m-1$, which is the most ordinary case, at least in the subsequent divisions, we assume $M=a^2$, the square of the 1st coefficient of the divisor T; when the quotient will be

$$Q=a(Ax+B)-Ab.$$

We shall at once form this quantity, and its product by T, which must be subtracted from a^2Z: the two first terms will disappear, there will in fact be no necessity to attend to them; and we shall thus arrive at the remainder R by a very easy process.

we shall find all the pairs of values of x and y required; but besides them these equations admit others also, which give $M=0$ and $T=0$, *values foreign to the question, and which have been introduced in the course of the calculation.* The problem, however, though it do contain these superfluous solutions, is become more simple, since we shall of course have continued the division of Z by T, until x has been reduced in R one degree lower than in T.

Let now T, or rather $M'T$, be divided by R, M' being a suitable factor; assume Q' for the integral quotient, and R' for the new remainder; we have then, $\quad M'T = Q'R + R' \ldots (2)$;
and it may be proved as before that the several pairs of values which reduce T and R to nothing, also give $R=0$ and $R'=0$, equations which admit all the solutions required; whilst, at the same time, they contain also the pairs of values which render M' and R nothing; so that taking these latter equations instead of those proposed, we shall have the solutions wanted, and besides them others which are foreign to the question, and reduce to nothing, either M' and R, or M and T.

And this calculation, which is precisely that for finding *the common divisor of the polynomials Z and T*, must be continued until the degree of x is so reduced in the dividend mV and in the divisor D, as to give a *remainder Y, independent of x*, viz.

$$mV = Dq + Y \ldots (3),$$

whence $$D=0 \text{ and } Y=0 \ldots (4).$$

These two equations then admit all the solutions required; but they have others also foreign to the question, and which we shall recognise from the circumstance of their reducing to zero one of the introduced factors $M, M' \ldots m$, at the same time with the *corresponding divisor* $T, R \ldots D$.

The equation $Y=0$ has but one unknown quantity y: we must investigate its roots, and substituting them in $D=0$, an equation generally of the 1st degree in x, we shall have the several values of x, which pair off respectively with those of y; only if D be of the 2nd degree, each root of y will be coupled with two values of x, and so on. Hence, *to eliminate x and y between the equations $Z=0$, and $T=0$, investigate the common divisor for Z and T arranged according to x; continue the calculation until you arrive at a divisor D, which gives a remainder Y independent of x; and, lastly, replace the equations proposed by $Y=0$ and $D=0$*, which are called the *Final Equations.*

Let the equations be

$$2x^2 - y^2 + 1 = 0,\ x^2 - 3xy + y^2 + 5 = 0:$$

dividing the 1st equation by the 2nd, the quotient is 2, and the remainder, disengaged of the factor 3, is $D = 2xy - y^2 - 3 = 0$; the 2nd equation being now multiplied by $4y^2$, and divided by D, the

quotient is $2xy - 5y^2 + 3$, and the remainder $Y = -y^4 + 8y^2 + 9 = 0$. We resolve this equation by making $y^2 = z$; whence $z^2 - 8z = 9$, $z = 9$, and -1; and consequently $y = \pm 3$, and $\pm\sqrt{-1}$: lastly, substituting in D, we obtain the corresponding values $x = \pm 2$, $\mp\sqrt{-1}$.

For the equations $x^2 + 2xy - 3y^2 + 1 = 0$, $x^2 - y^2 = 0$, the first remainder is $D = 2xy - 2y^2 + 1$, the second $Y = 4y^2 - 1 = 0$, and lastly $x = -y = \pm\frac{1}{2}$.

P, Q, p, q being the given functions of y, the equations

$$x^2 + Px + Q = 0,\ x^2 + px + q = 0$$

give the final equations

$$(P - p)x + Q - q = 0,\ (Q - q)^2 + q(P - p)^2 = p(Q - q)(P - p).$$

For $x^3 + x^2 - xy^2 - y^2 = 0$ and $2x^2 - x(4y - 1) - 2y^2 + y = 0$, the first remainder is $(16y^2 - 2y - 1)x + 8y^3 - 6y^2 - y = 0 = D$; and the last divisor being now multiplied by $(16y^2 - 2y - 1)^2$, and divided by D, we have the final equation $32y^3(4y^3 - 12y^2 + 3y + 1) = 0 = Y$. From this we deduce $y = 0$ and $\frac{1}{4}$ [N°. 518]; and Y being then reduced to the 2nd degree, we find $y = \frac{1}{4}(5 \pm \sqrt{33})$. Lastly, D gives the corresponding values $x = 0$, $\frac{1}{2}$, -1 and -1.

522. It now remains for us to distinguish, or rather to avoid, *the extraneous solutions*, those which reduce to nothing M and T, or M' and R, or &c.

Let $y = \phi$ be a root of the equation $M = 0$, which does not contain x; substituting in $T = 0$, we shall have an equation $T' = 0$ in x alone, which will at the highest be only of the degree $n - 1$, seeing that, from the very nature of the calculation, M must be a factor of the coefficient a in $T = ax^n + bx^{n-1}\ldots$. Thus, resolving the equation $T' = 0$, we shall deduce from it $n - 1$ values, $x = \psi$, corresponding to $y = \phi$, which render M and T each nothing at the same time. Substituting these values ϕ and ψ of y and x in the identical equations (1) and (2), it is obvious that we shall have R and R' each nothing; and we might consequently be led to conclude that, ϕ having been substituted for y in these two remainders, there are $n - 1$ values ψ of x which give $R = 0$ and $R' = 0$. But this is not possible, since R' must be of a lower degree than $n - 1$, that of R: thus, $y = \phi$ must render $R' = 0$, without the aid of any value of x; or, in other words, M divides R'. Hence *the factor M, introduced in the first division, is a divisor of the second remainder R'*, or $R' = Mr$.

Throwing out, therefore, the factor M from R', *i.e.* replacing R' by r, the operation will be freed from the extraneous roots arising from M; and the calculation for the common divisor must after this be directed to the quotient r, instead of R'. It will in like manner be

found that M' exactly divides the 3rd remainder R'', which must be replaced by the quotient of R'' divided by M', in order to suppress the extraneous roots introduced by M'; and so on, till we arrive at length at the final equation $Y=0$, which will thus be cleared of the whole of these extraneous roots. The last factor m of the equation (3) cannot introduce any, since every factor $y=\beta$ of m and of Y must also divide D [See N^{os}. 523, 4°. and 4th case].

Observe, that if we assume $y=\varphi$ in T and R, the polynomials in x which result must be identical, since they become nothing for the same $n-1$ values of $x=\psi$.

For example, let

$$x^3y-3x+1=0,\ x^2(y-1)+x-2=0:$$

multiplying the first by $(y-1)^2$, and dividing by the 2nd, we have

$$\text{1st. remainder...}\ -x(y^2-5y+3)+(y^2-4y+1)\dots(D);$$

multiplying the last divisor by $(y^2-5y+3)^2$, there results

$$\text{2nd. remainder...}\ y^5-10y^4+37y^3-64y^2+52y-16,$$

which must be divisible by $(y-1)^2$; and the quotient will be the final equation, clear of every extraneous root,

$$y^3-8y^2+20y-16=0\dots(Y).$$

The roots are $y=4$, 2 and 2; and $D=0$ gives $x=-1$, 1 and 1.

Should it appear, however, that the root $y=\varphi$ of $M=0$ reduced T to the degree $n-2$, M would no longer necessarily be a divisor of R', since the equations $R=0$ and $R'=0$ might admit the $n-2$ roots ψ of $T=0$, and the reasoning above would not now be applicable. We have an instance of this in the following example, where the factor y, introduced in the 1st division, does not divide the 2nd remainder, and shows itself again in the final equation:

$$(y-1)x^4-1=0,\ yx^3-x+1=0,$$

$$\text{1st. remainder...}\ x^2(y-1)-x(y-1)-y,$$

$$\text{2nd. ....}\ x(2y^2-2y+1)+y^2+y-1,$$

$$\text{3rd. ....}\ y(y^4-7y^3+14y^2-9y+2).$$

523. We have now some important remarks to make on this subject.

1°. If $x=\alpha$ and $y=\beta$ reduce both Z and T to nothing, making $x=\alpha$ in these polynomials, the results will no longer contain x, but will be each nothing for $y=\beta$; whence $y-\beta$ is a common factor. In order therefore to ascertain whether $x=\alpha$ forms a part of one of the solutions required, and to obtain the root $y=\beta$ corresponding to it, we must assume $x=\alpha$ in Z and T, investigate the common divisor of the results, and equate it to zero. If this factor be of the 2nd degree, α corresponds to two values of y, &c.

2°. If any combination of the polynomials Z and T give a more simple result, we may employ it in preference to Z; and it will be right also to arrange in respect to y, if the calculation becomes easier in consequence. Thus, in the 1st example of p. 55, adding the equations, and arranging relatively to y, which in the sum is only of the 1st degree, we arrive at once at the solutions.

When therefore Z and T are of the same degree m, by eliminating x^m as an unknown quantity, one of the equations may be reduced one degree lower.

3°. If Z is formed of two rational factors $Z = P \times Q$, it follows that, at the same time that $T = 0$, we must have either P, or Q nothing. Thus the problem proposed separates itself into two, and admits of the solutions of this double system:

$$T = 0 \text{ with } P = 0,\ T = 0 \text{ with } Q = 0.$$

And if T, P, or Q allow of being resolved into factors, the problem may be still farther sub-divided into others more simple.

In the 2nd example, p. 56, the equation $x^2 - y^2 = 0$ gives $x + y = 0$ and $x - y = 0$; whence we shall simply make $x = \pm y$ in the 1st equation.

4°. In the case when one of the polynomials that we meet with in the course of the calculation contains a factor which is a function of y (and it must be a factor of each term [N°. 102, 11]), we cannot now suppress this factor, as when our object was solely to obtain the greatest common divisor. According to what has been said, we must consider this equation separately, and equate it to zero; it contains a part of the solutions required.

For example, for $x^3 - 2x^2 + y^4 = 0$, $x^2(y-2) + xy = 0$, multiplying the 1st by $(y-2)^2$, and dividing by the 2nd, we have

1st remainder...... $y\,[x(3y-4) + y(y-2)^2] \ldots (D)$;

and before we take this remainder as a divisor, we must withdraw the common factor y, and put $y = 0$ in the 2nd equation, when we have $x = 0$. The other roots are then obtained by multiplying the 2nd equation by $(3y-4)^2$, and dividing, &c. The 2nd remainder must be divisible by $(y-2)^2$; which extraneous factor being suppressed, we arrive at the final equation

$$y^2(y^3 - 6y^2 + 9y - 4) = 0 \ldots (Y);$$

whence $y = 0, 1, 1, 4$; and consequently $x = 0, 1, 1, -2$.

In like manner the equations

$$x^2 + x(y-3) + y^2 - 3y + 2 = 0,\ x^2 - 2x + y^2 - y = 0$$

lead to the remainder $(y-1)(x-2)$; and we assume $y=1$ in the divisor: whence $x=0$ and 2. The calculation being now continued for the remainder $x-2$, the final remainder is found to be $y^2-y=0$, viz. $x=2$ with $y=0$ or 1.

5°. *If Z and T have a common factor F, $Z=PF$, $T=QF$*, the problem is resolved by assuming, either P and Q nothing, or $F=0$. The 1st system is treated in the usual manner, and gives different solutions; as to the equation $F=0$, since it cannot itself alone determine x and y, *the problem is indeterminate.* If F contain either x or y singly, this one of the unknown quantities is deduced from it, and the other is arbitrary; if $F=0$ contain both x and y, one may be assumed anything whatever, and the other must be deduced from it.

For example, taking the equations

$$(y-4)x^2-y+4=0,\ x^3-x^2-xy+y=0,$$

we find the common factor $x-1$, *i. e.* we have

$$(y-4)(x+1)(x-1)=0,\ (x^2-y)(x-1)=0;$$

and we shall therefore assume $x=1$ and y any thing whatever. Besides this infinite number of solutions, we shall have those also which result from the suppression of the factor $(x-1)$, viz. $y=1$ and 4; and $x=-1$ and ± 2.

The rule laid down N°. 521 presents four cases of exception; for it may possibly happen that the final remainder Y does not exist, or is a number; or that the last divisor D is in itself nothing, or is a number; in none of which cases can we assume Y and D each to be nothing.

1st Case. *The remainder Y not existing:* its terms must under these circumstances destroy each other, and Z and T will have a common factor, which is the last divisor. We have already examined this case (5°.), in which *the problem is indeterminate.*

2nd Case. *Y being a number:* since in the equation (3) V and D cannot now be both reduced to nothing at the same time, it follows from the analysis of N°. 521 that *the problem is absurd*, the equations proposed containing contradictory conditions. This exhibits itself in the equations

$$3x^2-6xy+3y^2-1=0,\ 2x^2-4xy+2y^2+1=0.$$

Assume any two equations with only one unknown quantity, as $3z^2-1=0$, $2z^2+1=0$; their coexistence is impossible, except in the case of a common factor N°. 501, 3°.; and if we now make $z=x+y$, or $x-y$, or any other such function of x and y, it is clear that the problem will be absurd.

3rd Case. *The last divisor D becoming a number α*, when in D we substitute for y a root β derived from $Y=0$. D being divided by

$y - \beta$, the quotient will be K in x and y, with the remainder L in y alone, or $D = (y - \beta)\ K + L$, in order that, making $y = \beta$, D may reduce itself to the value α which L then receives. But, that D' may be nothing, at the same time that $y = \beta$, x must evidently be infinite. For instance, the equations

$$y^3x^3 + xy^3(y-1) - 1 = 0,\ y^2x^2 + y^3 - y^2 - 1 = 0$$

have for their final equation $y^2(y-1) = 0$, and for the last divisor $xy - 1 = 0$; whence $y = 1$, $x = 1$, and $y = 0$, $x = \infty$.

4th Case. D *of itself becoming nothing*, when we make $y = \beta$, a value deduced from $Y = 0$: D will now [N°. 500] have the form $(y - \beta)K$; so that, according to equation (3), $y - \beta$ must also divide V, and this factor consequently ought to have been separately equated to zero [N°. 523, 4°.]. We must therefore repeat the calculation, having regard to this circumstance.

We will now show, by means of an example, how to eliminate three unknown quantities: let

$$x - 2y + z^2 = 0,\ x^2 + y^2 = 2,\ xz^2 = 1.$$

Eliminate y between these equations, 2 and 2; when you will have two final equations in x and z; and eliminating z between these, you will have an equation in x alone:

$$z^4 + 2xz^2 + 5x^2 = 8,\ z^2x = 1,\ 5x^4 - 6x^2 = -1.$$

Hence we have $x = \pm 1$, $5x^2 = 1$, and the four values of x are known; z results from the equation $xz^2 = 1$, &c.

EQUAL ROOTS.

524. A polynomial X, of the degree m, being resolved into its factors, it will appear under one or other of these forms,

$$X = (x-a)\ (x-b)\ (x-c)\ (x-d) \ldots\ (A),$$

or

$$X = (x-a)^n\ (x-b)^p\ (x-c)\ (x-d) \ldots\ (B).$$

In the latter of these cases, X is said to have n factors equal to $x - a$, p to $x = b$; or the equation $X = 0$ to have n roots $= a$, p roots $= b$; and our present object is to inquire into the means of ascertaining, without knowing these roots, whether X comes under the latter case, and of giving to this polynomial the form (B), supposing that we are competent to solve the equations when divested of the equal roots.

Since the equation (A) is identical, x may be replaced in it by $y + x$; and making this substitution, and developing the 1st side according to

the ascending powers of y, we shall have this identical equation [N°. 503],

$$X + X'y + \tfrac{1}{2}X''y^2 \ldots = (y + x - a)(y + x - b)(y + x - c)\ldots;$$

where X' is the derivative of X, X'' that of X'... The second side is formed of factors in each of which y is the 1st term; and the product therefore comes under the circumstances of p. 109, vol. I.; and the coefficients are the products, taken 1 and 1, 2 and 2, 3 and 3..., of the second parts $x - a$, $x - b$, $x - c$... Hence

1°. X is the product of these m binomials, or the polynomial (A).

2°. X' is the sum of their products taken $m - 1$ and $m - 1$; *i. e.* each factor, in the product (A), must be omitted successively, and the results added together.

And similarly for the other coefficients $\tfrac{1}{2}X''$, $\tfrac{1}{6}X'''$,...

This being premised, if X have but one factor $= x - a$, all the terms of X' will also contain this factor, except the term $R = (x - b)(x - c)\ldots$ in which $x - a$ was omitted in its turn. Thus X' has the form $R + (x - a)Q$, and is not divisible by $x - a$; and it may in like manner be proved that no one of the unequal factors of X can divide X'. Hence, *if X have all its factors unequal, X and X' have no common divisor.*

But if, in (A), we have $a = d = e\ldots$, as is the case for the equation (B), since, in order to form X', each of the n factors $x - a$ of X must be omitted in its turn, $x - a$ will enter into each of the results in the power $n - 1$, *i. e.* we shall have n terms equal to $(x - a)^{n-1}(x - b)^p(x - c)\ldots$; after which each of the other factors $(x - b)$, $(x - c)$... will also have to be successively omitted, and these last results will all contain $(x - a)^n$. Assuming therefore that $R = (x - b)^p(x - c)\ldots$, we have

$$X' = n(x-a)^{n-1}R + Q(x - a)^n = (x - a)^{n-1}[nR + (x - a)Q];$$

whence it appears that X is divisible by $(x - a)^n$, and X' by $(x - a)^{n-1}$; and consequently each multiple factor in X enters also into X', but is of a power precisely less by unity. Hence, *if X contain equal factors, X and X' have a common divisor, formed of the product of all these factors of X, each raised to a power one lower than that in X.*

Accordingly, a polynomial X being given, we must form its derivative X', and proceed to the investigation of the greatest common divisor between X and X'; if it be unity, X has no equal factors; whilst if there be a divisor F, a function of x, no one of the unequal factors of X will enter into it; but $\quad F = (x - a)^{n-1}(x - b)^{p-1}\ldots\ (C)$.

The calculation will give F under the form $x^k + p'x^{k-1} + \ldots + u'$;

which must then be decomposed into the form (C), in order to obtain the equal factors and their exponents.

Now, dividing the proposed equation (B) by F, *the quotient q will be formed of all the same factors as X, unaffected with the exponents,*

$$q = (x - a)(x - b)(x - c)(x - d) \ldots (D).$$

Let the equation $q = 0$ be solved; we shall then learn, by means of division, what are the factors $x - a$, $x \mp b \ldots$ of F, and what are the exponents; and these powers being each increased by unity, we shall have X brought under the form (B).

525. This theory may be exhibited under a more regular form. Let $[1]$ denote the product of all the unequal factors, $[2]^2$ that of the square factors, $[3]^3$ that of the cubic, &c.; also let F be the greatest divisor of X and X', and q the quotient of X divided by F; then the relations B, C, D will become

$$X = [1].[2]^2.[3]^3.[4]^4.[5]^5\ldots$$
$$F = \quad [2].[3]^2.[4]^3.[5]^4\ldots,\ q = [1].[2].[3].[4].[5]\ldots$$

Repeat on F the same course that has been pursued towards X, and let G be the greatest common divisor between F and F', and r the quotient of F divided by G; then

$$G = [3].[4]^2.[5]^3\ldots\ r = [2].[3].[4].[5]\ldots$$

In the same manner, let us take

$$H = \quad [4].[5]^2\ldots,\ s = [3].[4].[5]\ldots$$
$$I = \quad [5]\ldots,\ t = \quad [4].[5]\ldots;$$

and so on till we arrive at a polynomial N, for which the common divisor with N' is 1. It is evident then that, dividing q by r, the quotient is $[1]$; for r divided by s, it is $[2]$; for s by t, it is $[3]$,...; and thus, without knowing the factors of X, the polynomial will be subdivided into as many factors as there are different exponents, these factors being unaffected with their respective powers. The first $[1]$ will contain all the unequal factors; the second $[2]$ all the factors which were affected with the square, become unequal; the third $[3]$ all the cubic factors, reduced to the simple power, &c. If any one of these exponents be wanting in X, the corresponding quotient will be $= 1$; and, finally, the factors which have the highest exponent are given by the last N of the polynomials $G, H\ldots$, which is found to lead to the common divisor 1.

The operations therefore that are to be gone through may be thus tabulated:

Polynomials...	X,	F,	G,	H,	I...	N,
1st quotients...		q,	r,	s,	t...	N,
2nd quotients...		[1],	[2],	[3],	[4]...	N.

In the 1st line, each term is the common divisor of the preceding term and its derivative; the polynomials which compose the 2nd and 3rd line are the respective quotients of each term of the preceding line, divided by the term following in that line.

We shall now give some applications of this theory:

I. Let

$$X = x^5 - x^4 + 4x^3 - 4x^2 + 4x - 4;$$

we deduce from it

$$X' = 5x^4 - 4x^3 + 12x^2 - 8x + 4,$$

and the common divisor $F = x^2 + 2$; this has, with its derivative $2x$, the divisor 1; and thus the 1st line is terminated. Passing to the 2nd, X divided by F gives

$$q = x^3 - x^2 + 2x - 2, \text{ and we then find } r = x^2 + 2;$$

dividing q by r, we have $[1] = x - 1$, $[2] = x^2 + 2$, and lastly

$$X = (x - 1)(x^2 + 2)^2.$$

II. Let

$$X = x^6 + 4x^5 - 3x^4 - 16x^3 + 11x^2 + 12x - 9;$$

then

$$X' = 6x^5 + 20x^4 - 12x^3\ldots, \text{ and the common divisor}$$

$$F = x^3 + x^2 - 5x + 3; \text{ whence } F' = 3x^2 + 2x - 5,$$

and the common divisor $G = x - 1$, the derivative of which is 1. To form the 2nd line, we must divide X by F, and F by G, when we shall have

$$q = x^3 + 3x^2 - x - 3,\ r = x^2 + 2x - 3,\ s = x - 1;$$

and, lastly, dividing q by r, and r by s,

$$[1] = x + 1,\ [2] = x + 3,\ [3] = x - 1,$$

and consequently

$$X = (x + 1)(x + 3)^2(x - 1)^3.$$

III. Let us also assume the polynomial $X =$

$$x^8 - 12x^7 + 53x^6 - 92x^5 - 9x^4 + 212x^3 - 153x^2 - 108x + 108;$$

then

$$X' = 8x^7 - 84x^6\ldots,\ F = x^4 - 7x^3 + 13x^2 + 3x - 18,$$
$$G = x - 3,\ G' = 1.$$

The divisions of X by F, &c, give

$$q = x^4 - 5x^3 + 5x^2 + 5x - 6, \; r = x^3 - 4x^2 + x + 6;$$

whence

$$[1] = x - 1, \; [2] = x^2 - x - 2, \; [3] = x - 3,$$

and consequently

$$X = (x - 1)(x - 2)^2 (x + 1)^2 (x - 3)^3.$$

IV. For $x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 4$, we have

$$F = x^4 + x^3 - 3x^2 - 5x - 2, \; G = x^2 + 2x + 1, \; H = x + 1,$$

$$q = r = x^2 - x - 2, \; s = t = x + 1,$$

$$[1] = 1, \; [2] = x - 2, \; [3] = 1, \; [4] = x + 1;$$

and lastly

$$X = (x - 2)^2 (x + 1)^4.$$

INCOMMENSURABLE ROOTS.

526. *Newton's Method.* An equation being cleared of its equal and commensurable roots, there will remain only the irrational and imaginary ones; let it be proposed to find the former of these. Suppose that we have succeeded in obtaining an approximate value of one of the roots, that it is comprised between α and θ, and is the *only one* between these limits; making $x = \gamma$, an intermediate number to α and θ, we shall judge from the sign of the result [N°. 512], whether the root lies between α and γ, or between γ and θ. Let the first of these be the case. If now we make $x = \beta$, between α and γ, we shall find whether the root is between α and β, or β and γ, &c.; and continuing thus to contract the limits of x, we may approximate indefinitely to its real value.

This however is too laborious a process to be carried into effect for very high approximations, and is in general employed only *to obtain a number α approaching to x within at least the 10th of the true value.* Let the error be denoted by y; we have then $x = \alpha + y$, and introducing this binomial into the proposed equation

$$kx^m + px^{m-1} + \ldots + tx + u = X = 0,$$

it gives [503]

$$X + X'y + \tfrac{1}{2}X''y^2 + \ldots + ky^m = 0;$$

where y is by supposition a very small quantity, and α does not enter into the denominator of any of the coefficients, which are the values of the polynomial X and its derivatives, when we make $x = \alpha$. The rule of Newton consists in supposing the quantities $y^2, y^3 \ldots$ to be so small that

they may be neglected, which reduces the transformed equation to $X + X'y = 0$; whence

$$y = -\frac{X}{X'} = -\frac{k\alpha^m + p\alpha^{m-1} \ldots + t\alpha + u}{mk\alpha^{m-1} + p(m-1)\alpha^{m-2} \ldots + t}.$$

Let this fraction, or its approximate value, be represented by β; $y = \beta$ gives $x = \alpha + \beta$ for a second approximation; and making $\alpha + \beta = \alpha'$, and denoting the new correction by y', it will be given by the same fraction, observing only to replace α by α'; whence $x = \alpha + \beta + y' = \alpha' + y'$, and so on.

Take, for example, $x^3 - 2x - 5 = 0$: making $x = 2$ and 3, the results -1 and $+16$ testify that there is a root between 2 and 3, and that it is nearer to 2 than to 3; since also $x = 2{\cdot}1$ gives $0{\cdot}061$, we perceive that $2{\cdot}1$ is greater than x, and nearer to the root than 2 is; whence, making $\alpha = 2{\cdot}1$, the correction is

$$y = -\frac{\alpha^3 - 2\alpha - 5}{3\alpha^2 - 2} = -\frac{0{\cdot}061}{11{\cdot}23} = -0{\cdot}0054;$$

and limiting ourselves to the ten-thousandths, we have, for a first approximation, $x = 2{\cdot}0946$. Repeating the process on the assumption that $\alpha = 2{\cdot}0946$, we shall have

$$y = -\frac{0{\cdot}000541550536}{11{\cdot}16204748} = -0{\cdot}00004851;$$

thus our 4th decimal figure was faulty, and we have, for a more accurate value, $x = 2{\cdot}09455149$. And we might carry on the calculation still farther, and so correct the last decimals, or verify their exactness.

It will be observed that, retaining the y^2, we have $y = \dfrac{-X}{X' + \frac{1}{2}X''y}$; and if, after having determined the correction y, we substitute it in the denominator, we shall have a nearer approximation of y. Thus, in our example, $y = -0{\cdot}0054$, substituted in $\frac{1}{2}X''y$, gives $-0{\cdot}034$; and adding this to $11{\cdot}23$, the denominator becomes $11{\cdot}196$; whence $y = 0{\cdot}0054483$, a value in which the last decimal figure is the only one that is faulty.

Again, take the equation $x^3 - x^2 + 2x = 3$: it has a root between the numbers $1{\cdot}2$ and $1{\cdot}3$, which lead to the results $-0{\cdot}312$ and $+0{\cdot}107$; whence, making $\alpha = 1{\cdot}3$, we have

$$y = -\frac{\alpha^3 - \alpha^2 + 2\alpha - 3}{3\alpha^2 - 2\alpha + 2} = -\frac{0{\cdot}107}{4{\cdot}47} = -0{\cdot}02,$$

and the corresponding value $x = 1{\cdot}28$. Since also $\frac{1}{2}X''y =$

$(8\alpha - 1)y = 2\cdot 9 \times y$, the denominator, diminished by 0·058, becomes 4·412; whence $y = -0\cdot 0242$; and consequently $x = 1\cdot 2758$.

Assuming now that $\alpha = 1\cdot 276$, we find $y = -0\cdot 00031552$, and subsequently $y = -0\cdot 000315585$; whence $x = 1\cdot 275684415$; and so on.

It has not yet been clearly demonstrated that we are at liberty to reject the quantities $y^2, y^3 \ldots$; and, in fact, since $x = \alpha + y$, if $a, b, c \ldots$ be the roots of $X = 0$, the values of y are $a - \alpha, b - \alpha \ldots$, the product of which is $\pm X$; and consequently [N°. 502],

$$\mp X' = (b - \alpha)(c - \alpha)\ldots + (a - \alpha)(c - \alpha)\ldots + (a - \alpha)\ \&c.,$$

$$-\frac{X'}{X} = \frac{1}{a - \alpha} + \Sigma, \text{ assuming } \Sigma = \frac{1}{b - \alpha} + \frac{1}{c - \alpha} + \ldots;$$

and, finally,

$$-\frac{X}{X'} \text{ or } \beta = \frac{a - \alpha}{1 + (a - \alpha)\Sigma}.$$

Now, that Newton's method may be successful in its object, the value $\alpha + \beta$ given by the correction β must approximate to the root a, either by excess or defect, more nearly than α does; so that, in a *numerical* point of view, and leaving out of consideration the sign, the error $a - \alpha$ must be $> a - \alpha - \beta$. To do away with the effect of the signs, let this imparity be squared; suppressing then $(a - \alpha)^2$ on each side, and the factor β, we have $2(a - \alpha) > \beta$, *i. e.* from our value of β,

$$1 + 2(a - \alpha)\Sigma > 0, \text{ or positive};$$

and *accordingly as this imparity proves true or false, the method of Newton will be good or bad.*

But, when $a - \alpha$ and Σ have the same sign, this condition is fulfilled; and it is not, when these signs are different and the product is $> \frac{1}{2}$.

If now the proposed equation have two roots differing but slightly from each other, so that α, which is near to a, be so also to b, so long as we take α between a and b, the expressions $a - \alpha$ and Σ will have different signs, since $\dfrac{1}{b - \alpha}$ being the 1st and the greatest term of Σ, its sign will also be that of Σ; and in this case therefore the process may be at fault. But if a be the greatest or the least root, and we descend gradually towards it in the 1st case, or ascend in the 2nd, the process will not be faulty, since $a - \alpha$ and Σ will have the same sign, seeing that α is $>$ in the 1st case, and $<$ in the 2nd, than any of the roots.

The equation $x^4 - 4x^3 - 5x^2 + 18x + 20 = 0$ has a root between 3·3 and 3·5: but if we assume $\alpha = 3\cdot 3$, we find $y = -0\cdot 107$, and $x = 3\cdot 193$, a less approximate value than 3·3; and, in fact, 3·3 is com-

prised between two roots 3·236... and 3·449... approaching very nearly to each other.

If the proposed equation have two imaginary roots, as... $k \pm l\sqrt{-1}$, and it will be proved that these roots are all of this form [N°. 533], Σ will have two terms

$$\frac{1}{k-\alpha+l\sqrt{-1}}+\frac{1}{k-\alpha-l\sqrt{-1}}=\frac{2(k-\alpha)}{(k-\alpha)^2+l^2}$$

If now l be very small, this fraction will differ but little from $\frac{2}{k-\alpha}$; in which case, when α is near to k, the fraction will be very large, and Σ will take its sign. When therefore α is between k and a, the signs of $a-\alpha$ and Σ will be again different; and thus we see the possibility of another case of exception; which occurs when the proposed equation has some imaginary root $k \pm l\sqrt{-1}$, in which l is very small, and k very near to a.

The method of Newton therefore cannot be applied with any certainty, in as much as we ought only to make use of it, on the supposition of the fulfilment of a condition which it is impossible to verify, since it depends on roots that are not known. As, however, the cases of exception are rare, and when they do occur, become manifest of themselves in the course of the calculation, the method is very generally employed on account of its great facility.

527. *Method of Lagrange.* The most important thing to be ascertained, when we wish to resolve an equation, is *the locus of the real roots, i.e.* a series of limits between which each root is *singly* comprised; and this is, in fact, the true point of difficulty, and must be the essential object in every theory regarding the incommensurable roots; to which the preceding method is not an exception, as in it we are presumed to know, *à priori*, a number near to the root which we are investigating. When, with Bernoulli, whose method will be explained N°. 556, we have approximated to the greatest root a, the division will give a quotient more or less altered, in which some real roots may be lost, or some acquired at the expense of the imaginary ones; and since this belongs to the case in which the proposed equation has two roots very near to each other, we shall see the necessity of separating them, and generally of knowing their locus.

When, on substituting for x the numbers $-2, -1, 0, 1, 2, 3$..., we obtain as many results of different signs as the degree of the equation contains units, all the roots are real, and the locus of each is at once ascertained. But, with the exception of this case, we always remain uncertain as to the number of the real roots and their limits; for two

results of different signs may announce the presence of 1, 3, 5... roots between the numbers substituted, whilst two results with the same sign may intimate the existence of 2, 4... intermediate roots [N°. 513]. If, however, a series of substitutions can be selected, succeeding each other so closely that at the most we can fall in with only one root between any two of them, we shall then be certain that *every change of sign in the results points out a single root between the two numbers substituted; whilst there will be no intermediate root if the results have the same sign.*

If now two roots a and b be comprised between α and λ, the four numbers α, a, b and λ are written in order of magnitude; whence it follows that $\lambda - \alpha > b - a$; and if, on the contrary, $\lambda - \alpha < b - a$, the two roots a and b do not both lie between α and λ. Thus, let the numbers α and λ be so selected that they shall be less apart from each other than these roots, and this will be sufficient to satisfy us that there is either but one root between them, or none. Hence, *if δ be less than the least difference between the roots, and, commencing from the inferior limit l, we substitute the numbers l, $l + \delta$, $l + 2\delta$... up to the superior limit L, we shall obtain as many results with different signs, as there are real roots.* Each change of sign will prove the existence of a single root between the two numbers substituted; and there will be no intermediate ones, if the signs of the results be the same.

To obtain this number δ, we must form the equation the roots of which are the differences of all those of the one proposed, taken 2 and 2. For this purpose, make $x = a + y$; then $X = 0$ becomes $X + X'y + \frac{1}{2}X''y^2... = 0$; and if a be a root, the 1st term X vanishing, we shall have, dividing by y,

$$X = 0, \; X' + \tfrac{1}{2}X''y + \tfrac{1}{6}X'''y^2... + ky^{m-1} = 0.$$

These equations are between the unknown quantities a and y; and since $y = x - a$, *y is the difference between the root a and all the other roots.* Let a be eliminated [N°. 521], and the result will be an equation $Y = 0$, the unknown quantity y of which will be the difference between any two of the roots whatever; for this result, being independent of a, is the same with the one that would have been obtained by making $x = b + y$, $x = c + y$, &c. Thus, $Y = 0$ is *the equation of the differences.*

Since y is the difference between any one root and all the others, the degree of Y will be the number $m(m-1)$ of the arrangements 2 and 2 of the m roots x.

The differences $a - b$, $b - a$; $a - c$, $c - a$;... are equal 2 and 2, only with contrary signs; so that if $y = \alpha$, we have also $y = -\alpha$, and Y must become nothing in both cases; thus *Y can contain only even*

powers of y. This results also from the consideration that Y may be decomposed into factors all of the form $(y^2 - \alpha^2)(y^2 - \beta^2)\ldots$

We may therefore assume $y^2 = z$, without the risk of introducing radicals; and we shall thus have an equation $Z = 0$, in which the unknown quantity z is the square of all the differences of the roots, *i. e.* we shall have *the equation to the square of the differences.*

528. We are already acquainted with the means of determining a number i less than any of the positive roots of $Z = 0$ [N°. 510] ($i < z$ or y^2, $\sqrt{i} < y$); and $\sqrt{i}$, or any less quantity, may be taken for the difference δ between the numbers to be substituted. Y and Z having the same coefficients, i is the inferior limit also of y, or $i < y$; so that we are at liberty to assume either $\delta = i$ or $\delta = \sqrt{i}$. Since the less δ be, the more substitutions we shall have to effect between the limits l and L, we must assume δ as large as possible to avoid multiplying the operations unnecessarily. Thus, when $i > 1$, we shall assume $\delta = i$, or we may make $\delta = 1$, *i. e.* substitute the natural numbers 0, 1, 2, 3...; and if i be < 1, we shall take $\delta = \sqrt{i}$. But since it would in this case be tedious to substitute for x a series of irrational and fractional numbers, the following course is to be preferred:

1°. We can approximate to $\sqrt{i}$, within less than some specified fraction, as $\frac{1}{4}$ or $\frac{1}{5}$... [N°. 63]; and we shall therefore take $\sqrt{i}$ within less than $\frac{1}{h}$, by defect; whence $\delta = \frac{k}{h}$. Only, since the calculations must not be rendered complicated by assuming too large a number for h, nor the substitutions multiplied by falling much below $\sqrt{i}$, we must be careful to select h, according to circumstances, so as to meet both these difficulties.

2°. Instead of substituting $0, \frac{k}{h}, \frac{2k}{h}, \frac{3k}{h}\ldots$, we shall make the roots, and consequently their differences, h times greater [N°. 505], by assuming $hx = t$, and it will remain to substitute, for t, $0, k, 2k\ldots$, or, if we think proper, 0, 1, 2, 3.... Thus, *every equation may be transformed into another, having no more than one root comprised between any two successive integers whatever.*

It will be observed that i is deduced from the equation $Y = 0$, and that it is unnecessary to form Z. Moreover, by divesting X of its second term [N°. 504], the roots will all be increased by the same quantity, which will make no alteration in their difference; so that Y remains the same both for x and the transformed equation, by which the calculation is much simplified.

529. Let the equation, for example, be $x^3 - 2x = 5$, one root of which has been already found [N°. 526]; in order to convince ourselves that the two others are imaginary, change x into $x + y$; then $3x^2 - 2 + 3xy + y^2 = 0$; and eliminating x [N°. 521], there results the equation $y^6 - 12y^4 + 36y^2 + 643 = 0$. To obtain the inferior limit of y, make $y^2 = \frac{1}{v}$; this gives $643v^3 + 36v^2 \ldots = 0$, whence it appears that $v < 1 + \frac{12}{643}$, and we find, in fact, $v < 1$, and therefore $y > 1$. Thus $\delta = 1$ gives as many changes of signs as there are real roots; and therefore, &c.

The equation $x^3 - 12x^2 + 41x - 29 = 0$ gives

$$3x^2 - 24x + 41 + (3x - 12)\,2y + y^2 = 0;$$

and, therefore, getting quit of x, $y^6 - 42y^4 + 441y^2 = 49$. We make $y = \frac{1}{v}$, and there results $49v^6 - 441v^4 \ldots$; which gives $v < 10$, $y > \sqrt{\frac{1}{10}}$ or $\sqrt{\frac{1}{16}}$; and therefore $\delta = \frac{1}{4}$. Making $x = \frac{t}{4}$, we have

$$t^3 - 48t^2 + 656t = 1856;$$

which is the equation now to be solved. Assuming $t = 0, 1, 2 \ldots$, we shall see that t lies between 3 and 4, 21 and 22, 22 and 23; whence x is between $\frac{3}{4}$ and 1, $\frac{21}{4}$ and $\frac{22}{4}$, $\frac{22}{4}$ and $\frac{23}{4}$; so that x has two roots between 5 and 6, which could not have been discovered without this calculation. The roots are $x = 0{\cdot}95108\ldots$, $5{\cdot}35689\ldots$, $5{\cdot}69203$.

In like manner, $x^3 - 7x + 7 = 0$ gives $y^6 - 42y^4 + 441y^2 = 49$, whence $v < 9$ and $y > \frac{1}{9}$ and $\sqrt{\frac{1}{9}}$; $\delta = \frac{1}{3}$. Assuming $x = \frac{t}{3}$, &c. we readily perceive that there is one root between -3 and $-\frac{10}{3}$, one between $\frac{4}{3}$ and $\frac{5}{3}$, and, lastly, a third between $\frac{5}{3}$ and 2: viz.

$$x = -3{\cdot}04892\ldots, = 1{\cdot}35689\ldots, = 1{\cdot}69203\ldots$$

For the equation $x^3 - x^2 - 2x + 1 = 0$, since $x = 0, 1, 2$ gives $+1$, -1 and $+1$, we see at once that there is a root between 0 and 1, as also between 1 and 2; and $-x$ being now substituted for x, we find that the third root is between -1 and -2. Thus the equation of differences would be of no service. If, however, we investigated it, we should find $y^6 - 14y^4 + 49y^2 = 49$; and consequently $y > 1$, $\delta = 1$, which agrees with the statement just made.

These calculations can, in all cases, be carried into effect, and would leave nothing to be desired, did they not rise with the degree of the equation, till their complexity becomes so great as to render them at length impracticable [see N°. 557]; but, so far as regards the theory, it is clear,

complete, and free from embarrassment. It will remain to contract the limits of the roots in order to carry the approximation still farther; and Lagrange has given an additional process for effecting this, which we shall explain [N°. 573].

530. *Descartes' Rule.* When an equation $X = 0$ is arranged, we may frequently presume on the number of the positive and negative roots, simply from the inspection of the signs. When two successive signs are the same, the term *continuation* will be made use of to express it; and the term *variation*, if the signs are different. The theorem of Descartes consists in this: *Every equation has at the most as many positive roots as there are variations, and as many negative roots as there are continuations;* and this we have now to demonstrate.

Suppose, in order to fix our ideas, that the proposed equation presents this succession of signs:

$$+ - - + - - - + - + + + + - + - +.$$

If, in order to introduce a new negative root, we multiply by $x + a$, we shall have to multiply first by x, then by a, and to add the products; and these products will each present the same succession of signs, but the 2nd, in order to be in the same arrangement with the 1st, will have to be thrown one rank to the right, and written as under:

$$\begin{array}{cccccccccccccccccc}
+ & - & - & + & - & - & - & + & - & + & + & + & + & - & + & - & + & \\
 & + & - & - & + & - & - & - & + & - & + & + & + & + & - & + & - & + \\
\hline
+ & u & - & u & u & - & - & u & u & u & + & + & + & u & u & u & u & +
\end{array}$$

When the two corresponding signs are both the same, this sign will continue in the product; but, whenever this is not the case, we have put the letter u to indicate that an *uncertainty* exists as to the sign of the result, so long as the magnitude of the coefficients is not taken into account.

Now, since the two lines are composed of the same signs, the letter u will occur only when there is a variation; and an even number of successive variations will give an even number of letters u, situated between similar signs; whilst, on the other hand, when the variations are odd in number, the letters u will be so also, and fall between different signs. Hence, if we wish to dispose of all the coefficients so as to introduce *the greatest possible number of variations into the product*, the letters u must be changed into $+$ and $-$ alternately; and since each series of these letters lies between two similar or two different signs, accordingly as their number is even or odd, it is evident that we shall not be able to introduce more variations than there are letters u, or than there are variations in the equation proposed. But, the product

has one term more than the original equation; and, therefore, *there will be at least one more continuation.*

It is possible, however, that the letters u may not all give variations; in which case the product would have so many additional continuations, besides the one the existence of which has been just established. Thus, *the introduction of a negative root implies that of at least one continuation.*

Let the proposed equation be now multiplied by $x - a$, so as to introduce a positive root: the 2nd partial product, thrown back one rank to the right, will be composed of signs respectively opposite to those of the 1st; so that the letter u will in this case have to be written under each continuation in the original equation:

$$\begin{array}{ccccccccccccccccc} + & - & - & + & - & - & - & + & - & + & + & + & + & - & + & - & + \\ & - & + & + & - & + & + & + & - & + & - & - & - & - & + & - & + & - \\ \hline + & - & u & + & - & u & u & + & - & + & u & u & u & - & + & - & + & - \end{array}$$

And, since any succession of similar signs in the proposed equation must be terminated by a variation, each series of the letters u must be comprised between $+$ and $-$. Let these letters be so disposed of, by changing them all into $+$, or all into $-$, as to form *the greatest possible number of continuations;* there will be as many only as there are in the original equation, and the product having one more term than it, will consequently have *at least one variation more.* If the letters u do not all resolve themselves into continuations, there will be so many additional variations; and thus, *the introduction of a positive root implies that of at least one variation.*

Thus the proposed equation being the product of the binomial factors corresponding to the real roots, by a polynomial containing all the imaginary roots, each of the first factors will introduce at least one continuation or one variation, accordingly as the 2nd term of this factor is positive or negative; and hence the truth of the theorem enunciated follows as a necessary consequence.

531. Let the number of positive roots of an equation of the degree m be denoted by P, and that of the negative ones by N; the number of continuations by c, and that of the variations by v: it is demonstrated then that

$$1^{\circ}.\ P = \text{or} < v, \quad 2^{\circ}.\ N = \text{or} < c.$$

If now all the roots be real, we have $P + N = m$, and also $c + p = m$, since there are altogether $m + 1$ terms, and consequently

$$P + N = c + p.$$

Comparing P with c, these three cases present themselves, $P >$ or $<$ or $= c$: the 1st is proved to be impossible 1°.; if we assume the 2nd, the latter equation cannot subsist, unless, by way of compensation, we have $N > p$, which we have proved cannot be the case 2°. Hence, $P = c$ and $N = p$; *and when the roots of an equation are all real, it has precisely as many positive roots as there are variations of sign, and as many negative roots as there are continuations.*

532. The mere inspection of the signs of an equation is sufficient often to indicate that there are imaginary roots, and to relieve us from the tedious calculation of the equation of differences; as the following examples will serve to show:—

1°. *If one term be wanting, and the signs of the two adjacent terms be the same, the equation contains imaginary roots.* For, giving the coefficient ± 0 to the term that is wanting, we have three successive terms $Ax^{h+1} \pm 0x^h + Bx^{h-1}$; and, accordingly as we take $+ 0$ or $- 0$, we shall have two continuations or two variations. Thus, if all the roots were real, there would be, indifferently, two negative, or two positive roots, which is absurd. The equation $x^3 + 2x = 5$ has but one real root, which lies between 1 and 2. [see N°. 529].

2°. The three variations of $x^3 - 3x^2 + 12x - 4 = 0$ lead us to suppose that there are three positive roots. But let the equation be multiplied by $x + a$; then

$$x^4 + (a - 3)\,x^3 + (12 - 3a)\,x^2 + (12a - 4)\,x - 4a = 0;$$

and, selecting for a such a value as will introduce continuations, we see that $a > 3$ and < 4, for example, $a = 3\frac{1}{2}$, renders the four first terms positive; and, consequently, besides our three presumed positive roots, this equation should also have three negative ones, which is impossible. The proposed equation therefore has but one real root, which lies between 0 and 1.

3°. Change x successively into $y + h$ and $y' + h'$, and let the resulting equations be represented by $Y = 0$, $Y' = 0$. Suppose now that Y' have a less number of variations than Y: *if then all the values of x be real,* $Y = 0$ will have some one positive root α, which becomes a negative value $-\alpha'$ in $Y' = 0$, or $x = \alpha + h = -\alpha' + h'$; and thus x has one root $> h$ and $< h'$. The same may be said for each of the variations that have disappeared, and there should be as many values of x between h and h'; so that if the doctrine of limits shows that these roots of x do not all exist, we shall take it for granted that x has some imaginary values.

For example, $x^3 - 4x^2 - 2x + 17 = 0$ gives, when we change x into $y + 2$ and $y' + 3$,

$$y^3 + 2y^2 - 6y + 5 = 0, \; y'^3 + 5y'^2 + y' + 2 = 0;$$

and the two variations, which lead us to expect two positive roots of y, not existing for y', we consequently suppose that there are two values of x between 2 and 3. But, on the one hand, the inferior limit of y [N°. 510] is $y > \frac{5}{11}$; whilst, on the other, changing y' into $-y'$, the inferior limit is $\frac{2}{3}$; and since $y = y' + 1$, we find $1 - y > \frac{2}{3}$ and $y < \frac{1}{3}$. These two limits therefore being incompatible with each other, we conclude that x has two imaginary values. If the limits had not been contradictory, we should, it is true, have been still uncertain as to whether there are two roots of x between 2 and 3; but we should at least have contracted the interval within which they ought to be comprised.

IMAGINARY ROOTS.

533. Let $X = 0$ be an equation the roots of which are $a, b, c \ldots$; it will be easily seen that if one of the quantities $\alpha \pm \beta\sqrt{-1}$ satisfy this equation, the other must do so also. For, if we effect the substitution of one of them in X, the result will be of the form $P + Q\sqrt{-1}$. It follows also from the law of the development of powers that, if we now substitute the other binomial for x, the result will be the same, except as to the sign of the imaginary parts; and thus the double substitution gives $P \pm Q\sqrt{-1}$.

But, supposing that one of these results be nothing, since the real part cannot destroy the imaginary one, it follows that each must be separately nothing, *i. e.* $P = 0$, $Q = 0$. Consequently, this result must be nothing for both substitutions. *If therefore there be one root of the form $\alpha + \beta\sqrt{-1}$, there must be another $\alpha - \beta\sqrt{-1}$.* It must now be shown that all the imaginary roots have these forms.

Supposing in the first place that the degree m of X is the double $2i$ of some odd number $i (m = 6, 10, 14 \ldots)$, let us form the equation $V = 0$, which shall have for its unknown quantity $v = a + b + rab$, r being an arbitrary quantity. We shall have then to replace x in X by a and b, which will give two equations $X_1 = 0$, $X_2 = 0$, and to eliminate a and b between these three equations

$$X_1 = 0, \; X_2 = 0, \; v = a + b + rab.$$

Here, as for the equation of the squares of the differences [N°. 528], $V = 0$ will also have the roots $v = a + c + rac$, $v = b + c + rbc \ldots$, and the degree of V will be $n = \frac{1}{2} m(m - 1)$ the number of combina-

tions 2 and 2 of the m roots of x. But $n = i(2i-1)$ is necessarily odd, and v consequently will have at least one real root, as $v = a + b + rab$.

Now r may have an infinity of values assigned to it, which will lead to as many equations $V = 0$, each of them having some one real root as $a + c + r'ac$, or $b + c + r''bc\ldots$; and it is obvious that, after n values of r at the utmost, we must meet with an equation $V_1 = 0$, the real root of which is formed of a combination of the same two roots that enter into one of the preceding equations; for instance, $v_1 = a + b + r'ab$.

Thus we have proved that v and v_1 are real in the two expressions, which, making $a + b = A$ and $ab = B$, assume the form

$$v = A + rB,\ v_1 = A + r'B;$$

and consequently A and B, which appear in these equations as unknown quantities of the 1st degree, are also real. But the divisor of the 2nd degree, corresponding to the roots a and b of X, is the trinomial $x^2 - Ax + B$; and this divisor therefore will be real. Hence, in the case of $m = 2i$, *the proposed equation has at least one real factor of the 2nd degree*, and consequently the roots a and b have the form $\alpha \pm \beta\sqrt{-1}$.

This conclusion may be admitted for every value of m, provided we know that the equation $V = 0$ has one real root, whatever r be.

If $m = 4i$, i being always an odd number ($m = 4, 12, 20\ldots$), the degree n of V will then be $2i(4i-1)$, a number which answers to the supposition made in the foregoing case for m: thus $V = 0$ will have at the least one root of the form

$$\varphi = a + b + rab = \alpha + \beta\sqrt{-1} = A + rB;$$

and, changing the value of r, the reasoning just made use of will demonstrate the existence of the corresponding equation

$$v_1 = \alpha' + \beta'\sqrt{-1} = A + r'B.$$

A and B being eliminated between these two equations, they will no longer be real, as in the previous case, but of the form

$$A = a + b = \gamma + \delta\sqrt{-1},\ B = ab = \gamma' + \delta'\sqrt{-1}.$$

Thus, X has the factor of the 2nd degree $x^2 - Ax + B$; whence

$$x = \tfrac{1}{2}A \pm \tfrac{1}{2}\sqrt{(A^2 - 4B)};$$

and substituting for A and B their values, it is evident that $A^2 - 4B$ has the form $k \pm l\sqrt{-1}$, of which the square root is to be extracted. Assume

$$\psi = \sqrt{(k + l\sqrt{+1})} + \sqrt{(k - l\sqrt{-1})},$$
$$\omega = \sqrt{(k + l\sqrt{-1})} - \sqrt{(k - l\sqrt{-1})};$$

then

$$\psi^2 = 2k + 2\sqrt{(k^2 + l^2)},\ \omega^2 = 2k - 2\sqrt{(k^2 + l^2)},$$

and this last radical being $> k$, it gives its sign to ψ^2 and ω^2; whence one of them is positive, $\psi^2 = k'^2$; the other negative, $\omega^2 = -l'^2$; or. $\psi = k'$, $\omega = l'\sqrt{-1}$; and consequently

$$\psi \pm \omega = 2\sqrt{(k \pm l\sqrt{-1})} = k' \pm l'\sqrt{-1}.$$

And this is the form of $\sqrt{(A^2 - 4B)}$; whence it evidently follows that that of x is $p \pm q\sqrt{-1}$; and since these results must always exist together, X has in this case also the real factor of the 2nd degree $(x-p)^2 + q^2$. Thus, provided that V have a real factor of the 2nd degree, X will also have a similar one, whatever be the degree m.

If $m = 8i$, $n = 4i(8i-1)$, in which case it has already been demonstrated that V has a real factor, whence X has one also; and so on. Consequently,

1°. *Every equation of an even degree can be decomposed into real factors of the 2nd degree.*

2°. *The equations of an odd degree allow of a similar decomposition; only besides the factors of the 2nd degree there will be one real binomial factor of the 1st.*

3°. *The imaginary roots always enter in pairs under the form* $p \pm q\sqrt{-1}$.

4°. *Every imaginary algebraic function F is reducible to this form;* for whatever be the imaginary quantity, as

$$\sqrt[4]{-a},\ \sqrt[4]{a + \beta\sqrt{-1}},\ (a + \beta\sqrt{-1})^{m + n\sqrt{-1}},\ldots,$$

by equating the function to v, raising to the proper powers, and making the requisite transpositions, the imaginary expressions may always be made to disappear; and we shall thus arrive at an equation $V = 0$, in which the unknown quantity v has for one of its roots the value of the proposed function F. But it is proved that this value always has the form $p \pm q\sqrt{-1}$. Hence, &c.

534. Let therefore $x = a + \beta\sqrt{-1}$ be a root of the equation $X = 0$: substituting it for x, the equation will take the form $A + B\sqrt{-1} = 0$; and this separates itself into two, $A = 0$, $B = 0$, between which it will remain to eliminate the unknown quantities a and β. In regard to the latter of the equations, we must observe that, since $B\sqrt{-1}$ arises from the odd powers of $B\sqrt{-1}$, β will

be a factor of B; and this factor being done away with, there will remain only even powers of β in the equation $B = 0$; also, α does not enter into B in a higher degree than the $(m-1)^{th}$.

If now we assume $x = \alpha$ in X and its derivatives X', X''..., it will be easily seen [N°. 508] that our two equations correspond to

$$\left.\begin{array}{l} X - \frac{1}{2}\beta^2 X'' + \frac{1}{2.3.4}\beta^4 X^{IV} - \ldots = 0 \\ X' - \frac{1}{2.3}\beta^2 X''' + \frac{1}{2.3.4.5}\beta^4 X^{V} - \ldots = 0 \end{array}\right\} (A);$$

where the coefficients have for divisors the products 1.2.3.4... successively. Having eliminated α, we must take only the real values of β^2.

Suppose, for example, that $x^3 - 8x + 32 = 0$; we find

$$\alpha^3 - 8\alpha + 32 - 3\beta^2\alpha = 0,\ 3\alpha^2 - 8 - \beta^2 = 0;$$

and eliminating $\beta^2 = 3\alpha^2 - 8$, we have $\alpha^3 - 2\alpha - 4 = 0$, whence

$$\alpha = 2,\ \beta = \pm 2, \text{ and } x = 2 \pm 2\sqrt{-1}.$$

The other values of β are imaginary, and require no consideration. The proposed equation is $= (x + 4)(x^2 - 4x + 8)$.

535. Let a, b, c... be the real roots of the equation $X = 0$; $\alpha \pm \beta\sqrt{-1}$, $\gamma \pm \delta\sqrt{-1}$..., the imaginary ones; the differences then are of four descriptions:

1°. *Between two real roots*, $a - b$, $a - c$,..., the squares are positive: $(a - b)^2$...:

2°. *Between two imaginary roots of the same pair*, the squares are real and negative $-4\beta^2 - 4\delta^2$...:

3°. *Between a real and an imaginary root*, the squares are imaginary, as $(a - \alpha \pm \beta\sqrt{-1})^2$,...; unless we have $a = \alpha$; in which case the square is again real and negative $-\beta^2$: this square presents itself twice in consequence of the double sign $\pm$ of β, and is the fourth of the square of any other difference:

4°. *Between two imaginary roots of different pairs*, the square is again imaginary $[\alpha - \alpha' \pm (\beta - \beta')\sqrt{-1}]^2$: if however $\alpha = \alpha'$, the square is real and negative, being $= -(\beta - \beta')^2$; whilst, if $\beta = \beta'$, it is positive and $= (\alpha - \alpha')^2$. These squares also recur twice.

Hence the negative roots of the equation $Z = 0$ between the squares of the differences arise generally from the imaginary roots of corresponding pairs. To find these negative roots, change z into $-z$, and investigate the positive roots h, i...; assuming $h = 4\beta^2$, $i = 4\delta^2$..., we shall have $\beta = \frac{1}{2}\sqrt{h}$, $\delta = \frac{1}{2}\sqrt{i}$,...; and thus the imaginary part of

each pair will be known. These values of β being then substituted in the equations (A), the corresponding value of α must satisfy each of the equations, and they will have a common divisor in α (N°. 523, 1°.), which, equated to zero, will give α.

In the example of the preceding number, the equation $Z = 0$ is

$$z^3 - 48z^2 + 576z + 25600 = 0;$$

the alternate signs of which being changed, we find the only positive root to be 16; whence $\beta = \frac{1}{2}\sqrt{16} = 2$. Substituting this value in the equations (A), the results are $\alpha^3 - 20\alpha + 32$, and $\alpha^2 - 4$, which have $\alpha - 2$ for a common factor; whence

$$\alpha = 2,\ x = 2 \pm 2\sqrt{-1}.$$

For $x^4 + x^2 + 2x + 6 = 0$, the equation $Z = 0$ is

$$z^6 + 8z^5 + 70z^4 + 228z^3 - 3679z^2 - 1460z + 53792 = 0;$$

whence, changing the signs of the odd powers, we find $z = 8$ and $= 4$ and consequently $\beta = \sqrt{2}$ and 1. The equations (A) are

$$\alpha^4 + \alpha^2 + 2\alpha + 6 - (6\alpha^2 + 1)\beta^2 + \beta^4 = 0,$$
$$2\alpha^3 + \alpha + 1 - 2\alpha\beta^2 = 0;$$

and making $\beta = \sqrt{2}$ and 1, we find the common factors $\alpha - 1$ and $\alpha + 1$; whence $\alpha = 1$ and -1, and consequently

$$x = 1 \pm \sqrt{-2} \text{ and } -1 \pm \sqrt{-1}.$$

In the first example of N°. 529, we find $z = 5{\cdot}1614\,..$; whence $\beta = 1{\cdot}136$, $\alpha = -1{\cdot}0473$; and lastly,

$$x = -1{\cdot}0473 \pm 1{\cdot}136\sqrt{-1}.$$

Should the equations (A) have no common divisor, the conclusion will be that the negative root of z arises from one of the excepted cases, a presumption which will be verified by the calculation giving equal values of z, &c.

III. RESOLUTION OF SOME PARTICULAR EQUATIONS.

REDUCTION OF EQUATIONS.

536. *The degree of an equation $X = 0$ may frequently be reduced, if we know a relation, $f(a, b) = 0$, between two roots a and b.* For, substituting a and b for x successively, we shall have the three equations $f(a, b) = 0$, $A = 0$, $B = 0$; whence, eliminating b between the first and the third, we shall have a new divisor $F(a, b)$, and a final

equation in a alone, which must be co-existent with $A = 0$: these equations therefore must have a common divisor, a function of a, and which, equated to zero, will give a; and $F(a, b) = 0$ will subsequently give b. If this divisor is not to be found, the given relation $f(a, b) = 0$ cannot exist.

If, for example, we know that two of the roots x and a of the equation $x^3 - 37x = 84$ are such, that $1 = a + 2x$; eliminating a from $a^3 - 37a = 84$, there results $2x^3 - 3x^2 - 17x + 30 = 0$, which must have a common divisor with the equation proposed. And we find, in fact, that this factor is $x + 3$; whence $x = -3$ and consequently $a = 1 - 2x = 7$; which are the two roots in question; the third is $x = -4$.

Let $x^3 - 7x + 6 = 0$: if we still have $1 = a + 2x$, eliminating a from $a^3 - 7a + 6 = 0$, there results $(2x^2 - 3x - 2)\ 4x = 0$, the common divisor of which and the proposed equation is $x - 2$; and therefore $x = 2$, $a = -3$; and, lastly, $x = 1$.

Suppose that 2 is the sum of two of the roots of $x^4 - 2x^3 - 9x^2 + 22x = 22$; since it appears that $+ 2$ is also the sum of the four roots, the two others must give nothing for their sum, or $a = -x$. Substituting this value of a in $a^4 - 2a^3\ldots$, we have the original equation, except that the signs of the alternate terms are changed, or $x^4 + 2x^3 - 9x^2 - 22x\ldots$; and adding and subtracting these two equations in x, we find

$$x^4 - 9x^2 - 22 = 0,\ 2x^3 - 22x = 0;$$

equations which have $x^2 - 11$ for their common factor; whence $x = \pm\sqrt{11}$, and in the next place $x = 1 \pm \sqrt{-1}$.

537. The *reciprocal equations* are those in which the terms equally distant from the two extremes, have the same coefficient:

$$X = kx^n + px^{n-1} + qx^{n-2}\ldots + qx^2 + px + k = 0\ldots\ (1).$$

If a be one of the roots, $\frac{1}{a}$ will be so also, for on substituting these two values and getting quit of the denominators, we obtain identical results. *The roots therefore present themselves in pairs of reciprocal values*; and hence the name given to these equations.

1st Case. *Degree odd*: $n + 1$, which is the number of the terms in the equation (1), is in this case even; so that the middle coefficient P being repeated, it is evident that $x = -1$ satisfies the equation; and this is the only root that does not pair along with its reciprocal. To divide X by $x + 1$, we shall make use of the mode of calculation laid down in N°. 500. Let Q be the quotient, or $X = Q\ (x + 1)$; if now

we change x into $\frac{1}{x}$, and Q_1 represent the consequent value of Q, there results, multiplying the whole equation by x^n, $X = Q_1 (x+1)x^{n-1}$, since, by hypothesis, X will have undergone no change from this transformation. Thus $Q = Q_1 \times x^{n-1}$, which indicates that the quotient obtained is itself also a reciprocal equation $Q = 0$ of an even degree. Let the equation, for instance, be

$$x^9 + x^8 - 9x^7 + 3x^6 - 8x^5 - 8x^4 + 3x^3 - 9x^2 \text{ \&c.} = 0;$$

we have for the quotient

$$x^8 - 9x^6 + 12x^5 - 20x^4 + 12x^3 - 9x^2 + 1 = 0.$$

2nd. Case. *Degree even.* In this case the middle coefficient P is not repeated. Let n be changed into $2m$ in the equation (1); divide by x^m, and combine the terms having the same coefficient:

$$k(x^m + x^{-m}) + p(x^{m-1} + x^{-(m-1)}) + q(x^{m-2} + x^{-(m-2)}) \ldots + P = 0 \ldots (2);$$

then assume $z = x + x^{-1}$, and eliminate x. To effect this, we must form the integral power i of z; which, if i, A', A''... denote the coefficients in the development of the binomial, and the terms equidistant from the two extremes be combined, will be

$$z^i = (x^i + x^{-i}) + i(x^{i-2} + x^{-(i-2)}) + A'(x^{i-4} + x^{-(i-4)}) \ldots;$$

and, lastly, transposing, we have, whatever be the integer i,

$$x^i + x^{-i} = z^i - i(x^{i-2} + x^{-(i-2)}) - A'(x^{i-4} + x^{-(i-4)}) - \ldots$$

When i is even, the middle coefficient F is unique; and is repeated when i is odd; the last term of our formula being $-F$ in the 1st case, and $-F(x + x^{-1})$ in the 2nd. This formula gives the different terms of the equation (2): thus

$$\begin{aligned}
x + x^{-1} &= z, \\
x^2 + x^{-2} &= z^2 - 2, \\
x^3 + x^{-3} &= z^3 - 3(x + x^{-1}), \\
x^4 + x^{-4} &= z^4 - 4(x^2 + x^{-2}) - 6, \\
x^5 + x^{-5} &= z^5 - 5(x^3 + x^{-3}) - 10(x^1 + x^{-1}), \\
\text{\&c.} &= \text{\&c.}
\end{aligned}$$

It remains to assume $i = m$ and $m - 1$, substitute and reduce; to repeat the operation, making $i = m - 2$ and $m - 3$, and so on, the powers of i decreasing by 2 successively. It is obvious that the transformed equation in z will be reduced to the degree m, the half of n; and the roots of z being once determined, the equation $x + x^{-1} = z$ gives

$$x = \tfrac{1}{2}z \pm \sqrt{(\tfrac{1}{4}z^2 - 1)}.$$

The equation proposed above, $x^8 - 9x^6 + 12x^5 \ldots$, becomes

$$(x^4 + x^{-4}) - 9(x^2 + x^{-2}) + 12(x + x^{-1}) = 20;$$

whence

$$z^4 - 13z^2 + 12z = 0, \text{ and } z = 0, 1, 3 \text{ and } -4;$$

and consequently

$$x = \pm\sqrt{-1},\ \tfrac{1}{2}(1 \pm \sqrt{-3}),\ \tfrac{1}{2}(3 \pm \sqrt{5}), \text{ and } -2 \pm \sqrt{3}.$$

Thus the original equation of the 9th degree corresponds to

$$(x + 1)(x^2 + 1)(x^2 - x + 1)(x^2 - 3x + 1)(x^2 + 4x + 1) = 0.$$

In like manner, the equation

$$2x^5 - 3x^4 + x^3 + x^2 - 3x + 2 = 0$$

becomes, when divided by $x + 1$,

$$2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0,$$

or

$$2(x^2 + x^{-2}) - 5(x + x^{-1}) + 6 = 0.$$

Hence

$$2z^2 - 5z + 2 = 0, \text{ and } z = 2 \text{ and } \tfrac{1}{2};$$

consequently,

$$x = 1, \text{ a double root, and also } x = \tfrac{1}{4}(1 \pm \sqrt{-15});$$

and, lastly, the proposed equation is equivalent to

$$(x + 1)(x - 1)^2(2x^2 - x + 2) = 0.$$

EQUATIONS OF TWO TERMS: ROOTS OF UNITY.

538. Let the equation proposed for solution be $Ax^n = B$, A and B being positive. Suppose k to be the n^{th} root of $\frac{B}{A}$; then $k^n = \frac{B}{A}$, or $Ak^n = B$; whence, by substitution, $x^n - k^n = 0$; and making $x = ky$, it will remain to solve the equation $y^n - 1 = 0$, and to multiply the several values of y by k. Thus, *every number a has n different values for its n^{th} root; which are obtained by multiplying its arithmetical root by the n roots of unity.*

The equation $Ax^n + B = 0$ is reduced, by the same process, to $x^n + k^n = 0$, and subsequently to $y^n + 1 = 0$.

The equation $y^n - 1 = 0$ is evidently satisfied by $y = 1$; and, dividing by $y - 1$, we find

$$y^{n-1} + y^{n-2} + y^{n-3} \ldots + y + 1 = 0 \ldots (1),$$

a reciprocal equation, and consequently susceptible of reduction [N°. 537].

If now n be odd, since $y^n - 1 = 0$ can have no negative roots, and the equation (1) can have no positive ones, the proposed equation has only one real root.

If n be even, $y^n - 1 = 0$ is satisfied by $y = \pm 1$, and is divisible by $y^2 - 1$; whence $y^{n-2} + y^{n-4} \ldots + y^2 + 1 = 0$. And since, in this latter equation, the exponents are all even and the terms positive, it can have neither positive nor negative roots; so that the original equation has no other real roots than $y = \pm 1$.

Let $n = 2m$; we have then $y^{2m} - 1 = 0 = (y^m + 1)(y^m - 1)$; and the proposed equation separates itself into two others.

For example, $y^3 - 1 = 0$ gives $y^2 + y + 1 = 0$; whence

$$y = 1,\ y = -\tfrac{1}{2}(1 \pm \sqrt{-3}).$$

In like manner, $x^4 - k^4 = 0$ gives $y^4 - 1 = 0$; dividing by $y^2 - 1$, we find $y^2 + 1 = 0$; whence $y = \pm 1$, and $\pm \sqrt{-1}$; and, lastly, $x = \pm k$ and $\pm k\sqrt{-1}$.

539. Let α be one of the roots of the equation $y^n - 1 = 0$: since then $\alpha^n = 1$, we have $\alpha^{np} = 1$, whatever be the integer p, positive or negative; and thus the equation $y^n - 1 = 0$ is equally satisfied by $y = \alpha^p$, *i. e. if α be a root, α^p is so also.* Hence the following series, the numbers of which are all roots:

$$\ldots \alpha^{-4}, \alpha^{-3}, \alpha^{-2}, \alpha^{-1}, \alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4 \ldots \ (2).$$

1°. If we take $p > n$, dividing by n, p has the form $nq + i$, i being $< n$; so that

$$\alpha^p = \alpha^{nq+i} = \alpha^{nq} \times \alpha^i;$$

but $\alpha^{nq} = 1$ by supposition; and consequently $\alpha^p = \alpha^i$.

Thus, so soon as p exceeds n, we meet a second time with the same values, in the same order: whence this period

$$(\alpha^1, \alpha^2, \alpha^3 \ldots \alpha^n) \ldots (3).$$

2°. If p be negative, since $\alpha^n = 1$, we have $\alpha^{-p} = \alpha^{n-p} = \alpha^{2n-p} \ldots$; and the exponent $-p$ may therefore be replaced by $nk - p$; whence it appears that the negative exponents give a recurrence of the same numbers as the positive exponents, and in the same order.

Thus, *the values (2) are such, that if we take any one of them, and the $n - 1$ values which follow or precede it, we shall have a period which recurs indefinitely both ways.* Moreover, the equation $\alpha^p = \alpha^q$ is satisfied not only by $p = q$, but also by values of α which suppose p and q to be unequal. For, dividing by α^q, there results $\alpha^{p-q} - 1 = 0$; and it will suffice, for the existence of the equation $\alpha^p = \alpha^q$, that α be a root of the equation $y^{p-q} - 1 = 0$.

540. It remains to be seen if the n terms of the period (3) are in fact unequal; in other words, we must inquire if it be possible that $\alpha^p = \alpha^q$, p and q being each $< n$. Such can only be the case, on the condition that α, already a root of the equation $y^n - 1 = 0$, be so also of $y^m - 1 = 0$, making $p - q = m$; and this supposes these equations to have a common divisor, which, equated to zero, will give α. This factor we must investigate in the usual manner [N°. 103]. In the first place, dividing y^{n-1} by $y^m - 1$, we arrive at the remainders $y^{n-m} - 1$, $y^{n-2m} - 1, \ldots$, and lastly at $y^i - 1$, i being the excess of n above the multiples of m contained in it. We shall now divide $y^m - 1$ by this remainder, which will give the remainder $y^l - 1$, l being the excess of m above the greatest multiple of i, &c.; in a word, we shall proceed as though we were in search of the common factor between n and m.

1°. *If n be a prime number*, the only common divisor between n and m is 1, whence that of $y^n - 1$ and $y^m - 1$ is $y - 1$; $\alpha = 1$ is therefore the only value that can render $\alpha^p = \alpha^q$; and *all the terms of the period are unequal.* A single imaginary root α gives, by its powers ($\alpha^2, \alpha^3 \ldots \alpha^n$ or 1), all the other roots.

2°. *If n be the product of two prime factors l and h* ($n = lh$), assume the equations $y^l - 1 = 0$, $y^h - 1 = 0$, and let β and γ be roots of these equations other than $+1$, so that $\beta^l = 1$, $\gamma^h = 1$; and consequently $\beta^{lh} = \gamma^{lh} = (\beta\gamma)^{lh} = 1$. Since then β^n, γ^n and $(\beta\gamma)^n$ are each $= 1$, β, γ and $\beta\gamma$ are roots of $y^n - 1 = 0$; but ($\beta, \beta^2 \ldots \beta^l$) form l different numbers, which recur periodically; and thus the n powers of β form only l distinct numbers, which, in ($\beta, \beta^2 \ldots \beta^n$), recur h times. On the same principle, ($\gamma, \gamma^2 \ldots \gamma^n$) form l periods of h terms.

But ($\beta\gamma, \beta^2\gamma^2, \beta^3\gamma^3 \ldots \beta^n\gamma^n$) are all different, and constitute the period of the n roots required. For that we may have $(\beta\gamma)^p = (\beta\gamma)^q$, or $(\beta\gamma)^{p-q} - 1 = 0$, $\beta\gamma$ must be a root both of $y^{p-q} - 1 = 0$ and $y^n - 1 = 0$, equations which, since $n = lh$, can have no other factors than $y^l - 1$, or $y^h - 1$. Thus we should have $\beta^l\gamma^l = 1$; which, since $\beta^l = 1$, gives $\gamma^l = 1$; and as we have also $\gamma^h = 1$, l and h ought according to this to have some other factor besides unity, contrary to the hypothesis. Hence we shall conclude that if we assume $\alpha = \beta\gamma$, the period will be ($\alpha, \alpha^2, \alpha^3 \ldots \alpha^n$) consisting of n different terms.

The exponent p of $\beta^p\gamma^p$ may be reduced below l for β, and below h for γ; since $\beta^l = \gamma^h = 1$, and we may take from p all the multiples of l or h. Thus, $\beta^b\gamma^c$ represents all the terms of the period, b and c being the remainders from the division of p by l and h; and, consequently, to obtain all the roots of $y^n - 1 = 0$, we must investigate β and γ, *i.e.* one of the roots, other than $+1$, of each of the equations $y^l - 1 = 0$, $y^h - 1 = 0$; and then form $\beta^b\gamma^c$, taking for b and c all

the combinations of the numbers from 1 to l for β, and from 1 to h for γ.

When $l = 2$, we make $\beta = -1$.

When n is the product lhi of three prime numbers, it may be proved in the same manner that we must assume $y^l - 1 = 0$, $y^h - 1 = 0$, $y^i - 1 = 0$, deduce from each of these equations one other root than $+1$, form the product of these roots $\beta\gamma\delta$; and, lastly, take all the powers comprised in the form $\beta^b\gamma^c\delta^d$, b, c and d being the combinations of the numbers 1, 2, 3... up to l, h and i: and so on for the other cases.

3°. When the exponent n is of the form h^k, h being a prime number, we shall reason as in the following example: $y^{81} - 1 = 0$, where $81 = 3^4$. Assume $y^3 - 1 = 0$, and let θ be an imaginary root of this equation; take the 1st, 3rd, 9th and 27th roots of this value, viz. θ, $\sqrt[3]{\theta}$, $\sqrt[9]{\theta}$, $\sqrt[27]{\theta}$, and these will be as many solutions of the equation proposed, since the 81^{th} powers of them are powers of θ^3, which is $= 1$: the product $\theta . \sqrt[3]{\theta} . \sqrt[9]{\theta} . \sqrt[27]{\theta} = \alpha$ is also a root of y for the same reason. But $\alpha, \alpha^2, \alpha^3 ... \alpha^{81}$ are all different quantities, since otherwise α would be a root common to $y^{81} - 1 = 0$ and $y^i - 1 = 0$; and this supposes the equations to have a common factor, which can be no other than $y^3 - 1 = 0$; thus α would be a root of this latter equation, and $\alpha^3 = 1$, or $\theta^3 . \theta . \sqrt[3]{\theta} . \sqrt[9]{\theta} = 1$; whence, raising to the power 9, there results $\theta = 1$, contrary to the hypothesis. Consequently $\alpha, \alpha^2, \alpha^3 ... \alpha^{81}$ are the 81 roots of the equation proposed.

Generally, to resolve $y^n - 1 = 0$ when $n = h^k$, assume $y^h - 1 = 0$; and θ being one of its roots other than $+1$, extract the different roots of θ, the degrees of which are pointed out by $i = h^0, h^1, h^2 ... h^{k-1}$, so as to obtain the k results $\beta, \gamma ...$ denoted by $\sqrt[i]{\theta}$; these results will all be roots of $y^n - 1 = 0$, as will also their product $\alpha = \beta\gamma\delta ...$, and the terms $\alpha, \alpha^2, \alpha^3 ... \alpha^n$, all differing from each other, will constitute the n roots required.

It will in like manner appear that if $n = h^k l^i$, we must resolve $y^h - 1 = 0$ and $y^l - 1 = 0$, multiply the several roots of these equations together, and make this product $= \alpha$. Let β and γ be roots, others than $+1$, of each equation; and assume

$$\beta' = \sqrt[h]{\beta},\ \beta'' = \sqrt[h]{\beta'},\ \beta''' = \sqrt[h]{\beta''} ...,\ \gamma' = \sqrt[l]{\gamma},\ \gamma'' = \sqrt[l]{\gamma'} ...;$$

we shall have

$$\alpha = \beta\beta'\beta'' ... \times \gamma\gamma'\gamma'' ...$$

Let our example be $y^6 - 1 = 0$: we take $y^2 - 1 = 0$ and $y^3 - 1 = 0$; whence

$$\beta = -1,\ \gamma = -\tfrac{1}{2}(1 + \sqrt{-3}),$$

consequently

$$\alpha = \tfrac{1}{2}(1 + \sqrt{-3}),\ \alpha^2 = \tfrac{1}{2}(-1 + \sqrt{-3}),\ \alpha^3 = -1, \text{ \&c.},$$

and, finally,

$$y = \pm 1,\ \tfrac{1}{2}(1 \pm \sqrt{-3}),\ -\tfrac{1}{2}(1 \pm \sqrt{-3}).$$

For $y^{12} - 1 = 0$, make $y^4 - 1 = 0$ and $y^3 - 1 = 0$; for the 1st of these equations take -1 and $\sqrt{-1}$; their product is $-\sqrt{-1} = \beta$; γ is the same as in the preceding example, and we have

$$\alpha = \tfrac{1}{2}(\sqrt{-1} - \sqrt{3}),\ \alpha^2 = \tfrac{1}{2}(1 - \sqrt{-3}),\ \alpha^3 = \sqrt{-1}, \text{ \&c.};$$

whence

$$y = \pm 1,\ \pm\sqrt{-1},\ \pm\tfrac{1}{2}(1 \pm \sqrt{-3}),\ \pm\tfrac{1}{2}(\sqrt{-1} \pm \sqrt{3}).$$

541. Since $y = \alpha, \alpha^2, \alpha^3 \ldots$, the equation (1) [N°. 538] gives

$$1 + \alpha + \alpha^2 \ldots \alpha^{n-1} = 0,\ 1 + \alpha^2 + \alpha^4 \ldots \alpha^{2n-2} = 0,\ 1 + \alpha^3 + \alpha^6 \ldots = 0, \ldots;$$

or

$$\int_1 = \int_2 = \int_3 \ldots = \int_k = 0,\ \int_n = n,$$

denoting by $\int_k$ the sum of the powers k of all the roots, k being integral and not divisible by n.

542. The solution of the equation $y^n - 1 = 0$ has thus been reduced to the case in which n is a prime number. We shall now show the use that may be made of the trigonometrical lines, referring the reader for farther information to Note XIV *de la Resol. numer. des equ.*

Making cos. $x = p$, it has been seen, N°. 361, that the several cosines of the successive arcs $2x$, $3x$, $4x \ldots$ are obtained by multiplying the two preceding ones by $2p$ and -1, and taking the sum. To exhibit the law that these results observe, we shall have recourse to an analytical artifice. Assume $2 \cos. x = y + y^{-1}$; it follows from the rule specified that, to deduce cos. $2x$, we must multiply cos. x or $\tfrac{1}{2}(y + y^{-1})$ by $(y + y^{-1})$, the value of 2 cos. x, and subtract cos. 0 or 1; whence we shall find $2 \cos. 2x = y^2 + y^{-2}$. In the same manner we obtain

$$2 \cos. 3x = y^3 + y^{-3},\ 2 \cos. 4x = y^4 + y^{-4}, \text{ \&c.};$$

and it can be demonstrated that the results always follow the same law. For, suppose this law to be established for two consecutive numbers $n - 2$ and $n - 1$, or

$$2 \cos. (n-2)\, x = y^{n-2} + y^{-(n-2)},\ 2 \cos. (n-1)\, x = y^{n-1} + y^{-(n-1)};$$

multiplying the second of these equations by $y + y^{-1}$, and subtracting the first, there will result $2 \cos. nx = y^n + y^{-n}$; which proves the proposition.

Thus we have

$$2 \cos. x = y + \frac{1}{y},\ 2 \cos. nx = y^n + \frac{1}{y^n}$$

whence *

$$y^2 - 2y \cos. x + 1 = 0,\ y^{2n} - 2y^n \cos. nx + 1 = 0 \ldots (1).$$

If we know cos. x, these equations will give y and cos. nx, so that we shall be able to find cos. nx without successively investigating cos. $3x$, cos. $4x$... ; and this being the general term for the series of cosines, these equations might be employed for the composition of the tables; only the calculation would labour under the inconvenience of imaginary quantities.

If we suppose the tables of sines to be formed, and take the values of cos. x and cos. nx, our two equations, now containing only y, must have a common root a. But if we have $y = a$, we shall also have $y = \frac{1}{a}$, as may readily be perceived, the equations (1) being reciprocal; and they consequently have two roots common; in other words, the 1st divides the 2nd. Assume $nx = \phi$; it follows then that, whatever be the arc ϕ,

$$y^2 - 2y \cos. \left(\frac{\phi}{n}\right) + 1 \text{ divides } y^{2n} - 2y^n \cos. \phi + 1 \ldots (2).$$

543. To apply this theorem, which we owe to De Moivre, to the case under consideration, make $\phi = k\pi$, k being any integer, and π the semi-circumference; then cos. ϕ is $+ 1$ or $- 1$, accordingly as k is even or odd, and the 2nd trinomial becoming $y^{2n} \mp 2y^n + 1$, or $(y^n \mp 1)^2$, it appears that

$$y^2 - 2y \cos. \left(\frac{k\pi}{n}\right) + 1 \text{ divides } y^n \mp 1 \ldots (3),$$

k being any integer, *even for* $y^n - 1$, *odd when we have* $y^n + 1$. If the 1st trinomial be a square, we shall take its root as the divisor; that

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* Resolving these equations (1), we find

$$y = \cos. x \pm \sin. x. \sqrt{-1},\ y^n = \cos. nx \pm \sin. nx. \sqrt{-1};$$

whence

$$(\cos. x \pm \sin. x \sqrt{-1})^n = \cos. nx \pm \sin. nx \sqrt{-1}.$$

This beautiful property, of which frequent use is made in the higher branches of Algebra, though here demonstrated only for the case in which n is integral and positive, equally subsists in all cases.

We shall resume this subject in N°. 590.

this may be the case, the cosine must be ± 1; when k is $0, n, 2n\ldots$, and the factor reduces itself to $y \pm 1$.

The roots of $y^n \mp 1$ are comprised therefore in the formula

$$y = \cos.\left(\frac{k\pi}{n}\right) \pm \sin.\left(\frac{k\pi}{n}\right)\sqrt{-1}\ldots (4).$$

So long as the integer k does not exceed n, the arc $\frac{k\pi}{n}$ is a continually increasing fraction of the semi-circumference; the successive arcs have unequal cosines, and we obtain factors of the 2nd degree, all differing one from the other, which we shall represent by $A, B, C\ldots L, M$.

Since $n + i$ and $n - i$ are both even or odd together, let $k = n \pm i$, i being $< n$; we have then $\frac{k\pi}{n} = \pi \pm \frac{i\pi}{n}$, arcs the cosines of which are equal; whence it follows that the trinomial factor is the same for $k = n - i$ and $n + i$. Having therefore assumed for k all the numbers, even or odd, up to n, beyond this we shall again fall in with the same factors of the 2nd degree, in inverse order, $M, L\ldots C, B, A$.

Beyond $2n$, k has the form $2qn + i$, and the arc becomes $2q\pi + \frac{i\pi}{n}$, the cosine of which is still the same with that of a preceding arc $\frac{i\pi}{n}$; and thus we have a recurrence of the same factors in the same order $A, B\ldots L, M\ldots B, A$. We see, therefore, that it is *unnecessary to give to k values $> n$.*

1°. If n is *even*, $\frac{1}{2}n \pm i$ are even or odd together; and $k = \frac{1}{2}n \pm i$ gives the arcs $\frac{k\pi}{n} = \frac{1}{2}\pi \pm \frac{i\pi}{n}$, the cosines of which are the same only with opposite signs, viz. $= \mp \sin.\frac{i\pi}{n}$. Thus, *when n is even, we shall not make $k > \frac{1}{2}n$, but shall take the cosines with the sign $\pm$.*

2°. If n is *odd*, $k = n - i$ is odd when i is even, and the converse; and we are not at liberty, therefore, simultaneously to assume that $k = n - i$ and $k = i$. The first assumption, however, gives $\frac{k\pi}{n} = \pi - \frac{i\pi}{n}$, of which the part $\frac{i\pi}{n}$ will be $< \frac{1}{2}\pi$, if, in taking $k = n - i$, we suppose $i > \frac{1}{2}n$; and thus the semi-circumference being diminished by an arc less than the quadrant, the cosine will be negatively the same as for $\frac{i\pi}{n}$, i being among the integers which it is not allowable to take for k. We shall therefore make $k = 0, 1, 2, 3\ldots$, as far as $\frac{1}{2}n$; which will give

arcs $< \frac{1}{2}\pi$; part of them, taken from the series alternately, will correspond to theorem (3); and we must take the cosines of the others with a contrary sign.

Lastly, $y = \frac{x}{a}$ gives $x^2 - 2ax \cos.\left(\frac{k\pi}{n}\right) + a^2$ for the general formula of the factors of $x^n \mp a^n$.

For $y^4 + 1$, k must be odd; $k = 1$ gives the arc $\frac{1}{4}\pi$ or 45°, the cosine of which is $\frac{1}{2}\sqrt{2}$, and taking this with the sign $\pm$, we have the two factors $y^2 \pm y\sqrt{2} + 1$; thus

$$x^4 + a^4 = (x^2 + ax\sqrt{2} + a^2)(x^2 - ax\sqrt{2} + a^2).$$

For $y^6 + 1$, $k = 1$ gives the arc $\frac{1}{6}\pi$, the cosine of which is $\frac{1}{2}\sqrt{3}$, and which must be taken with the double sign; $k = 3$ then gives the cosine = zero; whence

$$y^6 + 1 = (y^2 + y\sqrt{3} + 1)(y^2 - y\sqrt{3} + 1)(y^2 + 1.)$$

Suppose we have $y^6 - 1$: making $k = 0$ and 2, the cosines of zero and $\frac{1}{3}\pi$ are 1 and $\frac{1}{2}$; which, taken + and −, give

$$y^6 - 1 = (y + 1)(y^2 + y + 1)(y^2 - y + 1)(y - 1).$$

Again, let $y^{13} \pm 1 = 0$: $k = 0, 1, 2 \ldots 6$ gives the arcs $0, \frac{1}{13}\pi, \frac{2}{13}\pi \ldots \frac{6}{13}\pi$, the cosines of which must be taken with the + and − sign alternately, the 1st being taken negatively for $y^{13} + 1$, and positively for $y^{13} - 1$.

544. The proposition (3) is known by the name of *Cotes' Theorem*; and was presented by that distinguished mathematician under the following geometric form. With the radius $RA = a$ (fig. 6), let the circle $ACHL$ be described, and draw the diameter AH passing through an arbitrary point O or O'; commencing from A, divide the circumference into $2n$ equal arcs $Aa, aB, Bb, \ldots$, each being the n^{th} of π, and draw radii vectores from the point O or O' to the points of division.

The radius drawn to any point C forms the triangle COP, from which, making the angle $CRA = \alpha$, $OR = x$, we deduce

$$CP = a.\sin.\alpha,\ RP = a.\cos.\alpha,\ OP = a\cos.\alpha - x,$$

and therefore

$$OC^2 = x^2 - 2ax.\cos.\alpha + a^2 = OC.OL.$$

If, now, the arc AC contain k divisions, we have $\alpha = \frac{k\pi}{n}$; and the above trinomial being thus a factor of $x^n \mp a^n$, accordingly as k is even or odd, the radii vectores, drawn to the alternate points of division, will

constitute the several factors: $OA = a - x$, $OH = a + x$ correspond to the real factors of the 1st degree.

Let Z, Z', Z''... denote the radii drawn to the even divisions, and z, z', z''... those drawn to the odd ones; we shall have

$z.z'.z''.... = a^n + x^n$, whether O be interior or exterior.
$Z.Z'.Z''.... = a^n - x^n$, when O is interior.
$Z.Z'.Z''.... = x^n - a^n$, when O is exterior.

EQUATIONS OF THREE TERMS.

545. Let the equation be $Ax^{2n} + Bx^n + C = 0$, where one of the exponents of x is double of the other; making $x^n = z$, the result is

$$Az^2 + Bz + C = 0.$$

1°. If the roots of z are real, as f and g, we shall have to resolve the equations of two terms $x^n = f$, $x^n = g$. For example, let it be required to find two numbers such, that their product shall be 10, and the sum of their cubes 133: the equation will be

$$x^3 + \left(\frac{10}{x}\right)^3 = 133, \text{ or } x^6 - 133\,x^3 + 1000 = 0.$$

Making $x^3 = z$, we get $z^2 - 133\,z + 1000 = 0$, whence $z = 8$ and 125; assuming then $x^3 = 8$ aud 125, there results $x = 2$ and 5, and, besides these values [No. 538], 2α and $5\alpha^2$, then 5α and $2\alpha^2$, α being an imaginary cube root of unity. And these are the three solutions of the problem.

2°. If the roots are equal, we have $B^2 - 4\,AC = 0$, the proposed equation is an exact square, $(ax^n + b)^2 = 0$, and the case becomes that of an equation of two terms. For example, to find a number such, that, its double being divided by 3, and 3 by its double, 2 shall be the sum of the 4th powers of the quotients, we have

$$\left(\frac{2x}{3}\right)^4 + \left(\frac{3}{2x}\right)^4 = 2, \text{ whence } (16x^4 - 81)^2 = 0;$$

and since $y^4 = 1$ has for roots ± 1 and $\pm \sqrt{-1}$, we have $x = \pm \frac{3}{2}$ and $\pm \frac{3}{2}\sqrt{-1}$.

3°. And lastly, when the roots are imaginary, or $B^2 - 4\,AC < 0$, we shall make $Ax^{2n} = Cy^{2n}$, and the proposed equation now becoming

$$y^{2n} + \frac{B}{\sqrt{(AC)}}\,y^n + 1 = 0,$$

it may be compared with (2) [No. 542]; for, since $B^2 < 4\,AC$, the

coefficient of y^n is < 2. There is, therefore, some arc φ which has this coefficient for the double of its cosine, and this arc we must determine by log. from the relation

$$\cos.\ \varphi = -\frac{B}{2\sqrt{(AC)}} \dots (5).$$

Our transformed equation will thus be divisible by $y^2 - 2y \cos. \left(\frac{\varphi}{n}\right) + 1 = 0$, taking for φ all the arcs of which the cosine is given by equ. (5), and which are, not only the arc $< 90°$ given by the table, but also $\varphi + 2\pi$, $\varphi + 4\pi \dots$, and generally $\varphi + 2k\pi$, k being any integer; so that if $\psi = \frac{\varphi + 2k\pi}{n}$, the factors required will all be comprised in the formula,

$$x^2 \sqrt[n]{A} - 2x^2 \sqrt[n]{(AC)}.\ \cos.\ \psi + \sqrt[n]{C} = 0 \dots (6).$$

It will be unnecessary, however, to take $k > n$, since $k = qn + i$ gives the arc $2q\pi + \frac{\varphi + 2i\pi}{n}$; and suppressing the entire circumferences $2q\pi$, it would remain to take the cosine of an arc which must have been already had for $k = i < n$; so that we should have a recurrence of the same factors.

It must be observed also, that the radius is here supposed $= 1$; so that if we make use of the common log. tables, 10 must be subtracted from the log. of all the cosines employed in the calculation.—[See Vol. I. p. 312].

Let the equation, for instance, be $x^6 - 2x^3 + 1 = 0$: then $A = C = 1$, $B = -2$, $n = 3$; we find $\cos.\ \varphi = 1$, which gives the arcs $\psi = 0°$, $120°$ and $240°$; the proposed equation, therefore, has its three factors of the form $x^2 - 2x \cos.\ \psi + 1$; and since the values of $\cos.\ \psi$ are 1, $-\sin.\ 30° = -\frac{1}{2}$ and $-\cos.\ 60° = -\frac{1}{2}$, we find $x^2 - 2x + 1$ and $x^2 + x + 1$, this last factor entering twice.

Thus, the proposed equation is the square of $(x - 1)(x^2 + x + 1)$, or of $x^3 - 1$.

Again, let $x^4 + x^2 + 25 = 0$: in this case $A = B = 1$, $C = 25$, $n = 2$, and $\cos.\ \varphi = -\frac{1}{10}$: the tables give, in consequence of the sign $-$, $\varphi = 95°\ 44'\ 20''$, of which the half ψ is $47°\ 52'\ 10''$; and adding $180°$, we shall form an arc the cosine of which is the same as the one preceding, but with the contrary sign. Substituting in the 2nd term of the general formula (6), the annexed calculation gives -3 for the coefficient of one of the factors; and thus our factors are $x^2 \pm 3x + 5$.

cos. ψ...	$\bar{1}$·8266074
2...	0·3010300—
$\sqrt{5}$...	0·3494850
3...	0·4771224—

Lastly, for $2x^6 + 3x^3 + 5 = 0$, we have $\cos.\ \phi = \frac{-3}{2\sqrt{10}}$

3...	$0{\cdot}4771213-$	2......	$0{\cdot}3010300-$
2...	$-\ 0{\cdot}3010300$	$\sqrt[6]{}\,10$......	$0{\cdot}1666667$
$\sqrt{}\,10$...	$-\ 0{\cdot}5000000$	$2\sqrt[6]{}\,10$......	$0{\cdot}4676967-$
cos. ϕ......	$\bar{1}{\cdot}6760913-$		

We find $\phi = 61° 41'$, or rather $118° 19'$, taking the supplement, on account of the sign $-$. The third of this is $\psi = 39° 26' 20''$; and adding $120°$ twice successively, and taking the cos. ψ, we have cos. $39° 26' 20''$, $-$ sin. $69° 26' 20''$, and sin. $9° 26' 20''$.

Hence,

$2\sqrt[6]{}\,10$...	$0{\cdot}46770-$	$0{\cdot}46770-$	$0{\cdot}46770-$
cos. ψ...	$\bar{1}{\cdot}88779$	$\bar{1}{\cdot}97141-$	$\bar{1}{\cdot}21483$
	$0{\cdot}35549-$	$0{\cdot}43911+$	$\bar{1}{\cdot}68253-$

and assuming $\alpha = -\ 2{\cdot}2672,\ +\ 2{\cdot}7486,\ -\ 0{\cdot}4814$, our three factors are of the form $x^2 \sqrt[3]{2} + \alpha x + \sqrt{5}$.

ROOTS OF EXPRESSIONS INVOLVING RADICALS.

546. Supposing $a + \sqrt{b}$ to be a square, let us investigate its root, which must be of the form $\sqrt{x} + \sqrt{y}$; should it be $f + \sqrt{y}$, we should have $x = f^2$. Assume therefore,

$$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y};$$

the square gives

$$a + \sqrt{b} = x + y + 2\sqrt{xy};$$

and separating this equation into two, as in No. 533, we have

$$x + y = a,\ 2\sqrt{xy} = \sqrt{b}.$$

To deduce x and y from these equations, form the squares and subtract, when you will have

$$x^2 - 2xy + y^2 = (x - y)^2 = a^2 - b;$$

and since x and y are rational by supposition, $a^2 - b$ must be a known perfect square. Representing this square by k^2, $x - y = k$ and $x + y = a$ give the required solution

$$x = \tfrac{1}{2}(a + k),\ y = \tfrac{1}{2}(a - k),\ k = \sqrt{(a^2 - b)}.$$

Let the expression be $\sqrt{(4 + 2\sqrt{3})}$: we have $a = 4, b = 12$;

whence $a^2 - b = k^2 = 4$; consequently $k = 2$, and $x = 3$, $y = 1$; and the root required is $1 + \sqrt{3}$. That of $4 - 2\sqrt{3}$ is $1 - \sqrt{3}$.

For $\sqrt{(-1 + 2\sqrt{-2})}$, $a^2 - b = 9$, $k = 3$, $x = 1$, $y = -2$; and we have $\pm(1 + \sqrt{-2})$ for the root.

If $a + \sqrt{b}$ be an exact cube, we shall assume

$$\sqrt[3]{(a + \sqrt{b})} = (x + \sqrt{y})\sqrt[3]{z},$$

z being an indeterminate quantity to be disposed of at pleasure, so as to facilitate the calculation. Raising to the cube and comparing the rational terms, we find

$$a = z(x^3 + 3xy),\ \sqrt{b} = z\sqrt{y}\ (3x^2 + y);$$

and squaring these equations and subtracting, we have

$$a^2 - b = z^2[(x^3 + 3xy)^2 - (3x^2\sqrt{y} + y\sqrt{y})^2];$$

where the factor of z^2 is the difference of two squares, and is evidently equivalent to $(x + \sqrt{y})^3 \times (x - \sqrt{y})^3$, or $(x^2 - y)^3$; and consequently

$$\frac{a^2 - b}{z^2} = (x^2 - y)^3.$$

But x and y are supposed to be rational; so that the 1st side must be an exact cube; and we shall always be able to determine z so as to fulfil this condition, be it only by making $z = (a^2 - b)^2$. If $a^2 - b$ be itself a cube, we shall make $z = 1$; and, generally, we must decompose $a^2 - b$ into its prime factors, and we shall readily see what factors ought to be introduced or suppressed, in order that we may have an exact cube. Thus, z and k will be known in the relations

$$k = \sqrt[3]{\left(\frac{a^2 - b}{z^2}\right)},\ x^2 - y = k,\ a = zx(x^2 + 3y);$$

whence

$$y = x^2 - k,\ 4zx^3 - 3kxz = a.$$

The last of these equations gives x, our attention being directed only to the rational roots; the preceding one then makes known y, and we have the root required.

For $10 + 6\sqrt{3}$ we have $a = 10$, $b = 108$, $a^2 - b = -8$; so that $z = 1$, and $k = -2$. Consequently,

$$4x^3 + 6x = 10, \text{ whence } x = 1; \text{ this leads to } y = 3;$$

and lastly

$$\sqrt[3]{(10 + 6\sqrt{3})} = 1 + \sqrt{3}.$$

Again, let the expression be $8 + 4\sqrt{5}$: we have now $a^2 - b = -16$; we shall therefore make $z = 4$, and $k = -1$; whence

$$4x^3 + 3x = 2, \text{ which gives } x = \tfrac{1}{2}, \text{ then } y = \tfrac{5}{4};$$

and, lastly, $\frac{1}{2}\sqrt[3]{4}\cdot(1 + \sqrt{5})$ for the cube root of $8 + 4\sqrt{5}$.

Assuming $\sqrt[n]{(a + \sqrt{b})} = (x + \sqrt{y})\sqrt[n]{z}$, and making use of the same reasoning, we shall be able to determine x, y, z, in all cases in which $a + \sqrt{b}$ is an exact n^{th} power.

547. In any other formula, in which radical quantities appear, it will not be enough to substitute merely their approximate values, as we should thus pass over all the imaginary values of which these quantities are susceptible: $\sqrt[n]{A}$ must be replaced by $\alpha\sqrt[n]{A}$, $\alpha^2\sqrt[n]{A}$..., supposing $1, \alpha, \alpha^2$... to be the roots of the equation $y^n - 1 = 0$.

If we have $x = a\sqrt[n]{g} + b\sqrt[n]{g^2} + c\sqrt[n]{g^3}$..., it will suffice to assume

$$y^n = g, \quad x = ay + by^2 + cy^3 \ldots,$$

and to eliminate y between these two equations; the several roots of the final equation in x will be the required values of x.

When we have a function X involving radical quantities $\sqrt[n]{A}$, $\sqrt[m]{B}$..., to obtain all the values of X, assume $y^n = A$, $t^m = B$..., and introduce, in place of the radicals, the n values of y, the m of t,..., combined one with the other in all the ways possible.

An equation $X = 0$ is cleared of the radicals which enter into it in the same manner; X is changed into a function of x, y, t..., and we must then eliminate y, t... by means of $y^n = A$, $t^m = B$...; the final equation in x will be the one required.

For example, let $x = \sqrt[3]{A} + \sqrt[3]{B}$: we shall assume $y^3 = A$, $t^3 = B$, and therefore $x = y + t$; substitute for y the three values $y, \alpha y, \alpha^2 y$, and in like manner $t, \alpha t, \alpha^2 t$ for t; and, combining these substitutions 2 and 2, we shall have the nine values of x; or we may eliminate y and t between the three equations, and the final equation in x will have for its roots all the values required.

EQUATIONS OF THE THIRD DEGREE.

548. To solve the equation $kx^3 + ax^2 + bx + c = 0$, getting quit of the 2nd term and the coefficient of the 1st by assuming [p. 38] $x = \dfrac{x' - a}{3k}$, we have

$$x'^3 + 3x'(3kb - a^2) + 2a^3 - 9abk + 27ck^2 = 0;$$

and thus every equation of the 3rd degree is reducible to

$$x^3 + px + q = 0 \ldots (1).$$

If now we assume $x = y + z$, this gives $x^3 = 3yz(y + z) + y^3 + z^3$; and thus the proposed equation becomes

$$(3yz + p)(y + z) + y^3 + z^3 + q = 0.$$

But this partition of x into two numbers y and z may be effected in an infinity of ways, and we are at liberty to assign the product of these parts, or their difference, or their ratio, &c. Assume, therefore, that the 1st factor is 0, or that

$$yz = -\tfrac{1}{3}p, \text{ and consequently } y^3 + z^3 = -q.$$

Taking the cube of the 1st of these equations, $y^3 z^3 = -(\tfrac{1}{3}p)^3$, it appears that y^3 and z^3 have $-q$ for their sum and $-(\tfrac{1}{3}p)^3$ for their product, *i. e.* the unknown terms y^3 and z^3 are the roots t and t' of the equation of the 2nd degree [N°. 137, 5°.]

$$t^2 + qt = (\tfrac{1}{3}p)^3 \ldots (2),$$

which is called the *reduced* equation. Having determined t and t', we have $y^3 = t$, $z^3 = t'$; and $1, \alpha, \alpha^2$ being the three cube roots of unity [N°. 540], these equations give

$$y = \sqrt[3]{t},\ \alpha\sqrt{t},\ \alpha^2\sqrt[3]{t};\ z = \sqrt[3]{t'},\ \alpha\sqrt[3]{t'},\ \alpha^2\sqrt[3]{t'}.$$

We must not, however, in order to obtain $x = y + z$, take the sums of all these values 2 and 2; which would give us 9 roots instead of 3. When, in lieu of the equation $yz = -\tfrac{1}{3}p$, we employed its cube, the number of the roots was tripled; but of these values of y and z, those only must be added together, the product of which is really $= -\tfrac{1}{3}p$, or $= \sqrt[3]{tt'}$; since, the 2nd side of the equation (2) being $-tt'$, the cube root of this quantity is $= \tfrac{1}{3}p$. Thus, α^3 being $= 1$, it will be easily seen that, of the 9 combinations, we can admit, besides $x = \sqrt[3]{t} + \sqrt[3]{t'}$, only the two

$$x = \alpha\sqrt[3]{t} + \alpha^2\sqrt[3]{t'}, \text{ and } x = \alpha^2\sqrt[3]{t} + \alpha\sqrt[3]{t}.$$

The values of α, α^2 are $-\tfrac{1}{2}(1 \pm \sqrt{-3})$ N°. 538; substituting which, and supposing, for conciseness, that

$$\left.\begin{array}{l} s = \sqrt[3]{t} + \sqrt[3]{t'},\ d = \sqrt[3]{t} - \sqrt[3]{t'}, \\ \text{we have } x = s,\ x = -\tfrac{1}{2}(s \pm d\sqrt{-3}). \end{array}\right\} \ldots (3).$$

Hence, to resolve the equation (1) of the 3rd degree, we must first of all solve the reduced equation (2); and having thus determined t and t', introduce their values in the formulæ (3).

For example, $x^3 + 6x = 7$ gives $p = 6$, $q = -7$, and the reduced

equation is $t^2 - 7t = 8$; whence $t = \frac{7}{2} \pm \frac{9}{2}$, $t = 8$, $t' = -1$; and the cube roots of these values being 2 and -1, we have

$$s = 1, d = 3, x = 1 \text{ and } -\tfrac{1}{2}(1 \pm 3\sqrt{-3}).$$

Let $y^3 - 3y^2 + 12y = 4$: assuming $y = x + 1$, in order to remove the 2nd term, we have $x^3 + 9x + 6 = 0$; whence $p = 9$, $q = 6$, and the reduced equation is $t^2 + 6t = 27$; this gives $t = 3$, $t' = -9$; consequently

$$s = \sqrt[3]{3} - \sqrt[3]{9} = -0{\cdot}637835 = x, \; d = 3{\cdot}522333;$$

and, lastly,

$$y = 0{\cdot}362165, \text{ and } = 1{\cdot}318918 \pm 1{\cdot}761167\sqrt{-3}.$$

The equation $x^3 - 3x = 18$ gives $t^2 - 18t + 1 = 0$, whence $t = 9 \pm 4\sqrt{5}$; the cube root of which [p. 91] is $\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$; thus, $s = 3$, $d = \sqrt{5}$, and, lastly, $x = 3$ and $-\frac{1}{2}(3 \pm \sqrt{-15})$. $x^3 - 27x + 54 = 0$ gives $t^2 + 54t + 729 = 0$, or $(t + 27)^2 = 0$, whence $t = -27$: thus $x = -6$ and 3 (a double root).

549. So long as *the two roots t and t' of the reduced equation are real*, $\sqrt[3]{t}$, $\sqrt[3]{t'}$, and therefore s and d are so also; in which case it follows from the formulæ (3), that the *proposed equation has but one real root.*

At the same time, if $t = t'$, we have $d = 0$, and all the three values of x are real, two of them being equal to the half of the 3rd taken with the contrary sign. Of this we have an instance in the last example.

But *if the reduced equation have its roots imaginary* (p must in this case be negative, and also $4p^3 > 27q^2$), the expressions (3) being all of them intermixed with imaginary quantities, it would appear that not one of the roots is real, contrary to what has been proved elsewhere [N°. 515, 1°.]. This paradox, which long engaged the attention of algebraists, has, from its circumstances, been denominated the *irreducible case.* We shall proceed to show that, in fact, *all the three roots are in this case real.* The imaginary values of t and t' being represented by $a \pm b\sqrt{-1}$, the cube roots, or the powers $\frac{1}{3}$, of these values may be obtained in the form of two series [p. 15]. Without, however, effecting this development, it is evident that imaginary quantities cannot present themselves except in the terms in which $b\sqrt{-1}$ is affected with odd exponents; and since one of these series is deducible from the other by changing b into $-b$, it is obvious that they are both comprised in the form $P \pm Q\sqrt{-1}$, the sum of which is $s = 2P$, and the difference $d = 2Q\sqrt{-1}$. Thus, the formulæ (3) reduce themselves to the real expressions

$$x = 2P \text{ and } -P \pm Q\sqrt{3} \ldots (4);$$

and our roots consequently are real, precisely when the equations (3) present them under an imaginary form.

The origin of this singular case is that, in the assumptions $x = y + z$ and $yz = -\frac{1}{3}p$, there is nothing to express that y and z are actually real; and, in fact, our calculation proves that they are imaginary when the three roots are all real. To obtain these roots, we must develop the power $\frac{1}{3}$ of $a + b\sqrt{-1}$ under the form $P + Q\sqrt{-1}$; when P and Q will be known in the equations (4).

550. As this however would require the series to be convergent, we prefer making use of the following process. It follows from the theorem (2), N°. 542, that, making $n = 3$, and supposing the radius to be unity,

$$y^2 - 2y\cos.\tfrac{1}{3}\phi + 1 \text{ divides } y^6 - 2y^3\cos.\phi + 1.$$

Now, make $x = m(y + y^{-1})$ in $x^3 - px + q = 0$; then

$$m^3(y^3 + y^{-3}) + (3m^3 - mp)(y + y^{-1}) + q = 0;$$

and getting quit of the 2nd term by assuming $3m^2 = p$, or $m = \sqrt{\frac{1}{3}p}$, we arrive at

$$y^6 + \frac{qy^3}{(\frac{1}{3}p)^{\frac{3}{2}}} + 1 = 0.$$

But, in the case now under consideration, t is supposed to be imaginary in the equation (2), or $(\frac{1}{2}q)^2 < (\frac{1}{3}p)^3$; so that, the half of the factor of y^3 being < 1, we can find an arc ϕ the cosine of which is equal to this half, or

$$\cos.\phi = \frac{-q}{2.\frac{1}{3}p\sqrt{(\frac{1}{3}p)}} \ldots (5);$$

and then the proposed equation, being reduced to our 2nd trinomial, is divisible by $y^2 - 2y\cos.\frac{1}{3}\phi + 1 = 0$. Hence, dividing by y, we have $y + y^{-1} = 2\cos.\frac{1}{3}\phi$; and since $x = m(y + y^{-1})$, where $m = \sqrt{\frac{1}{3}p}$, we find

$$x = 2\sqrt{(\tfrac{1}{3}p)}.\cos.\tfrac{1}{3}\phi \ldots (6).$$

A logarithmic computation will give the arc ϕ, of which we must take the third; and to this we shall add 120° and 240°; since, besides the arc found in the table, we may also take the arcs $\phi + 2\pi$, $\phi + 4\pi$, which have the same cosine. The equation (6), in which $\cos.\frac{1}{3}\phi$ thus takes three values, will determine the three real roots.

Taking, for example, $x^3 - 5x - 3 = 0$; we have $p = 5$, $q = -3$, $\cos.\phi = \dfrac{3}{2.\frac{5}{3}\sqrt{\frac{5}{3}}}$.

The annexed calculation gives $\phi = 45^\circ 48' 9''$, the third of which is $15^\circ 16' 3''$; and to this we must add 120° and 240°, and take the cosines, which are $\cos. 15^\circ 16' 3''$, $-\sin. 45^\circ 16' 3''$, $-\cos. 75^\circ 16' 3''$.

5......	0·6989700
3......	0·4771213
diff......	0·2218487
half......	0·1109243
2......	0·3010300
denom...	— 0·6338030
3...	+ 0·4771213
cos. ϕ...	$\bar{1}$·8433183

We have now

$2\sqrt{\frac{5}{3}}$......	0·4119543,	0·4119543,	0·4119543
cos. $\frac{1}{3}\phi$......	$\bar{1}$·9843955,	$\bar{1}$·8515032 —	$\bar{1}$·4053576 —
x......	0·3963498,	0·2634575 —	$\bar{1}$·8173119 —
$x =$	2·490862	— 1·834245	— 0·6566166

For the equation $x^3 - 5x + 3 = 0$, only change x into $-x$, and it will become the same with the one preceding, and we shall therefore have the same roots with contrary signs. Otherwise, treating this example directly, since the equation (5) now gives $\cos.\phi$ negative, the arc ϕ is $> 90^\circ$, and is the supplement of the preceding value of ϕ; in other respects the calculation is the same.

Let the equation be $x^3 - 4x + 1 = 0$: then $\cos.\phi = \dfrac{-1}{2.\frac{4}{3}\sqrt{\frac{4}{3}}}$; the calculation gives $\phi = 108^\circ 57' 3''.5$; and we finally obtain

$$x = 1{\cdot}860807\ldots,\ -2{\cdot}114907,\ 0{\cdot}254099.$$

EQUATIONS OF THE FOURTH DEGREE.

551. Let the equation proposed be $x^4 + px^2 + qx + r = 0$; to resolve it, we shall follow the same course as for the 3rd degree, and in the first place consider x as composed of two parts y and z, $x = y + z$; whence

$$\begin{gathered} y^4 + (6z^2 + p)\,y^2 + (z^4 + pz^2 + qz + r) \\ + 4zy^3 + (4z^3 + 2pz + q)\,y = 0. \end{gathered}$$

For the additional relation that we are at liberty to assume between y and z, we shall equate the 2nd line to zero; which gives

$$y^2 = -z^2 - \frac{p}{2} - \frac{q}{4z} \ldots (1);$$

and, y^2 being eliminated between this and the 1st line, the transformed equation becomes

$$z^6 + \tfrac{1}{2}pz^4 + \tfrac{1}{16}(p^2 - 4r)\,z^2 - \tfrac{1}{64}q^2 = 0,$$

an equation which contains only even powers of z. Simplifying it, by making $z^2 = \frac{1}{4}t$, we shall have

$$t^3 + 2pt^2 + (p^2 - 4r)\,t - q^2 = 0 \ldots\ldots (A);$$

and this is the *reduced equation* which is of the 3rd degree, and necessarily has at least one real and positive root.*

Let this root be denoted by t; we have then $z = \pm \frac{1}{2}\sqrt{t}$, where the sign is arbitrary; whence, substituting in $x = y + z$ and in (1), there results

$$x = y \pm \tfrac{1}{2}\sqrt{t},\ y^2 = -\frac{t}{4} - \frac{p}{2} \mp \frac{q}{2\sqrt{t}} \ldots (2);$$

and eliminating y, with due regard to the corresponding signs, we find lastly

$$\left.\begin{aligned} x &= \frac{\sqrt{t}}{2} \pm \sqrt{\left(-\frac{t}{4} - \frac{p}{2} - \frac{q}{2\sqrt{t}}\right)} \\ x &= -\frac{\sqrt{t}}{2} \pm \sqrt{\left(-\frac{t}{4} - \frac{p}{2} + \frac{q}{2\sqrt{t}}\right)} \end{aligned}\right\} \ldots (B)$$

Thus, we must resolve the reduced equation (A); take its positive root t, and substitute it in the formulæ (B), which will give the four values of x.

Take, for example, $2x^4 - 19x^2 + 24x = \frac{23}{8}$: $p = -\frac{19}{2}$, $q = 12$, &c.; and the reduced equation is $t^3 - 19t^2 + 96t = 144$. The positive root $t = 3$ gives

$$x = \tfrac{1}{2}\sqrt{3} \pm \sqrt{(4 - 2\sqrt{3})}, \text{ and } -\tfrac{1}{2}\sqrt{3} \pm \sqrt{(4 + 2\sqrt{3})};$$

and since [p. 91] $\sqrt{(4 \pm 2\sqrt{3})} = 1 \pm \sqrt{3}$, we have

$$x = 1 \pm \tfrac{1}{2}\sqrt{3},\ x = -1 \pm \tfrac{3}{2}\sqrt{3}.$$

The equation $x^4 - 25x^2 + 60x - 36 = 0$ has, for the one of reduction, $t^3 - 50t^2 + 769t = 3600$; and taking $t = 9$, we shall have $x = 3, 2, 1$ and -6.

For $x^4 - x + 1 = 0$, we have $t^3 - 4t = 1$; whence $t = 2{\cdot}114907\ldots$, [see p. 97]; and we deduce

$$x = -\,0{\cdot}7271360 \pm 0{\cdot}9340992\sqrt{-1},$$
$$x = +\,0{\cdot}727236 \pm 0{\cdot}4300139\sqrt{-1}.$$

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* This equation must be cleared of its 2nd term, by assuming $t = \frac{1}{3}(u - 2p)$; whence

$$u^3 - 9u\,(p^2 + 4r) + 72pr - 2p^3 - 27q^2 = 0.$$

Lastly, the equation $x^4 - 3x^3 - 42x = 40$ gives

$$t^3 - 6t^2 + 169t = 1764;$$

whence

$$t = 9, \text{ and consequently } x = 4, -1 \text{ and } \tfrac{1}{2}(3 \pm \sqrt{-31}).$$

552. We shall now examine the different cases which may present themselves. Denoting the roots by t, t', t'', it appears from the equation (A) that

$$t + t' + t'' = -2p, \quad t.t'.t'' = q^2;$$

the 1st of which gives... $-t - 2p = t' + t''$... (3),

the 2nd .... $\sqrt{(t'.t'')} = \dfrac{q}{\sqrt{t}}$ ... (4).

In consequence of the extraction of the roots, it might seem necessary to introduce the sign $\pm$; but it will be sufficient to observe that, if q be positive, $\sqrt{(t'.t'')}$ and $\sqrt{t}$ have the same sign, whilst the contrary is the case if q be negative. Since the reduced equation (A) contains, not q, but q^2, it remains the same, whatever be the sign of q, which requires us to distinguish two cases, both of them comprised in the equation (4).

If q be positive in the proposed equation, substituting the values (3) and (4) in the equation (2), in which regard will have been already had to the sign $\pm$ of $\sqrt{t}$; we shall have

$$y^2 = \tfrac{1}{4}t' + \tfrac{1}{4}t'' \mp \tfrac{1}{2}\sqrt{(t'.t'')} = \tfrac{1}{4}(\sqrt{t'} \mp \sqrt{t''})^2;$$

whence

$$y = \tfrac{1}{2}(\sqrt{t'} \mp \sqrt{t''}) \text{ and } -\tfrac{1}{2}(\sqrt{t'} \mp \sqrt{t''}).$$

The double sign $\pm$ must agree with that of the equation $x = y \pm \frac{1}{2}\sqrt{t}$, as it results from the calculation above.

Consequently, q being positive, we have

$$x = \tfrac{1}{2}\sqrt{t} \pm \tfrac{1}{2}(\sqrt{t'} - \sqrt{t''}) \text{ and } -\tfrac{1}{2}\sqrt{t} \pm \tfrac{1}{2}(\sqrt{t'} + \sqrt{t''}):$$

If q be negative,* $\sqrt{(t'.t'')} = -\dfrac{q}{\sqrt{t}}$; the substitution of which in the equation (2) causes only the modification of sign of the last term;

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* It is not necessary to make this distinction in the formulæ (B), because in them we always substitute, for p, q, r, their given values, affected with the signs which belong to them, and it is obvious that if q be negative, the sign of the last term of the equations (B) changes of itself.

the calculation therefore continues the same, except as to this sign, and the values of y take only $\pm \sqrt{t''}$ instead of $\mp \sqrt{t''}$:

$$x = \tfrac{1}{2}\sqrt{t} \pm \tfrac{1}{2}(\sqrt{t'} + \sqrt{t''}) \text{ and } -\tfrac{1}{2}\sqrt{t} \pm (\sqrt{t'} - \sqrt{t''}).$$

1°. *If the reduced equation have its three roots real,* there can be but two cases: since their product $t.t'.t'' = q^2$ is positive, either two of the roots are negative, or not one of them is so.

In the latter case, $\sqrt{t}$, $\sqrt{t'}$, $\sqrt{t''}$ are real, and our four roots of x are therefore real also. In the other case, on the contrary, $\sqrt{t'}$ and $\sqrt{t''}$ are imaginary, and the four roots of x will be so too. Hence, *when the equation of reduction comes under the irreducible case, the proposed equation has its four roots all real or all imaginary, accordingly as t has three positive values or only one.* Of this we have examples above.

If, however, it occur, in this second case, that $t' = t''$, since two of our values of x contain the difference of the radicals $\sqrt{t'}$, $\sqrt{t''}$, these imaginary parts will destroy each other, and the proposed equation will thus have two real and equal roots, and two imaginary ones.

2°. *If the reduced equation have but one real root t,* since t is then positive, $\sqrt{t}$ is real. Likewise, denoting t', t'' by $a \pm b\sqrt{-1}$, we have

$$\sqrt{t'} \pm \sqrt{t''} = \sqrt{(a + b\sqrt{-1})} \pm \sqrt{(a - b\sqrt{-1})};$$

the square of which gives

$$(\sqrt{t'} \pm \sqrt{t''})^2 = 2a \pm 2\sqrt{(a^2 + b^2)},$$

where the radical part is obviously real and $> a$. Thus, our square has two real values, one positive, the other negative; and consequently, on extracting the root, which is $\sqrt{t'} \pm \sqrt{t''}$, we shall have on the one hand a real quantity $\sqrt{A}$, and on the other an imaginary result $\sqrt{-B}$. Hence, recurring to the foregoing values of x, it will be clearly seen that, *if the reduced equation have but one real root t, this is positive, and the proposed equation has two roots real and two imaginary.*

IV. SYMMETRICAL FUNCTIONS.

POWERS OF THE ROOTS OF EQUATIONS.

553. A function is said to be *symmetrical* or *invariable*, when it undergoes no alteration on any two of the letters which enter into it being changed one into the other: such are the functions a^2+b^2, $\sqrt{a}+\sqrt{b}$, $a+b+\sin. a \sin. b$, &c., which continue the same when b is put for a and a for b. The coefficients of the several terms of an equation $X=0$ are symmetrical functions of the roots $a, b, c\ldots$ [N°. 502].

For the future, we shall represent by $[a^\alpha b^\beta c^\gamma \ldots]$, the symmetrical function of which $a^\alpha b^\beta c^\gamma \ldots$ is one term, and the other terms of which are obtained by changing each root $a, b, c\ldots$ into all the others successively; whilst $\int_m$ will denote the m^{th} powers of these roots, or $\int_m = a^m + b^m + c^m \ldots$. And we shall now prove that, though these roots be not known, we can, whatever be the integers $m, \alpha, \beta, \gamma\ldots$, always find the quantities $\int_m$ and $[a^\alpha b^\beta c^\gamma \ldots]$, in functions of the coefficients p, q, $r\ldots$ of the proposed equation

$$X = x^m + px^{m-1} + qx^{m-2} \ldots + tx + u = 0.$$

X is identical with $(x-a)(x-b)(x-c)\ldots$, and it has been seen [N°. 524, 2°] that the derivative X' is

$$mx^{m-1}+(m-1)px^{m-2}\ldots+t=(x-b)(x-c)\ldots+(x-a)(x-c)\ldots+\&c.;$$

whence, dividing by X, we find

$$\frac{mx^{m-1}+(m-1)px^{m-2}\ldots+t}{x^m+px^{m-1}+qx^{m-2}\ldots+u} = \frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\ldots.$$

Now, developing $(x-a)^{-1}$, we have [p. 14, 1]

$$\frac{1}{x-a} = \frac{1}{x}+\frac{a}{x^2}+\frac{a^2}{x^3}+\frac{a^3}{x^4}+\ldots;$$

so that, changing a into $b, c\ldots$, and taking the sum of these results, our 2nd side is

$$= \frac{m}{x}+\frac{\int_1}{x^2}+\frac{\int_2}{x^3}+\frac{\int_3}{x^4}+\ldots \&c.;$$

and consequently, multiplying the equation by $x^m+px^{m-1}+qx^{m-2}+\ldots$, we have

$$mx^{m-1} + (m-1)\,px^{m-2} + (m-2)\,qx^{m-3} \ldots + t =$$

$$\begin{array}{l|l|l|l|l|l}
mx^{m-1} + \int_1 & x^{m-2} + \int_2 & x^{m-3} + \int_3 & x^{m-4} \ldots \text{\&c.} & \ldots + \int_l & x^{m-l-1} \ldots \\
\quad + mp & \quad + p\int_1 & \quad + p\int_2 & \ldots\ldots & \ldots + p\int_{l-1} & \ldots\ldots \\
 & \quad + mq & \quad + q\int_1 & \ldots\ldots & \ldots + q\int_{l-2} & \ldots\ldots \\
 & & \quad + mr & \ldots\ldots & \ldots + r\int_{l-3} & \ldots\ldots \\
 & & & \ldots\ldots & \quad \text{\&c.} & \ldots\ldots
\end{array}$$

The 1st side has m terms; the 2nd extends to infinity, each line having its first term thrown one rank farther to the right than in the preceding; the number of these lines is $m + 1$. Hence, comparing the coefficients of the same powers of x in this identical expression, we obtain an infinite number of equations. The m first, each containing one more term than the preceding, are (suppressing mp, mq... on both sides)

$$\int_1 + p = 0,\ \int_2 + p\int_1 + 2q = 0,\ \int_3 + p\int_2 + q\int_1 + 3r = 0, \ldots,$$
$$\int_k + p\int_{k-1} + q\int_{k-2} + r\int_{k-3} \ldots kv = 0 \ldots (A),$$

k being an integer $< m$, and v the coefficient of x^{m-k} in X.

After these m equations, the 1st side no longer gives any term to be compared with those of the 2nd, and we therefore find

$$\int_l + p\int_{l-1} + q\int_{l-2} + r\int_{l-3} \ldots + u\int_{l-m} = 0 \ldots (B),$$

l being an integer $=$ or $> m$. We have $\int_0 = a^0 + b^0 \ldots = m$.

554. These equations are due to Newton: to exemplify their use, the 1st gives $\int_1 = -p$, a value which, introduced in the 2nd, gives $\int_2$; we have then $\int_3$...

$$\int_1 = -p,\ \int_2 = -p\int_1 - 2q,\ \int_3 = -p\int_2 - q\int_1 - 3r \ldots;$$

and so on step by step. The value of $\int_l$ leads to this general rule: under the m terms which, in the series of the successive sums $\int$, precede the one that we wish to calculate, write the coefficients of X in inverse order, with their signs changed; multiply each term by the one below it, add the results, and you will have the following term $\int_l$:

$$\begin{array}{cccccc}
\int_{l-m}, & \int_{l-m-1} \ldots, & \int_{l-3}, & \int_{l-2}, & \int_{l-1}, \\
-u, & -t \ldots, & -r, & -q, & -p.
\end{array}$$

Take, for example, the equation $x^3 - 3x^2 + 2x - 1 = 0$: here $p = -3, q = 2, r = -1$; and the factors therefore are 1, -2 and 3. We find in the first place $\int_0 = 3, \int_1 = 3, \int_2 = 5$; and the series of the $\int$ is continued as follows, each term being formed of the product of the three which precede it, multiplied respectively by 1, -2 and 3,

3, 3, 5, 12, 29, 68, 158, 367, 853, 1983, 4610, 10717, 24914...

For $x^3 - 3x^2 + 12x = 4$, the factors are 4, -12 and 3, and we obtain

$$3, 3, -15, -69, -15, 723, 2073, -2517, -29535, \ldots$$

Lastly, for $x^3 - 2x = 5$, the multipliers are 5, 2, 0; and we have

$$3, 0, 4, 15, 8, 50, 91, 140, 432\ldots$$

Applying this theorem to $x^m - 1 = 0$, we obtain, as in p. 85,

$$\int_1 = \int_2 = \int_3 = \ldots = 0, \int_m = \int_{2m} = \int_{3m} = \ldots = m.$$

It is an easy matter, therefore, to obtain the sum of all the integral powers of the roots of an equation without knowing these roots. Should the negative powers be required, we must change x into $\frac{1}{y}$, and apply our formulæ to the transformed equation in y; when we shall have the powers in question. For the equation $x^3 - 3x^2 + 2x = 1$, the factors corresponding to the transformed equation will be $1 - 3$ and 2; whence the sums of its positive powers, and therefore those of the negative powers of the original equation, are

$$3, 2, -2, -7, -6, 7, 25, 23, -22, -88\ldots$$

555. Let us now investigate *an expression for any symmetrical function* $[a^\alpha b^\beta c^\gamma \ldots]$, *in terms of* $\int_1 \int_2 \int_3 \ldots$, the number of the roots $a, b, c\ldots$ comprised in each term being n. This function may be obtained by taking all the permutations, n and n, of the m letters $a, b, c\ldots$, and giving to the 1st letter the exponent α, to the 2nd $\beta, \ldots$; the number of the terms will be $m\,Pn$. In case, however, that two of the exponents were equal, as $\alpha = \beta$, since the initial letters ab, ba would now produce no change in the resulting term, the number of the terms would only be one-half of its previous value; it would be the 6th of that, should three of the exponents be equal, &c. [See N°. 493].

To obtain the value of $[a^\alpha b^\beta]$, in which the terms contain only two of the m roots, we must form the permutations, as in N°. 492, by multiplying

$$\int_\alpha = a^\alpha + b^\alpha + c^\alpha \ldots \text{ by } \int_\beta = a^\beta + b^\beta + c^\beta \ldots:$$

when the partial factors contain the same root, the partial product will have the form $a^{\alpha+\beta}$; otherwise, this product will be similar to $a^\alpha b^\beta$. Thus the result will be $\int_{\alpha+\beta} + [a^\alpha b^\beta]$; and therefore

$$[a^\alpha b^\beta] = \int_\alpha \times \int_\beta - \int_{\alpha+\beta} \ldots (C).$$

In like mannner, for the function $[a^\alpha b^\beta c^\gamma]$, multiplying $[a^\alpha b^\beta]$ by $\int_\gamma$, (C) will become $= \int_\alpha \times \int_\beta \times \int_\gamma - \int_{\alpha+\beta} \times \int_\gamma$.

Now, to form the product

$$(a^\alpha b^\beta + a^\alpha c^\beta + b^\alpha c^\beta + \ldots) \times (a^\gamma + b^\gamma + c^\gamma \ldots),$$

1°. If the partial factors have not a common root, the partial product is such as $a^\alpha\, b^\beta\, c^\gamma$; and these results combined, form the function $[a^\alpha b^\beta c^\gamma]$ the value of which we are investigating.

2°. When the partial factors do contain a common root, the term will correspond to $a^{\alpha+\gamma}\, b^\beta$, or $a^\alpha b^{\beta+\gamma}$, accordingly as this common root is the 1st factor or the 2nd; and hence result the functions $[a^{\alpha+\gamma} b^\beta]$, $[a^\alpha b^{\beta+\gamma}]$, of which the equation (C) gives the values:

$$\int_{\alpha+\gamma} \times \int_\beta - \int_{\alpha+\beta+\gamma},\ \int_\alpha + \int_{\beta+\gamma} - \int_{\alpha+\beta+\gamma}$$

Thus, collecting our results, we have... (D)

$$[a^\alpha b^\beta c^\gamma] = \int_\alpha . \int_\beta . \int_\gamma - \int_{\alpha+\beta} . \int_\gamma - \int_{\alpha+\gamma} . \int_\beta - \int_{\beta+\gamma} . \int_\alpha + 2\int_{\alpha+\beta+\gamma}.$$

The spirit of this species of calculation will readily be comprehended; and we may apply it to symmetrical functions of four or more factors. Thus, we shall be able to determine the values of these functions simply by means of the coefficients of the proposed equation, the sums $\int$ being known in terms of those coefficients from the rules of the preceding N°.

Should the symmetrical function proposed be fractional, by reduction to a common denominator, it will form a fraction, each term of which will be an invariable function : thus

$$\left[\frac{a}{b}\right], \text{ or } \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b} \ldots, = \frac{[a^2 b]}{abc\ldots}.$$

We shall now proceed to apply these general formulæ.

ON THE NUMERICAL SOLUTION OF EQUATIONS.

556. Assuming the different values of $\int$ to be already known, the greater a be in respect to the other roots $b, c\ldots$, the more closely will $\int_k$ approach to an equality with its 1st term a^k, and $\int_{k-1}$ with a^{k-1}; whence, by division, we have the approximate value $a = \int_k : \int_{k-1}$. Thus, having formed the series of the numbers $\int_0, \int_1, \int_2 \ldots$, the quotient of each term by the one preceding will become more and more nearly equal to the superior root a, as the index of $\int$ rises gradually higher. And the transformation of N°. 506 will enable us by the same means to approximate to the least root.

In the case of imaginary roots, it will be necessary to modify our proposition. For let $x = \alpha \pm \beta\sqrt{-1}$: making $\alpha = \lambda \cos.\, \phi$, $\beta = \lambda \sin.\, \phi$ (and this supposition is always allowable, for there results from it

$$\lambda^2 = \alpha^2 + \beta^2, \tan.\, \phi = \frac{\beta}{\alpha},$$

equations whence λ and the arc ϕ may be deduced in all cases), we have $x = \lambda\,(\cos.\, \phi \pm \sin.\, \phi\sqrt{-1})$; and consequently [note p. 86]

$$(\alpha \pm \beta\sqrt{-1})^k = \lambda^k\,(\cos.\, k\phi \pm \sin.\, k\phi . \sqrt{-1}).$$

Thus the supposition of our two imaginary roots introduces into $\int_k$ the term $2\lambda^k \cos.\, k\phi$; and in order, therefore, that the preceding theorem may be established in this case, we must have λ or $\sqrt{(\alpha^2 + \beta^2)}$ less than the greatest root a.

For the 1st ex. of N°. 554, we have $\int_{13} = 57918$, $\int_{12} = 24914$; and the quotient $\frac{57918}{24914} = 2{\cdot}324717$ is an approximate value of x.

557. Let it be proposed to form the equation of the square of the differences

$$Z = z^n + Pz^{n-1} + Qz^{n-2} \ldots + U = 0,$$

where the unknown quantities are $P, Q \ldots U$. We have

$$(x-a)^l = x^l - lax^{l-1} + A'a^2 x^{l-2} - A''a^3 x^{l-3} \ldots \pm a^l,$$
$$(x-b)^l = x^l - lbx^{l-1} + A'b^2 x^{l-2} - A''b^3 x^{l-3} \ldots \pm b^l,$$
$$(x-c)^l = x^l - lcx^{l-1} + A'c^2 x^{l-2} \text{ \&c.}$$

$l, A', A''\ldots$ being the coefficients of the binomial for the power l. Let these m equations be added together; the 2nd side will be

$$mx^l - l\int_1 x^{l-1} + A'\int_2 x^{l-2} - A''\int_3 x^{l-3} \ldots \pm \int_l;$$

and x being changed successively into $a, b, c\ldots$, we get

$$(a-b)^l + (a-c)^l \ldots = ma^l - l\int_1 a^{l-1} + \ldots \pm \int_l,$$
$$(b-a)^l + (b-c)^l \ldots = mb^l - l\int_1 b^{l-1} + \ldots \pm \int_l,$$
$$(c-a)^l \text{ \&c.}$$

And these equations being also added together, the 1st side will be *the sum of the powers l of the differences of the roots, each of them subtracted from all the others:* the 2nd side will be

$$m\int_l - l\int_1 \int_{l-1} + A'\int_2 \int_{l-2} - A''\int_3 \int_{l-3} + \ldots \pm m\int_l.$$

If, now, l be odd, nothing can be gained from this formula; for, on the 1st side, the several pairs of differences are equal, and of contrary signs, and their powers l destroy each other; whilst the 2nd side is composed of terms, of which those that are equally distant from the extremes have the same coefficient, and the same index for $\int$, with opposite signs; and these terms, therefore, also destroy each other: thus the result is nothing more than $0 = 0$.

But if l be even, $(a-b)^l, (b-a)^l; (a-c)^l, (c-a)^l;\ldots$ are equal two and two, with the same sign, and the terms on the 1st side form themselves into pairs; also, the parts of the 2nd are still equal two and two, and have now the same sign; they, therefore, also pair one with another, except the middle term, which has no other corresponding to it. Taking the half of these two sides, each term becomes single, and the middle term must be reduced to its half. Thus, on the one hand, making $l = 2i$, the 1st side becomes the sum of the powers $2i$ of the differences of the roots, or that of the powers i of the squares of those differences, a sum which we shall represent by S_i. On the other hand, the coefficients of the binomial for the exponent $2i$ being denoted by $2i, A', A''\ldots$, there results

$$S_i = m\int_{2i} - 2i\int_1 . \int_{(2i-1)} + A'\int_2\int_{(2i-2)} - A''\int_3\int_{(2i-3)}\cdots$$
$$\pm\frac{1}{2}.\frac{2i\,(2i-1)\,(2i-2)\ldots\,(i+1)}{2.3.4\ldots\,i}\times(\int_i)^2\ldots\ (N).$$

The values of the coefficients $2i, A', A''\ldots$ are the numbers of the line $2i$ in the table, p. 6; stopping at the middle term, of which we must take the half. These factors are for

$i = 1\ldots\ 1, 1$
$i = 2\ldots\ 1, 4, 3$
$i = 3\ldots\ 1, 6, 15, 10$
$i = 4\ldots\ 1, 8, 28, 56, 35$
$i = 5\ldots\ 1, 10, 45, 120, 210, 126$
$i = 6\ldots\ 1, 12, 66, 220, 495, 792, 462$, &c.

And hence we deduce

$S_1 = m\int_2 - (\int_1)^2$, $\qquad S_4 = m\int_8 - 8\int_1\int_7\ldots + 35(\int_4)^2$,
$S_2 = m\int_4 - 4\int_1\int_3 + 3(\int_2)^2$, $\qquad S_5 = m\int_{10} - 10\int_1\int_9\ldots - 126(\int_5)^2$,
$S_3 = m\int_6 - 6\int_1\int_5 + 15\int_2\int_4 - 10(\int_3)^2$, $\qquad S_6 = m\int_{12} - 12\int_1\int_{11}\ldots + 462(\int_6)^2$.

This being premised, when we have calculated the series $\int_0 \int_1 \int_2\ldots$, we shall be able, making $i = 1$, to deduce from our equation the value of $(a-b)^2 + (a-c)^2\ldots$, which will be the sum S_1 of the simple powers of

the roots of $Z=0$; $i=2$ will in like manner give $(a-b)^4+(a-c)^4\ldots$, or S_2, &c.; and, generally, the equation (N) will give the sum S_i of the powers i of the roots of the equation between the squares of the differences. But, according to the equations (A), [p. 102] we have, for this equation,

$$P=-S_1,\ Q=-\tfrac{1}{2}(PS_1+S_2),\ R=-\tfrac{1}{3}(QS_1+PS_2+S_3)\ldots$$

The calculation for S must be continued up to the index $n=\frac{1}{2}m(m-1)$, the degree of Z; and that for f, to an index the double of this.

For $x^3+qx+r=0$, the values of $f_0, f_1\ldots$ are

$$3,\ 0,\ -2q,\ -3r,\ 2q^2,\ 5qr,\ -2q^3+3r^2;$$

whence

$$\begin{aligned} S_1 &= -6q,\ S_2=18q^2,\ S_3=-66q^3-81r^2,\\ P &= \ \ 6q,\ Q=\ 9q^2,\ R=\ \ 27r^2+\ 4q^3; \end{aligned}$$

and these are the coefficients of the equation of the squares of the differences for the 3rd degree. The formulæ for the 4th and 5th degree will be found in the *Resol. numer.* of Lagrange, Nos. 38, 39, and Note 111.

EQUATIONS OF THE SECOND DEGREE.

558. The equation $x^2+px+q=0$ having a and b for its unknown roots, let $z=a+mb$, m being an arbitrary number; since then $a+b=-p$, these two equations will serve to determine a and b, when z is known. But this value of $a+mb$ cannot be determined, without our, at the same time, obtaining that of $b+ma$; so that z, having these two roots, is given by another equation of the 2nd degree

$$[z-(a+mb)]\times[z-(b+ma)]=0.$$

Thus, so long as m continues indeterminate, nothing can be gained from this calculation. But if this equation in z be deprived of its 2nd term, which will be done by making $m=-1$, we have

$$z^2=(a-b)^2=a^2+b^2-2ab=f_2-2q;$$

and since [p. 102] $f_2=p^2-2q$, we find

$$z=a-b=\pm\sqrt{(p^2-4q)},\ a+b=-p,$$

whence, lastly, we deduce the two roots a and b.

EQUATIONS OF THE THIRD DEGREE.

559. The roots of $x^3 + px + q = 0$ being a, b, c, the quantity $z = a + mb + nc$, where m and n are any numbers whatever, is susceptible of six values [see equ. 2 below]; and since not one of these values can be found, without the calculation giving at the same time the five others, z must be a root of an equation of the 6th degree: under these circumstances therefore we cannot hope to obtain z before x. Admitting, however, that m and n can have such values assigned to them that the equation in z shall be of the form $z^6 + Az^3 + B = 0$, this equation will be resolvable by the method of the 2nd degree [N°. 545], and we shall from it readily deduce z, and then x. In effect, assuming $z^3 = u$, we have

$$u = -\tfrac{1}{2}A \pm \sqrt{(\tfrac{1}{4}A^2 - B)} = z^3 \dots (1).$$

Now, denoting the two cube roots of u by z', z'', and those of unity by 1, α, α^2 [N°. 539], the six values of z must result from the several alternations of position between a, b, c in the trinomial $a + mb + nc$: assume therefore

$$\left.\begin{array}{ll} z' = a + mb + nc, & z'' = a + nb + mc \dots \\ \alpha z' = b + mc + na, & \alpha z'' = b + nc + ma \dots \\ \alpha^2 z' = c + ma + nb, & \alpha^2 z'' = c + na + mb \dots \end{array}\right\} (2);$$

where, in respect to the position of a, b, c, the 1st term in one equation takes the last place in the following, and each of the other two letters passes from its previous place to the one on its left.

It remains to determine the arbitrary quantities m and n so that these six equations may be realized. For this purpose, multiply $\alpha z'$ by α^2; the result, since $\alpha^3 = 1$, is

$$z' = \alpha^2 b + m\alpha^2 c + n\alpha^2 a = a + mb + nc,$$

and the supposed identity of these equations requires that the respective coefficients of a, b, c be equal, or $\alpha^2 = m$, $m\alpha^2 = n$, $n\alpha^2 = 1$; and therefore $m = \alpha^2$, $n = \alpha$.

Substituting these values in the six equations (2), it appears that they are a consequence of

$$z' = a + \alpha c + \alpha^2 b,\ z'' = a + \alpha b + \alpha^2 c \dots (3).$$

Thus, taking $m = \alpha^2$, $n = \alpha$, our trinomial has six values, which form only two different cubes z'^3, z''^3; for, multiplying the equations (3) by 1, α and α^2, the results are the six equations (2), and the 1st sides of these evidently have only z'^3 and z''^3 for their cubes.

It is now proved therefore that the six values of z are roots of an equation of the form $z^6 + Az^3 + B = 0$, or

$$(z^3 - z'^3)(z^3 - z''^3) = z^6 - (z'^3 + z''^3)z^3 + z'^3 z''^3 = 0;$$

and it only remains to determine A and B, viz.

$$A = -(z'^3 + z''^3),\ B = (z'.z'')^3;$$

for, A and B being known in functions of the coefficients p and q, the equation (1) will give the values of z^3, of which the cube roots z' and z'' will thus be known; and the equations (3) will then give a, b, c, as we shall show below.

In the mean time, developing the cube of $z' = a + \alpha c + \alpha^2 b$, and substituting 1 for α^3, whenever it occurs, we have

$$z'^3 = \int_3 + 6abc + 3\alpha(a^2c + b^2a + c^2b) + 3\alpha^2(a^2b + c^2a + b^2c);$$

and z''^3 is obtained from this by changing b into c.

Let the two results be added together; since $abc = -q$, and $\alpha + \alpha^2 = -1$, they give

$$-A = 2\int_3 - 12q - 3[a^2b],$$

and the formula C [p. 108] giving $[a^2b] = \int_1 \int_2 - \int_3$, of which $\int_1 = 0$, there results

$$-A = 2\int_3 - 12q + 3\int_3 = 5\int_3 - 12q,$$

and, $\int_3$ being $= -3q$, we finally have $A = 27q$.

On the other hand,

$$z'z'' = \int_2 + (\alpha + \alpha^2)[ab];$$

which, since $\int_2 = -2p$, $[ab] = p$, $\alpha + \alpha^2 = -1$, becomes $z'z'' = -3p$; and the cube is $B = -27p^3$.

Thus,

$$u = -27\left(\tfrac{1}{2}q \pm \sqrt{\tfrac{1}{4}q^2 + \tfrac{1}{27}p^3}\right) = z^3;$$

and since the factors of 27 in this expression are the roots t' and t'' of the equation $t^2 + qt = (\frac{1}{3}p)^3$, we have $z^3 = 27t$.

Eliminating a, b, c between the equations (3) and the one $a + b + c = 0$, which exists in consequence of the proposed equation being devoid of the 2nd term, we have

$$3a = z' + z'',\ 3b = \alpha z' + \alpha^2 z'',\ 3c = \alpha^2 z' + \alpha z'';$$

and since $z' = 3\sqrt[3]{t'}$, $z'' = 3\sqrt[3]{t''}$, we arrive again at the values of No. 548.

EQUATIONS OF THE FOURTH DEGREE.

560. To resolve the equation $x^4 + px^2 + qx + r = 0$, we shall not attempt to form the values of $z = a + lb + mc + nd$, which are 24 in number; but of $z = a + b + m(c + d)$, which has only six values; and, in fact, making $m = -1$, we shall assume $z = a + b - c - d$, the six values of which are equal two and two with opposite signs. The root z will then be given by an equation of the 6th degree, as $z^6 + Az^4 + Bz^2 + C = 0$, containing only even powers, and consequently these six values will have but three different squares. Assuming $z^2 = t$, we shall reduce our equation to one of the 3rd degree; this will give t, and we shall thence deduce z, and lastly x.

Developing the square, we have

$$(a + b - c - d)^2 = (a + b + c + d)^2 - 4(ac + ad + bc + bd);$$

but, since the 2nd term is wanting in the proposed equation, the 1st part of the 2nd side of this is $= 0$; consequently, adding and subtracting $4(ab + cd)$, we obtain

$$(a + b - c - d)^2 = -4[ab] + 4(ab + cd);$$

and, $[ab]$ being $= p$, we have, changing b into c, and then into d,

$$(a + c - b - d)^2 = -4p + 4(ac + bd),$$
$$(a + d - c - b)^2 = -4p + 4(ad + bc);$$

and these are the values of our three squares z^2. The calculations will be simplified, if we assume for the unknown quantity $u = \frac{1}{4}z^2 + p$, since the values of u will thus be

$$ab + cd,\ ac + bd,\ ad + bc.$$

To form the equation which has these three roots, since

$$\int_1 = 0,\ \int_2 = -2p,\ \int_3 = -3q,\ \int_4 = 2p^2 - 4r,$$
$$\int_5 = 5pq,\ \int_6 = -2p^3 + 6pr + 3q^2,$$

we find, from formula (D), and dividing, when it is possible, by 2 or 6, that

1°. The sum of the binomials is $[ab] = p$;

2°. The sum of their products 2 and 2 is

$$[a^2bc] = \int_4 - \tfrac{1}{2}(\int_2)^2 = -4r;$$

3°. The product of the three binomials is $abcd \times \int_2 + [a^2b^2c^2]$, or

$$r\int_2^1 + \tfrac{1}{6}(\int_2)^3 - \tfrac{1}{2}\int_4 \cdot \int_2 + \tfrac{1}{3}\int_6 = -4pr + q^2.$$

Thus, we have

$$u^3 - pu^2 - 4ru + 4pr - q^2 = 0,$$

or, substituting $\frac{1}{4}z^2 + p$ for u,

$$z^6 + 8pz^4 + 16z^2(p^2 - 4r) - 64q^2 = 0.$$

Having from this determined the three values of z^2, and then their roots $\pm (z, z', z'')$, we must deduce a, b, c, d from the equations

$$f_1 = a + b + c + d = 0, \qquad a + c - b - d = z',$$
$$a + b - c - d = z, \qquad a + d - b - c = z''.$$

These equations, added 2 and 2, give

$$a + b = \tfrac{1}{2}z,\ a + c = \tfrac{1}{2}z',\ a + d = \tfrac{1}{2}z'';$$

from the sum of these we find $a = \frac{1}{4}(z + z' + z'')$, and the values of b, c and d follow of course.

But z, z', z'' being taken with the sign $\pm$, we have 8 roots instead of 4; and, in fact, the equation in z being dependent on q^2, and not on q, our calculation leaves the sign of q arbitrary. Now the product of the three last equations is

$$\tfrac{1}{8}zz'z'' = a^3 + a^2(b + c + d) + [abc];$$

and, since $-a = b + c + d$, this becomes $\frac{1}{8}zz'z'' = -q$.

The sign therefore of the product $zz'z''$ is the opposite to that of q; and consequently, as in p. 98, we have the two systems

$$q \text{ positive},\ x = \tfrac{1}{2}(z \pm z' \mp z''), \text{ and } \tfrac{1}{2}(-z \pm z' \pm z'');$$
$$q \text{ negative},\ x = \tfrac{1}{2}(z \pm z' \pm z''), \text{ and } \tfrac{1}{2}(-z \mp z' \pm z'').$$

ELIMINATION.

561. Let $\quad Z = 0$, or $kx^m + px^{m-1} + \ldots + u = 0$,
$\quad T = 0$, or $k'x^m + p'x^{n-1} + \ldots + u' = 0$,

be two equations in x and y. Supposing the 2nd of these equations to be solved in respect to x, and the results to be $x = fy, \phi y, \psi y \ldots$, if these functions of y be substituted for x in $Z = 0$, we shall obtain as many equations $A = 0$, $B = 0$, $C = 0 \ldots$, in y alone. And if the first of these be solved, the values $y = \alpha, \alpha', \alpha'' \ldots$, substituted in $x = fy$, will give the corresponding values $x = \beta, \beta', \beta'' \ldots$; whence we have the pairs $(\alpha, \beta), (\alpha', \beta') \ldots$ which will render Z and T simultaneously $= 0$. And the same will apply to $B = 0$ and $x = \phi y$, $C = 0$ and $x = \psi y \ldots$ If now we assume the product $A \times B \times C = 0$, this equation will

have for roots all the values of y thus obtained; and it will therefore be the *final equation* in y, clear of every extraneous root. Our object at present is to compose this product $A.B.C...$

Let fy, φy, $\psi y...$ be denoted by $a, b, c...$; if then x be changed in Z into $a, b, c...$ successively, we shall have different polynomials $Z, Z', Z''...$; and forming their product, it would be the one required, did it not contain $a, b, c....$ But since the product $Z.Z'.Z''...$ is not to vary when a is changed into b, into $c,...$, the coefficients must be symmetrical functions of these letters, which we suppose to be roots of the equation $T=0$, resolved in respect to x. We shall be able therefore to express these coefficients in terms of $\int_1, \int_2, \int_3...$ deduced from $T=0$, that is to say, in terms of the coefficients of T, which are functions of y. And thus the product $Z.Z'.Z''$, being cleared in the first place of x, and then of $a, b, c...$, will contain only the unknown quantity y, and be the required product $A.B.C...$

Hence, substitute for x successively, in $Z=0$, the letters $a, b, c...$, their number being that of the degree of x in T; multiply the resulting polynomials, and the coefficients of the product will be symmetrical functions of $a, b, c...$; then deduce from $T=0$ the values of $\int_1, \int_2...$ in y; express your symmetrical functions in terms of these values, and you will have the final equation required.

Let the equations be

$$x^3y-3x+1=0,\ x^2(y-1)+x-2=0;$$

we take, according to our rule,

$$(a^3y-3a+1)(b^3y-3b+1)=0,$$

whence

$$a^3b^3y^2+y\int_3-3aby\int_2+9ab-3\int_1+1=0.$$

But from the 2nd of the proposed equations we find

$$\int_1=\frac{-1}{y-1},\ ab=\frac{-2}{y-1},\ \int_2=\frac{1}{(y-1)^2}+\frac{4}{y-1};$$

and substituting, we obtain the same final equation as in p. 57.

Adding the exponents which, in each term of Z, affect x and y, let the greatest of these sums be denoted by m; m then is said to be the *degree* of the equation $Z=0$: y can at the most enter but in the 1st degree into the coefficient p of x^{m-1}, in the 2nd degree into that q of $x^{m-2}...$ &c.

Let n be the degree of $T=0$; we will now prove that *the degree of the final equation cannot exceed the product mn of the degrees of the equations proposed.*

In the first place, we know that the value of $\int_1$ contains no other co-efficient than p', that of $\int_2$ contains also q'... ; and thus $\int_1, \int_2, \int_3$... have their highest degree in y expressed by their respective indices. On the other hand, a term of the product $Z.Z'.Z''$..., as $y^i\,[a^\alpha b^\beta c^\gamma]$, has its degree $i+\alpha+\beta+\gamma... = mn$ at the most, since any term of Z is at the most but of the degree m, and there are only n factors Z, Z'.... It follows also from the formulæ of N°. 555, which give the expression for invariable functions, that $[a^\alpha b^\beta c^\gamma ...]$ will have its degree in $y = \alpha+\beta+\gamma$... Hence, the term itself will not, at the most, have its degree higher than mn.

[See a Memoir by M. Poisson, 11^e *Journal Polytechnique.*]

V. CONTINUED FRACTIONS.

GENERATION AND PROPERTIES.

562. Suppose that, in approximating to the unknown quantity x of an equation $X=0$, we have arrived at the integer y immediately less, we shall have then $x = y + \frac{1}{x'}$, x' being a new unknown quantity >1; and substituting this value in $X=0$, we shall obtain a transformed equation in x' of the same degree. Repeating the operation on this equation, and investigating the integer y' contained in x', we shall make $x' = y' + \frac{1}{x''}$, then $x'' = y'' + \frac{1}{x'''}$..., x'', x'''... being >1; and thus we shall obtain the series of equations (A); whence, by substitution, will result the value of x under the form (B), which is denominated a *continued fraction*:

$$
\left.\begin{aligned}
x &= y + \frac{1}{x'}\\
x' &= y' + \frac{1}{x''}\\
x'' &= y'' + \frac{1}{x'''}\\
x''' &= y''' + \frac{1}{x^{IV}}\\
\&c. &= \&c.
\end{aligned}\right\}\ (A)
\qquad
x = y + \cfrac{1}{y' + \cfrac{1}{y'' + \cfrac{1}{y''' + \cfrac{1}{y^{IV} + \cfrac{1}{y^{V} + \&c.}}}}}\quad (B)
$$

The integers y, y', y'', y'''... are the *terms* of the continued fraction, which we shall write under the abridged form

$$x = y, y', y'', y''' \ldots$$

The value of x in the form of an ordinary fraction is obtained by a process similar to the following. Let the fraction be

$$x = 2, 1, 3, 2, 4 = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{4}}}}$$

Commencing from the extremity, $2 + \frac{1}{4}$ reduces itself to $\frac{9}{4}$; the unit, divided by $\frac{9}{4}$, gives $\frac{4}{9}$, and x becomes

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{4}{9}}}$$

In the same manner, $3 + \frac{4}{9} = \frac{31}{9}$, $1 : \frac{31}{9} = \frac{9}{31}$; whence $x = 2 + \dfrac{1}{1 + \frac{9}{31}} = 2 + 1 : \frac{40}{31} = 2 + \frac{31}{40} = \dfrac{111}{40}$, the value required.

This calculation evidently corresponds to that of N°. 30; the 1st line containing the terms of the continued fraction, and the 2nd being constructed by this rule: *multiply each term of the 1st line by the number subscribed below it, and to the product add the number which stands on the right of the one subscribed; then place the sum in the next rank on the left.*

$$\begin{array}{r} x = 2,\ 1,\ 3,\ 2,\ 4 \\ 111, 40, 31, 9, 4, 1 \end{array} \quad \Bigg\| \quad \begin{array}{r} x = 3,\ \ 2,\ 1,\ 1,\ 3, 2, 4 \\ 617, 182, 71, 40, 31, 9, 4, 1 \end{array}$$

For $x = 3, 2, 1, 1, 3, 2, 4$ we have $x = \frac{617}{182}$. When the continued fraction extends to infinity, an approximate value of it is obtained by neglecting all the terms after some specified one. Let x^{IV} be neglected in the 4th of the equations (A); we have then $x''' = y'''$, and x''' is thus rendered *too small*; consequently, on substituting this value, $x'' = y'' + \dfrac{1}{x'''}$ becomes *too great*; whilst the next value x' is *too small*, &c. Generally, *the continued fraction, when broken off at a term of an odd order, is $< x$; and $> x$ in the opposite case;* and if the fraction be successively limited to the 1st term y, to the 2nd y', the 3rd y''..., the values will be successively $<$ and $>$ x, which is therefore

comprised between any two of these results taken consecutively. These results, which are called *convergent fractions,* we shall represent by

$$\frac{a}{a'}, \frac{b}{b'}, \frac{c}{c'}, \frac{d}{d'} \dots \frac{m}{m'}, \frac{n}{n'}, \frac{p}{p'} \dots (C),$$

taking

$$y, y', y'', y''' \dots y^{i-2}, y^{i-1}, y^{i} \dots$$

for the term to which the fraction is limited.

We have thus

$$\frac{a}{a'} = \frac{y}{1}, \frac{b}{b'} = \frac{y\,y' + 1}{y'}, \frac{c}{c'} = \frac{y\,y'\,y'' + y'' + y}{y'\,y'' + 1} \dots$$

and the last of these fractions is evidently equivalent to $\frac{c}{c'} = \frac{b\,y'' + a}{b'\,y'' + a'}$.

To obtain $\frac{d}{d'}$, y'' must be replaced in this expression by $y'' + \frac{1}{y'''}$, since $x = y, y', y''$ on the introduction of this fraction becomes $x = y, y', y'', y'''$. But the numerator is changed by this into

$$b\,y'' + a + \frac{b}{y'''} = c + \frac{b}{y'''} = \frac{c\,y''' + b}{y'''};$$

whilst the denominator becomes $\frac{c'\,y''' + b'}{y'''}$; and therefore $\frac{d}{d'} = \frac{c\,y''' + b}{c'\,y''' + b'}$.

From a comparison of these values of $\frac{c}{c'}$ and $\frac{d}{d'}$ we infer this law: *the numerator of a convergent fraction is deduced from the two preceding numerators, by multiplying them respectively by* 1 *and by the integer which terminates the continued fraction, and taking the sum of the products. The denominator observes the same law,* which extends to the whole series C of the convergents, being the result of a calculation which subsists for each particular fraction. Thus

$$p = n\,y^{(i)} + m,\ p' = n'\,y^{(i)} + m' \dots (D),$$

and

$$\frac{p}{p'} = \frac{n\,y^{i} + m}{n'\,y^{(i)} + m'} \dots (E).$$

Having, therefore, formed the two first convergents, we shall from them be able to deduce all the others consecutively.

In this manner we find, for

$$x = 2, 1, 3, 2, 4, \text{ the fractions } \tfrac{2}{1}, \tfrac{3}{1}, \tfrac{11}{4}, \tfrac{25}{9}, \tfrac{111}{40}.$$

These fractions are alternately $<$ and $>$ x, until the last, which is the value itself of x; this theorem, therefore, gives us a second mode of obtaining the value of x.

If, in the equation (E), y^i be replaced by the total value z of the continued fraction, taken from the term y^i to the end, $z = y^i, y^{i+1}, y^{i+2}\ldots$, it is clear that instead of a convergent, we shall have the exact value of x, viz.

$$x = \frac{n\,z + m}{n'\,z + m'} \ldots (F);$$

and this is denominated a *complete fraction.*

563. Eliminating y^i between the equations (D), there results

$$p\,n' - p'\,n = -(n\,m' - m\,n');$$

i. e. *the difference of the products of the terms, taken cross-wise, of two consecutive convergents is constantly the same, or varies only in sign;* and since for the two first, which are y and $\frac{y\,y' + 1}{y'}$, *this difference is* 1, we conclude that

$$p\,n' - p'\,n = \pm 1, \frac{p}{p'} - \frac{n}{n'} = \pm \frac{1}{p'\,n'} \ldots (G).$$

The sign $+$ must be taken when y^i and $\frac{p}{p'}$ are of an even order, $\frac{p}{p'}$ being then $> \frac{n}{n'}$; and the sign $-$ in the contrary case.

From this theorem we derive many consequences:

1°. Since every divisor of p and p' must also divide 1, it appears that p and p' are prime to each other; and the same is the case for p and n, p' and n'. Thus *the convergents are all irreducible.*

2°. Let $\frac{n}{n'}$, and $\frac{p}{p'}$ be successively subtracted from $x = \frac{p\,z + n}{p'\,z + n'}$, the complete fraction, in which $z = y^{i+1}, y^{i+2}\ldots$; the differences are

$$\frac{\pm z}{n'\,(p'\,z + n')}, \; \frac{\mp 1}{p'\,(p'\,z + n')} = \delta;$$

and of these the 1st exceeds the 2nd; for $p'\,z + n'$ is a common divisor,

$n' < p'$ (a', b', c'..., from their composition, becoming greater and greater), and $z > 1$ (y^{i+1} is contained in z). Thus, x is nearer to $\frac{p}{p'}$ than to $\frac{n}{n'}$; and the signs $\pm$ and $\mp$ arise from x lying between these two convergents. *The fractions (C), therefore, are nearer and nearer approximations, by defect and by excess alternately, to the true value of x;* and hence their name of *convergents.*

Moreover, δ is the error incurred by limiting the continued fraction to the integer y^i, *i. e.* by taking $x = \frac{p'}{p}$. Put 1 for z, and also neglect n'; we find then

$$\delta < \frac{1}{p'} \times \frac{1}{p' + n'}, \text{ and } \delta < \frac{1}{p'^2} \dots (H);$$

and these are limits of the error incurred; we should have one still lower by assuming $z = y^{i+1}$, the integer contained in z. *For any convergent, therefore, the error does not amount to* 1 *divided by the square of its denominator.* This results also from the fact that

$$\frac{p}{p'} - \frac{n}{n'} = \frac{1}{p'n'} < \frac{1}{n'^2}.$$

In our last example, [illegible] is not in error by [illegible], nor even by [illegible] or [illegible].*

3°. Let $\frac{h}{h'}$, $\frac{k}{k'}$, $\frac{l}{l'}$ be any three increasing fractions; the difference between the extremes exceeds that of either of them with the intermediate one. Suppose, likewise, that h, h', l and l' have been so selected that $lh' - l'h = 1$; we shall have then

$$\frac{l}{l'} - \frac{h}{h'} = \frac{1}{l'h'} > \frac{kh' - k'h}{k'h'} \text{ and } \frac{lk' - kl'}{k'l'}.$$

---

* The successive differences between the convergents are

$$\frac{b}{b'} - \frac{a}{a'} = \frac{1}{a'b'},\ \frac{c}{c'} - \frac{b}{b'} = \frac{-1}{b'c'} \dots \frac{p}{p'} - \frac{n}{n'} = \frac{\pm 1}{p'n'}.$$

The sum of all these equations gives

$$\frac{p}{p'} = \frac{a}{a'} + \frac{1}{a'b'} - \frac{1}{b'c'} + \frac{1}{c'd'} - \dots \pm \frac{1}{p'n'};$$

and we thus obtain a developed expression for the exact value of x, when $\frac{p}{p'}$ is the last convergent, and for an approximate value in any other case. In our example, we find

[illegible] = x = [illegible] + [illegible] − [illegible] + [illegible] − [illegible]

But these numerators, being integral and positive, must, at the least, be 1; let them be replaced by 1; we find then $l'h' < k'h'$ and $k'l'$, *i. e.* suppressing the common factors, $k' > l'$ and h': k', therefore, is the greatest of the three denominators; and, reversing the three fractions, it may be proved in the same manner that $k > h$ and l. Thus, the middle fraction is composed of higher terms than the extremes.

Now, x lies between $\frac{n}{n'}$ and $\frac{p}{p'}$; and in order, therefore, that the fraction $\frac{k}{k'}$ may be nearer to x than either of the convergents, it must fall between them, and consequently be composed of higher terms. Hence, *every convergent approaches more nearly to x than any other fraction conceived in lower terms.*

4°. From $\frac{m}{m'}$, $\frac{n}{n'}$, form the two fractions

$$\frac{h}{h'} = \frac{m + (t-1)n}{m' + (t-1)n'},\ \frac{l}{l'} = \frac{m + tn}{m' + tn'};$$

t then being changed successively into 1, 2, 3... y^i, y^i being the integer contained in the following convergent, we have

$$\frac{m}{m'},\ \frac{m+n}{m'+n'},\ \frac{m+2n}{m'+2n'} \dots \frac{m+y^i n}{m'+y^i n'} = \frac{p}{p'} \dots (I).$$

But $\frac{h}{h'} - \frac{l}{l'} = \frac{\pm 1}{h'l'}$, whatever be the integer t. Hence, these fractions are irreducible [1°]; they approach more nearly to x than any other fraction in lower terms [3°]; their consecutive differences having the same sign, they go on increasing from the first to the last; they are all $< x$, if the extremes occupy odd ranks; and they descend towards x in the contrary case; lastly, the error δ, incurred in taking one of them $\frac{l}{l'}$ for x, is less than $\frac{n}{n'} - \frac{l}{l'} = \frac{1}{n'l'}$, since x lies between these two fractions.

Thus it appears that, between our *principal* convergents, we may insert $y^i - 1$ fractions possessing the same properties, and so obtain a number of *intermediate* convergents. These fractions altogether form two series; in the one, the fractions, derived from the odd ranks, ascend towards x; in the other, they descend towards that quantity. The fractions themselves are formed by adding the respective terms of the successive convergents $\frac{m}{m'}$, $\frac{n}{n'}$, this addition being repeated y^i times.

In our example, we have $x = 2, 1, 3, 2, 4$:

principal convergents ... $\frac{2}{1}$, $\frac{3}{1}$, $\frac{11}{4}$, $\frac{25}{9}$, $\frac{111}{40}$.

Commencing with $\frac{2}{1}$ and $\frac{3}{1}$, we deduce $\frac{5}{2}$, $\frac{8}{3}$, $\frac{11}{4}$, the last of which is the third convergent, at which we have arrived after three operations, y' being $= 3$. Taking now $\frac{11}{4}$ and $\frac{25}{9}$, we find $\frac{36}{13}$, $\frac{61}{22}$, $\frac{86}{31}$, $\frac{111}{40}$; and thus we have

$$\left(\tfrac{2}{1}\right),\ \tfrac{5}{2},\ \tfrac{8}{3},\ \left(\tfrac{11}{4}\right),\ \tfrac{36}{13},\ \tfrac{61}{22},\ \tfrac{86}{31},\ < x = \tfrac{111}{40}.$$

The fractions derived in the same manner from the even ranks present this series (it is unlimited)

$$\left(\tfrac{3}{1}\right),\ \tfrac{14}{5},\ \left(\tfrac{25}{9}\right),\ \tfrac{136}{49},\ \tfrac{247}{89},\ \tfrac{358}{129},\ \tfrac{469}{169}\ldots > x.$$

The series of the principal convergents (C) might be made to commence from $\frac{0}{1}$ and $\frac{1}{0}$, which fulfil all the specified conditions.

DETERMINATE EQUATIONS OF THE FIRST DEGREE.

564. To reduce the value of x in the equation $Ax = B$ to the form of a continued fraction, we must, according to the principles of N°. 562, extract the integer y contained in $x = \dfrac{B}{A} = y + \dfrac{R}{A} = y + \dfrac{1}{x'}$, R being the remainder from the division of B by A; and repeat the operation on

$$x' = \frac{A}{R} = y' + \frac{R'}{R},\ x'' = \frac{R}{R'} = y'' + \frac{R''}{R'},\ x''' = \&c.:$$

then

$$x = y + \frac{R}{A} = y + \cfrac{1}{y' + \cfrac{R'}{R}} = y + \cfrac{1}{y' + \cfrac{1}{y'' + \&c.}}$$

This operation gives, for the *terms of the continued fraction, the successive quotients obtained in the calculation for the greatest common divisor between A and B*, viz.

$x = y, y', y'', y'''\ldots$ This expression is always finite.

Thus, for the equation $2645\,x = 9752$, we have

$$9752\,\Big|\,\frac{2645}{3}\,\Big|\,\frac{1817}{1}\,\Big|\,\frac{828}{2}\,\Big|\,\frac{161}{5}\,\Big|\,\frac{23}{7},\quad x = \frac{424}{115} = 3, 1, 2, 5, 7.$$

Hence, by the calculations (E) and (I), may be deduced the principal and intermediate convergents; and we thus obtain

$$\left(\tfrac{0}{1}\right),\ \tfrac{1}{1},\ \tfrac{2}{1},\ \left(\tfrac{3}{1}\right),\ \tfrac{7}{2},\ \left(\tfrac{11}{3}\right),\ \tfrac{70}{19},\ \tfrac{129}{35},\ \tfrac{188}{51}\ldots < x = \tfrac{424}{115};$$

$$\left(\tfrac{1}{0}\right),\ \left(\tfrac{4}{1}\right),\ \tfrac{15}{4},\ \tfrac{26}{7},\ \tfrac{37}{10},\ \tfrac{48}{13},\ \left(\tfrac{59}{16}\right),\ \tfrac{483}{131},\ \tfrac{907}{246}\ldots > x.$$

Any one of these fractions, as $\frac{26}{7}$, is a nearer approximation than any other of a more simple form, and does not differ from x by $\frac{1}{49}$.

In like manner, we find, for $x = \frac{409}{119} = 3, 2, 3, 2, 7,$

$$\left(\tfrac{3}{1}\right), \tfrac{10}{3}, \tfrac{17}{5}, \left(\tfrac{24}{7}\right), \tfrac{79}{23}, \tfrac{134}{39} \ldots < x; \; \left(\tfrac{7}{2}\right), \tfrac{31}{9}, \left(\tfrac{55}{16}\right), \tfrac{464}{135} \ldots > x.$$

We can therefore solve this problem: *A fraction being given, to find others more simple, and which are more nearly equal to it than any value composed of lower terms.*

The following are some important applications of this theory.

I. We have found, N°. 248, for *the ratio of the diameter to the circumference*, $\pi = 3{\cdot}1415926$, or $\frac{31415926}{10000000}$; which, reduced to a continued fraction, gives $\pi = 3, 7, 15, 1, 243, 1, 2 \ldots$: and hence result the principal and intermediate convergents.

$$\left(\tfrac{3}{1}\right), \tfrac{25}{8}, \tfrac{47}{15}, \tfrac{69}{22}, \tfrac{91}{29} \ldots \left(\tfrac{333}{106}\right) < x; \; \left(\tfrac{22}{7}\right), \left(\tfrac{355}{113}\right) \ldots > x.$$

All these fractions are approximate values of x, simpler than any other; among them are comprised the ratios given by Archimedes and Adrian Metius.

II. *The solar tropical year*, or the time that the sun employs for returning to the same equinox is about $365^d.242264$ [See *l'Uranographie*, N°. 33]. Thus, assigning only 365 days to the civil year, the equinox would, very nearly every four years, recur one day too late, and thus pass gradually through the whole of the calendar; but, to restore the agreement between the civil and natural divisions of time, the civil year is made, at stated epochs, to consist of 366 days. These years of 366 days, which are denominated *Bissextile*, recurred, in the calendar of Julius Cæsar, every four years. This intercalation, however, supposed the solar year to be $365^d.25$; so that it was thus exceeded in a very small degree by the civil year. Let us see how these differences may be balanced, so as to arrive at greater exactness,

Reducing 0.24226419 or $\frac{24226419}{100000000}$ to a continued fraction, we have

$$x = 0, 4, 7, 1, 4, 2, 1, \ldots \tfrac{1}{4}, \tfrac{7}{29}, \tfrac{8}{33}, \tfrac{39}{161}, \tfrac{86}{355}, \ldots$$

If, now, we take for the value of x one of these principal convergents, as $\frac{8}{33}$, this supposes the solar year to be $365\frac{8}{33}$ days, so that it will exceed the common year by $\frac{8}{33}^d$ annually; and consequently, in 33 years there will be 8 days to be intercalated; every fourth year therefore should have 366 days, only after 7 *bissextiles*, the 8th should be deferred to the 5th year; and we must then recommence with a new period of 33 years. This was the arrangement of the common year among the ancient Persians.

The *Julian* correction was established on the fraction $\frac{1}{4}$, the bissextiles recurring every four years: in the Gregorian calender, the same plan is adopted; but only one secular bissextile year is retained out of four, *i.e.* 97 days are intercalated in 400 years. The fraction $\frac{97}{400}$ not being found among our convergents, it is not so exact as others that might have been taken; but the variation is too small to be of any importance [See *l'Uranographie*].

III. The lunar month consists of $29^d.5305887$, the solar month of $30^d.4368535$; the ratio of which numbers, $x = \frac{304368535}{295305887}$, being converted into a continued fraction, we deduce the convergents

$$\left(\tfrac{1}{1}\right), \left(\tfrac{34}{33}\right), \tfrac{101}{98}, \left(\tfrac{168}{163}\right) \ldots < x\,; \left(\tfrac{33}{32}\right), \left(\tfrac{67}{65}\right), \tfrac{235}{228}, \ldots > x.$$

Assume, for this ratio, $x = \frac{235}{228}$, and it will follow that, in 235 lunar months, there have elapsed only 228 solar months, or 19 times 12 solar months; the difference between which numbers is 7. Hence, in 19 solar years, there is an excess of 7 lunar months, which must be intercalated; and after this time the sun and the moon will be found in the same positions as at first, and will begin to present a recurrence of their aspects in the same order. If therefore 19 tables be formed recording the epochs of the lunar phases; in all following years, we shall be able to predict the return of any one of these phases, by referring to the time in the tables at which it occurred in its periodic order. All this was taught by Meton to the Greeks, whose calendar was luni-solar, and who gave the name of *Solar Cycle* or *Golden Numbers* to the numbers which marked the order of recurrence of each year in the period of 19 years.

INDETERMINATE EQUATIONS OF THE FIRST DEGREE.

565. It has been shown [N°. 118] that, having deduced one integral solution, $x = \alpha$, $y = \beta$, of the equation $ax + by = c$, we shall from it be able to deduce all the others; the values of x and y forming equidifferences, the ratio of which is b for x and $-a$ for y, viz. $x = \alpha + bt$, $y = \beta - at$.

The processes however which have been made use of for determining this solution are less elegant and less expeditious than the one which we shall now derive from the principle of continued fractions.

Resolve $\frac{a}{b}$ into its convergents, and let $\frac{p}{p'}$ be the last but one, that which precedes the fraction proposed: it has been seen then [D, N°. 562] that

$$ap' - bp = \pm 1\,; \text{ whence } ap'c - bpc = \pm c.$$

The sign + is to be used when the continued fraction, taken through its whole extent, is composed of an *even number* of terms; and the sign − in the contrary case. Comparing this equation with $ax + by = c$, it is clear that, if the second sides have the same sign, the latter equation will be satisfied by assuming $x = \alpha = p'c$, $y = \beta = -pc$; whilst, if the signs of c be different, we must make $x = \alpha = -p'c$, $y = \beta = pc$.

Nothing is easier therefore than to obtain an integral solution of the equation $ax + by = c$; *we must resolve* $\frac{a}{b}$ *into a continued fraction; take the convergent* $\frac{p}{p'}$, *which results from this fraction, when its last term is neglected; then form the equation* $ap' - pb = \pm 1$, *and multiply it by* c. *It will remain only to compare this equation, term by term, with the one proposed.*

Take, for example, the equation $105x - 43y = 17$; the method of the common divisor gives $\frac{105}{43} = 2, 2, 3, 1, 4$; and $\frac{22}{9} = 2, 2, 3, 1$.

$$\begin{array}{c|c|c|c|c|c} 105 & \frac{43}{9} & \frac{19}{9} & \frac{5}{9} & \frac{4}{1} & \frac{1}{4} \\ 22, & 9, & 4, & 1, & 1. & \end{array}$$

This latter fraction is obtained by neglecting the term 4, and making use of the process described p. 114; and subtracting $\frac{22}{9}$ from $\frac{105}{43}$, we find $105 \times 9 - 43 \times 22 = -1$ (the sign − is used because the continued fraction has 5 terms; besides that, from the products of merely the figures of the units, it is at once obvious that the difference is negative). This equation being now multiplied by −17, and compared with the one proposed, we have $x = -9 \times 17$, $y = -22 \times 17$; and consequently

$$x = -153 + 43t, \quad y = -374 + 105t.$$

In like manner, for the equation $424x + 115y = 539$, we have $\frac{424}{115} = 3, 1, 2, 5, 7$; suppressing the final 7, there results $\frac{59}{16}$; subtracting these fractions, we find $424 \times 16 - 115 \times 59 = -1$; and multiplying this equation by −539, and comparing it then with the one proposed, we have $x = -16 \times 539$, $y = 59 \times 539$; viz.

$$x = -8624 + 115t, \quad y = 31801 - 424t.$$

These equations will be simplified by changing t into $t + 75$; which will make it requisite to subtract from 8624 and 31801, the products of 115 and 424 by 75 respectively; and we shall thus find

$$x = 1 + 115t, \quad y = 1 - 424t.$$

$19x + 7y = 117$ gives $\frac{19}{7} = 2, 1, 2, 2$; whence $\frac{8}{3} = 2, 1, 2$; and, by subtraction, $19 \times 3 - 8 \times 7 = 1$; this being multiplied by 117, &c., we find

$$x = 3 \times 117 - 7t, \; y = -8 \times 117 + 19t, \text{ or } x = 1 - 7t, \; y = 14 + 19t.$$

The problem in Chronology, which consists in finding the year x of which the *solar cycle* is c, and the *golden number* n, comes to the same thing with finding the integer x, which, divided by 28 and 19, gives for remainders $c - 9$ and $n - 1$ respectively. The method of N°. 121 gives, for this year,

$$x = 56(c - n) + c + 75 + 532t.$$

Thus the same numbers c and n recur together periodically every 532 years; an interval which is called the *Dionysian Period*. [See *l'Uranographie*, N°. 75.]

And if it be likewise required that the year x in question should have i for its *indiction*, *i. e.* that x divided by 15 should give the remainder $i - 3$, we have the period of 7980 years, styled *Julian*, and devised by Scaliger; we find

$$x = 4845c + 4200n - 1064i + 3267 + 7980t.$$

The examples of N°. 120 will serve for additional practice.

DETERMINATE EQUATIONS OF THE SECOND DEGREE.

566. Let us now reduce to the form of continued fractions the roots of the equation

$$Ax^2 - 2ax = k,$$

in which A, a, k are integral, and A positive. We shall suppose the irrational root to be positive; for, should x be negative, we have only to change a into $-a$, to give to this root the sign $+$. If the coefficient of the 2nd term be not an even number, the whole equation must be multiplied by 2. We shall have then

$$x = \frac{\pm\sqrt{t} + a}{A} \ldots (1),$$

supposing that

$$t = a^2 + Ak \ldots (2),$$

a value we presume to be known, positive and not a square. Let $\sqrt{t}$ be taken first with the sign $+$, and let y denote the greatest integer contained in x, viz.

$$x = \frac{\sqrt{t} + a}{A} = y + \frac{1}{x'}, \text{ where } x' = \frac{A}{\sqrt{t} + a - Ay}.$$

Making

$$\beta = Ay - a \ldots (3),$$

and multiplying the value of x', above and below, by $\sqrt{t} + \beta$, we shall have

$$x' = \frac{A(\sqrt{t} + \beta)}{t - \beta^2}.$$

But it follows from the equations (2) and (3), that $t - \beta^2 = A(k - Ay^2 + 2\alpha y)$, so that A is a common factor of the terms of x'; and assuming

$$k - Ay^2 + 2\alpha y = B \ldots (4),$$

there results

$$t - \beta^2 = AB, \text{ and } x' = \frac{\sqrt{t} + \beta}{B} \ldots (5).$$

This value of x' being of the same form as x, we may now extract the integer y' contained in x' by means of a similar process, which will give $x'' = \dfrac{\sqrt{t} + \gamma}{C}$; then $x''' = \dfrac{\sqrt{t} + \delta}{D}$, &c.; and we shall thus have

$$t = \alpha^2 + Ak = \beta^2 + AB = \gamma^2 + BC = \delta^2 + CD = \ldots (a),$$

$$x = \frac{\sqrt{t} + \alpha}{A}, x' = \frac{\sqrt{t} + \beta}{B}, x'' = \frac{\sqrt{t} + \gamma}{C}, x''' = \frac{\sqrt{t} + \delta}{D} \ldots (b),$$

$$\beta = Ay - \alpha, \gamma = By' - \beta, \delta = Cy'' - \gamma \ldots (c).$$

Instead of applying the calculation directly to the complete fractions (1), (5), such as they are given in the example proposed, we may successively deduce $\beta, \gamma \ldots, B, C \ldots$ from the equations (c) and (a), which will successively give these complete fractions, and consequently the integers, $y', y'' \ldots$ contained in them.

567. Let one of the complete fractions be $z = \dfrac{\sqrt{t} + \pi}{P} = y^{(i)} + \ldots,$ the corresponding convergent being $\dfrac{p}{p'} = y, y', y'' \ldots y^{(i)}$, and $\dfrac{m}{m'}, \dfrac{n}{n'}$ being the two convergents preceding.

We know then [p. 116] that $x = \dfrac{nz + m}{n'z + m'}$; and substituting, for x and z, the complete fractions which they represent, there results

$$\frac{\sqrt{t} + \alpha}{A} = \frac{n(\sqrt{t} + \pi) + Pm}{n'(\sqrt{t} + \pi) + Pm'}.$$

This equation being reduced to a common denominator, and then divided into two, in consequence of the irrational parts separately destroying each other, we have

$$\pi n' = (An - \alpha n') - Pm, \ \pi(An - \alpha n') = P\alpha m' - APm + n't;$$

whence, eliminating ϖ, and observing that $m'n - mn' = \pm 1$, there results

$$(An - \alpha n')^2 = \pm PA + n'^2 t,$$

or

$$A\left(\frac{n}{n'}\right)^2 - 2\alpha\left(\frac{n}{n'}\right) - k = \pm \frac{P}{n'^2} \ldots (f).$$

But this process is equivalent to the elimination of m and m' between the three equations above; and consequently *the equation* (f) *expresses that the fraction* $\frac{n}{n'}$ *is some one of the convergents towards* x: the sign $+$ indicates that this fraction is of an even rank; the $-$, that its rank is odd.

Two cases here present themselves:

1°. If z be of an odd rank, the sign $+$ must be adopted. But in this case $\frac{n}{n'}$ is of an even rank and $> x$; so that this convergent, substituted for x in $Ax^2 - 2\alpha x - k$, must give a positive result; and in order therefore that this condition may be fulfilled, P must have the sign $+$. Thus, *the denominators of the complete fractions of the odd ranks are all positive.*

2°. When z is of an even rank, the sign $-$ must be taken. But, if $\frac{n}{n'}$ be comprised between the two roots of x, the first side of (f) is negative, which requires P to be positive, as in the 1st case; whilst, when this convergent is less than both roots, the contrary is the case. *The denominators therefore of the even ranks are not negative, except when the convergents of the odd ranks are at the same time less than both roots.*

The denominators therefore of the complete fractions $x, x', x'' \ldots$ can be negative only for the odd ranks; and not for them, unless also the roots of x have such a proximity to each other that the convergents shall not fall between them; in which case the continued fractions of the two roots will have the same initial letters. But it cannot be long before we arrive at such an approximation to the greater root, that the convergents of the odd ranks shall fall between it and the least; and after that negative denominators will be no more to be met with, onwards to infinity.

Each complete fraction being > 1, if the denominator P have the sign $-$, the numerator $\sqrt{t} + \varpi$ must have it also; thus ϖ must be negative and $> \sqrt{t}$, and the fraction consequently will have the form $\frac{\sqrt{t} - \varpi}{-P}$.

Let $\frac{\sqrt{t}+\delta}{D}$, $\frac{\sqrt{t}+\varepsilon}{E}$, $\frac{\sqrt{t}+\varphi}{F}$... be fractions taken among those which have not negative denominators (and this will be the case from the very first, when the two roots of x have not a common integer); then the equations (a) give $DE+\varepsilon^2=t$; whence D, E and ε^2 are $< t$; and therefore

$$D, E, F... < t;\ \varepsilon, \varphi... < \sqrt{t}.$$

If now it be possible, let $\frac{\sqrt{t}-\varphi}{F}$ be one of our fractions; then, according to the equations (a and c), $EF=t-\varphi^2$,... $Ey^{IV}=\varepsilon-\varphi$: but the 1st of these gives

$$EF=(\sqrt{t}+\varphi)(\sqrt{t}-\varphi), \frac{\sqrt{t}-\varphi}{F}=\frac{E}{\sqrt{t}+\varphi};$$

whilst, from the 2nd, $Ey^{IV} < \varepsilon$; whence $E < \sqrt{t}$, and our complete fraction would therefore be < 1, which is impossible. Hence, it is possible that, so long as we have negative denominators, the parts α, β... may be negative also; but, beyond that point, they will be all positive, and then $Ey^{IV}=\varepsilon+\varphi$ gives $E < \varepsilon+\varphi$, or $E < 2\sqrt{t}$: *the denominators therefore cannot attain to the double of* $\sqrt{t}$.

Since then these constants ε, φ..., D, E... are all integral, positive and infinite in number, whilst, at the same time, they cannot exceed certain limits; we must, sooner or later, fall in a second time with some complete fraction that has been already obtained; and we shall then consequently have a repetition, in the same order, of the complete fractions subsequent to it, and of the terms of the continued fraction, which will recur periodically.

Thus, *after a certain number of initial terms, we must discover a period.* We shall write this continued fraction under the form $x=y, y'...(u, u', u''...)$, inclosing the periodic part in a parenthesis, in order to exhibit it in a clear and succinct manner.

As to the second root $x=\frac{\alpha-\sqrt{t}}{A}$ supposed to be positive, we must commence the calculation in the same manner as for the first: v being the integer nearest to x, we shall find

$$x'=\frac{A}{\alpha-Av-\sqrt{t}}=\frac{A(\alpha-Av+\sqrt{t})}{(\alpha-Av)^2-t},$$

the second expression being multiplied, above and below, by $\alpha-Av+\sqrt{t}$. It may now be proved as before that A is a common factor in the last fraction; and x will take the form $\frac{\sqrt{t}+\beta'}{B'}$, where $\sqrt{t}$ has

the sign $+$. And this brings us again to the theory that has been just explained.

Take, for example, the equation $59x^2 - 319x + 431 = 0$: doubling it, in order to render the 2nd coefficient even, we have these successive results:

$$x = \frac{319+\sqrt{45}}{118} = 2+, \frac{\sqrt{45}-83}{-58} = 1+, \frac{\sqrt{45}+25}{10} = 3+,$$

$$\frac{{}^*\sqrt{45}+5}{2} = 5+, \frac{\sqrt{45}+5}{10} = 1+, \frac{{}^*\sqrt{45}+5}{2} \text{ \&c.}$$

and hence, $x = 2, 1, 3, (5, 1)$. For the 2nd root

$$x = \frac{319-\sqrt{45}}{118} = 2+, \frac{\sqrt{45}+83}{58} = 1+, \frac{\sqrt{45}-25}{-10} = 1+,$$

$$\frac{\sqrt{45}+15}{18} = 1+, \frac{\sqrt{45}+3}{2} = 4+, \frac{\sqrt{45}+5}{10} = \text{\&c.}$$

the last of these fractions has been met with already, and we have $x = 2, 1, 1, 1, 4, (1, 5)$.

For $2x^2 - 14x + 17 = 0$, $x = \dfrac{7 \pm \sqrt{15}}{2}$;

$$\frac{\sqrt{15}+7}{2} = 5+, \frac{{}^*\sqrt{15}+3}{3} = 2+, \frac{\sqrt{15}+3}{2} = 3+, \frac{{}^*\sqrt{15}+3}{3} \text{ \&c.}$$

$$\frac{-\sqrt{15}+7}{2} = 1+, \frac{\sqrt{15}+5}{5} = 1+, \frac{\sqrt{15}+0}{3} = 1+, \frac{\sqrt{15}+3}{2} \text{ \&c.}$$

Thus $x = 5, (2, 3)$, and $= 1, 1, 1, (3, 2)$.

Lastly, the equation $1801x^2 - 3991x + 2211 = 0$ gives, representing $\sqrt{37}$ simply by $\sqrt{\ }$,

$$x = \frac{3991+\sqrt{37}}{3602} = 1+, \frac{\sqrt{\ }-389}{-42} = 9+, \frac{\sqrt{\ }+11}{2} = 8+,$$

$$\frac{{}^*\sqrt{\ }+5}{6} = 1+, \frac{\sqrt{\ }+1}{6} = 1+, \frac{\sqrt{\ }+5}{2} = 5+, \frac{{}^*\sqrt{\ }+5}{6} \text{ \&c.}$$

Hence, $x = 1, 9, 8, (1, 1, 5)$: the other root is obtained in the same manner, and we find $x = 1, 9, 2, 2, (5, 1, 1)$. It appears, moreover, from these examples, that *the two fractions have their periods formed of the same terms in inverse order.* [See M. Legendre's *Theorie des Nombres.*]

The continued fraction being once found, it is easy from it to deduce a series of convergents approaching more and more nearly to the root, and possessing the properties peculiar to that species of expressions [N°. 563].

When the equation of the 2nd degree is $x^2 = t$, all the same operations may be put in practice for developing $\sqrt{t}$ in the form of a continued fraction. We shall only observe, in regard to this fraction, that it fulfils the following conditions: 1°. The period commences from the 2nd term; 2°. the last term of this period is $2y$, the double of the initial y which does not make a part of the period; 3°. except as to the last term, the period is *symmetrical, i. e.* it remains the same when taken in the reverse order: thus $\sqrt{t} = y, (y', y'' \dots y'', y', 2y)$.

We find, for instance, for $x^2 = 61$, $\sqrt{61} = 7 + \dots$

$$\frac{\sqrt{}+7}{12} = 1,\ \frac{\sqrt{}+5}{3} = 4,\ \frac{\sqrt{}+7}{4} = 3,\ \frac{\sqrt{}+5}{9} = 1,\ \frac{\sqrt{}+4}{5} = 2,$$

$$\frac{\sqrt{}+6}{5} = 2,\ \frac{\sqrt{}+4}{9} = 1,\ \frac{\sqrt{}+5}{4} = 3,\ \frac{\sqrt{}+7}{3} = 4,\ \frac{\sqrt{}+5}{12} = \&c.$$

whence $x = 7, (1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14)$.

The table I. [p. 130] gives the periods for all the integers $t < 79$; for the most part only the semi-period has been set down, the middle term being marked by ″, when it is to be repeated twice, and by ′, when it is unique; we have also frequently dispensed with putting down the initial term y, or the greatest integer contained in $\sqrt{t}$.

568. A periodic continued fraction being given, let it be proposed to trace it back to the equation of which it is a root.

1st Case. The period commencing from the first term, or $z = (u, u', u'' \dots\dots u^{(l)})$. Find the two final convergents of the period:

$$\frac{h}{h'} = u, u', u'' \dots u^{(l-1)},\ \frac{i}{i'} = u, u', u'' \dots u^{(l)};$$

we have then [F, p. 116]

$$z = \frac{iz + h}{i'z + h'},$$

whence, making $2\omega = i - h'$, $t = \omega^2 + hi'$, there results

$$i'z^2 - 2\omega z = h,\ z = \frac{\omega + \sqrt{t}}{i'}.$$

For example, $z = (1, 1, 2, 1)$ gives $1, 1, 2 = \frac{5}{3}$; $1, 1, 2, 1 = \frac{7}{4}$; whence $h = 5$, $h' = 3$, $i = 7$, $i' = 4$, $\omega = 2$; and lastly, $4z^2 - 4z = 5$.

2nd Case. If the period is preceded by an irregular part, as $x = y, y', y'' \dots (u, u' \dots u^{(l)})$, take the two convergents $\frac{m}{m'}$, $\frac{n}{n'}$, which

terminate this part $y, y', y''\ldots$, and assume $z = (u, u'\ldots u^{(l)})$; then $x = y, y'\ldots y^{(l)}, z$; whence

$$x = \frac{nz + m}{n'z + m'} \text{ and } z = -\frac{m'x - m}{n'x - n};$$

and substituting this value of z in the equation $i'z^2 - 2\omega z = h$, we shall obtain the equation in x, one of the roots of which is equivalent to the continued fraction proposed.

For example, $x = 1, 1, (1, 1, 2, 1)$ gives the convergents $\frac{1}{1}$ and $\frac{2}{1}$; thus $n = 2$, $n' = m = m' = 1$, and $z = -\dfrac{x-1}{x-2}$; which being substituted in $4z^2 - 4z = 5$, there results, for the equation required,

$$3x^2 = 8.$$

The fraction proposed is also equivalent to $x = 1, (1, 1, 1, 2)$, and we may withdraw from the period any number of terms we think proper.

INDETERMINATE EQUATIONS OF THE SECOND DEGREE.

569. Let it be proposed first to resolve, in integral numbers, the equation $my = x^2 \pm a$, *i. e.* to render integral the quantity $\dfrac{x^2 - r}{m}$, r being the negative remainder $< m$ from the division of a by m.

If we assume $x = 1, 2, 3, 4\ldots$, and x^2 be divided by m, the remainders will exhibit a very remarkable property.

If m be *even*, assume $x = \frac{1}{2}m \pm \alpha$; then

$$\frac{x^2}{m} = \frac{\frac{1}{4}m^2 \pm m\alpha + \alpha^2}{m} = \pm\,\alpha + \frac{\frac{1}{4}m^2 + \alpha^2}{m};$$

and the remainders, therefore, of $\dfrac{x^2}{m}$ are the same for both of the numbers $x = \frac{1}{2}m \pm \alpha$: thus, from $x = \frac{1}{2}m$, onwards to $x = m$, we find the same remainders recur in inverse order.

Thus, for the divisor 14, we find the following remainders:

1. 4. 9. 2. 11. 8. 7. 8. 11. 2. 9. 4. 1.

If m is *odd*, the numbers $\frac{1}{2}(m \pm 1)$ are integral; and making $x = \frac{1}{2}(m \mp 1) \mp \alpha$, the remainders from x^2 divided by m are still equal; so that, past $x = \frac{1}{2}(m - 1)$, we shall again find a recurrence of the same remainders in inverse order. The middle term is in this case repeated.

We find, for example, that, for the divisor 17, the successive remainders are

$$1.4.9.16.8.2.15.13.13.15.2.8.16.9.4.1.$$

When $x > m$, viz. $x = tm + a$, since

$$\frac{x^2}{m} = t^2 m + 2at + \frac{a^2}{m},$$

the remainder is the same as though we had taken $x = a < m$: and hence we conclude that

1°. *If we take $x = 1, 2, 3\ldots$ ad. inf., the remainders from the division of x^2 by m will recur and form a symmetrical period of m terms.*

Table II. gives these periods for the more simple divisors.

TABLE I.—PERIODS OF $\sqrt{t}$. [*See p.* 128]

t.	Period.	*t*.	Period.	*t*.	Period.	*t*.	Period.	*t*.	Period.
2	1(2)	19	2.1.3′	34	(1.4.1.10)	50	7(14)	65	8(16)
3	1(1.2)	20	4(2.8)	35	5(1.10)	51	7(7.14)	66	8(8.16)
5	2(4)	21	1.1.2′	37	6(12)	52	4.1.2′	67	5.2.1.1.7′
6	2(2.4)	22	1.2.4′	38	6(6.12)	53	(3.1 3.14)	68	8(4.16)
7	(1.1.1.4)	23	1 3′	39	6(4.12)	54	2.1.6′	69	3.3.1.4′
8	2(1.4)	24	4(1.8)	40	6(3.12)	55	(2.2.2.14)	70	2.1.2′
10	3(6)	26	5(10)	41	6(2.2.12)	56	7(2.14)	71	2.2.1.7′
11	3(3.6).	27	5(5.10)	42	6(2.12)	57	1.1.4′	72	8(2.16)
12	3(2.6)	28	3.2′	43	1.1.3.1.5′	58	1.1.1″	73	1.1.5″
13	(1.1.1.1.6)	29	2.1″	44	1.1.1.2′	59	1.2.7′	74	(1.1.1.1.16)
14	(1.2.1.6)	30	5(2.10)	45	1.2.2′	60	1.2′	75	(1.1.1.16)
15	3(1.6).	31	1.1.3.5′	46	1.3.1.1.2.6′	61	1.4.3.1.2″	76	1.2.1.1.5.4′
17	4(8)	32	1.1′	47	6(1.5.1.12)	62	7(1.6.1.14)	77	1.3.2′
18	4(4.8)	33	1.2′	48	6(1.12)	63	7(1.14)	78	(1.4.1.16)

TABLE II.—PERIODS OF THE REMAINDERS OF $x^2 : m$.

m.	Periods.	*m*.	Per. 1.4.9.16.	*m*.	Per. 1.4.9.16.25.36.
5	(1.4.4.1.0)	17	8.2.15.13″...	37	12.27.7.26.10.33.21.
6	(1.4.3.4.1.0)	19	6.17.11.7.5″...		11.3.34.30.28″...
7	1.4.2″...	21	4.15.7.1.18.16″...	41	8.23.40.18.39.21.5.32.
8	1.4.1.0′...	23	2.13.3.18.12.8.6″...		20.10.2.37.33.31″...
9	1.4.0.7″...	25	0.11.24.14.6.0.	43	6.21.38.14.35.15.40.24.
10	1.4.9.6.5′.		21.19″...		10.41.31.23.17.13.11″...
11	1.4.9.5.3″...	27	25.9.22.10.0.19.	47	2.17.34.6.27.3.28.8.37.
12	1.4.9.4.1.0′...		13.9.7″...		21.7.42.32.24.18.14.12″...
13	1.4.9.3.12.10″...	29	25.7.20.6.23.13.	49	0.15.32.2.23.46.22.0.
14	1.4.9.2.11.8.7′...		5.28.24.22″...		29.11.44.30.18.8.0.43.
					39.37″...
15	1.4.9.1.10.6.4″...	31	25.5.18.2.19.7.	53	49.11.28.47.15.38.
16	1.4.9.0.9.4.1.0′...		28.20.14.10.8″...		10.37.13.44.24.6.43.
					29.17.7.52.46.42.40″...

2°. $\frac{x^2 - r}{m}$ cannot be rendered integral, unless r be one of the terms of this period; if α be the rank of this term, $x = \alpha$ gives r for the remainder from the division of x^2 by m; and we have the indefinite number of solutions comprised in the form $x = tm \pm \alpha$, t being any integer whatever. Each time that r enters into the period, we have a new value of α, and a similar equation giving a system of solutions. It will not be necessary, however, to extend our examination beyond the semi-period, as the coincidence in respect to the remainder r takes place in the ranks α and $m - \alpha$, equally distant from the extremes, and the latter of these values of solution does not lead to any such recurrence.

For example, $13y = x^2 + 40$ gives $\frac{x^2 + 40}{13}$ or $\frac{x^2 - 12}{13}$ = an integer. In the semi-period of the divisor 13, the remainder 12 occurs only in the 5th rank; and thus $x = 13\,t \pm 5$.

The equation $x^2 = 17\,y + 7$ is impossible in integral numbers, 7 not being found in the period of the divisor 17.

Lastly, for $x^2 - 4 = 12y$, since 4 appears in the 2nd and 4th ranks of the semi-period for the divisor 12, we have

$$x = 12\,t \pm 2 \text{ and } \pm 4.$$

Should the divisor m be a product $p\,p'$, it is evident that $x^2 - r$ is not divisible by m unless it be so also by p and by p', and $\frac{x^2 - r}{p}$, $\frac{x^2 - r}{p'}$ must therefore be rendered integral by values such as $x = t\,p \pm \alpha$, $x = t'\,p' \pm \alpha'$. And it will then remain to adjust these solutions in respect to each other, for the values of t and t' must be so selected as to give the same number for x. Thus, we shall assume [N°. 121]

$$\frac{x^2 - r}{p}, \frac{x^2 - r}{p'}, \text{ and } \frac{x \pm \alpha}{p}, \frac{x \pm \alpha'}{p'} = \text{integers.}$$

If p be also decomposable into two factors, the 1st fraction may be replaced by two others, and so on.

For example, to obtain integral solutions of the equation $315y = x^2 - 46$, since $315 = 9.\ 7.5$, $x^2 - 6$ must be rendered divisible by 9, 7 and 5; *i. e.* extracting the integers, we must have

$$\frac{x^2 - 1}{9}, \frac{x^2 - 4}{7}, \frac{x^2 - 1}{5} \text{ each } = \text{an integer.}$$

The periods of these divisors give $\alpha = 1$, $\alpha' = 2$, $\alpha'' = 1$; thus (without any mutual dependence between the sign $\pm$), we must render

$$\frac{x \pm 1}{9}, \frac{x \pm 2}{7}, \frac{x \pm 1}{5} = \text{integers;}$$

and we shall finally find that if k denote any one of the four numbers 19, 89, 26 and 44, we have $x = 315\,t \pm k$, whence

$$\pm\, x = 19, 26, 44, 89, 226, 271\ldots,\ y = 1, 2, 6, 25, 162, 233\ldots$$

For the equation $my = ax^2 + 2\,bx + c$, multiplying by a, and assuming $ax + b = z$, $b^2 - ac = D$, we have

$$ay = \frac{(ax+b)^2 - (b^2 - ac)}{m} = \frac{z^2 - D}{m};$$

and to obtain integral solutions of the equation, we must first investigate the solutions $z = mt \pm \alpha$, which will render this fraction $\dfrac{z^2 - D}{m}$ integral; and then solve the equation of the 1st degree $ax + b = mt \pm \alpha$, *i. e.* take only those integral values of t, which will render x also integral. If a and m are prime to each other, $z^2 - D$ will be a multiple of a and of m (since we have multiplied by a), and dividing the result by a, we shall have y. When a and m have a common factor θ, it must be a factor also of $2\,bx + c$; we must therefore investigate the general form of the values of x, which fulfil this condition $x = \theta\,x' + \gamma$, and substituting in the equation proposed, θ will disappear.

Let $7y = 3x^2 - 5\,x + 2$: this being first multiplied by 2, in order that the coefficient of x may be even, we have $a = 6, b = -5, c = 4$, $D = 1$. Also $z = 6\,x - 5$; $z^2 - 1$ is rendered a multiple of 7 by assuming $z = 7\,u \pm 1$; and we deduce

$$x = 7\,t + 1 \text{ and } + 3.$$

The equation $11\,y = 3\,x^2 - 5\,x + 6$ is absurd, if the solutions are to be integral.

For $15\,y = 6\,x^2 - 2\,x + 1$, we first render $2x - 1$ a multiple of the factor 3, common to 15 and 6, by assuming $x = 3\,x' + 2$, whence $5y = 18\,x'^2 + 22\,x' + 7$; extracting the integers, it remains to make $3x'^2 + 2\,x' + 2$ a multiple of 5; we find $z = 5\,t = 3\,x' + 1$; and consequently $x' = 3, x = 11$, and generally, $x = 15\,t' + 11$.

570. Let the equation be

$$az^2 + 2bzy + cy^2 = M,$$

of which integral solutions are required.

1st Case, $b^2 - ac = 0$: multiplying the 1st side by a, it becomes an exact square, $(az + by)^2 = a\,M$; thus $a\,M$ must also be a square h^2, otherwise the problem would be absurd; and it will remain, therefore, to obtain integral solutions of the equation $az + by = h$. We must take z and y with the sign $\pm$, since h ought to be so affected.

From $4z^2 - 20zy + 25y^2 = 49$, we have $2z - 5y = \pm 7$; whence $y = 2t \mp 1$, $z = 5t \pm 1$.

2nd Case, $b^2 - ac < 0$: the proposed equation is now equivalent to

$$(az + by)^2 + Dy^2 = aM, \text{ or } u^2 + Dy^2 = aM,$$

making $b^2 - ac = -D$, and $az + by = u$. Thus, M must be positive. We shall assume $y = 0, 1, 2\ldots$, and retain those values only which render $aM - Dy^2$ a square. These trials will be limited in number, Dy^2 being $< aM$; and having thus determined y and u, we shall take those only of the results which will render z integral.

For $3z^2 - 2zy + 7y^2 = 27$, we find

$$(3z - y)^2 + 20y^2 = 81, \text{ or } u^2 = 81 - 20y^2, \text{ where } 3z - y = u;$$

whence

$$\pm y = 0 \text{ and } 2, \pm u = 9 \text{ and } 1,$$
$$\pm z = 3 \text{ and } 1.$$

3rd Case. If $b^2 - ac$ be a positive square k^2, still multiplying by a, and equating the 1st side to zero, in order to obtain its factors, we find

$$[az + y(b + k)] \cdot [az + y(b - k)] = aM.$$

Let f and g be two factors producing aM; if we assume them equal to those of the 1st side, there will result

$$y = \frac{f - g}{2k}, \quad z = \frac{f - y(b + k)}{a}.$$

Thus, having decomposed aM into two factors in every way possible, we must take them successively, one for f, the other for g, and retain those systems only by which first y, and then z, are rendered integral. We shall take y and z with the sign $\pm$, since we might give to f and g the sign $+$ or $-$. For instance, $2z^2 - 9yz + 7y^2 = 38$, being first doubled in order to render the coefficient of yz even, gives $a = 4$, $b = 9$, $c = 14$, $k = 5$, $aM = 304$; the several pairs of factors of 304 are $2 \times 152 = 8 \times 38 = 4 \times 76 = 1 \times 304 = 16 \times 19$; the two first systems alone answer the conditions and give

$$\pm y = 15 \text{ and } 3, \mp z = 53 \text{ and } 1.$$

4th Case. $b^2 - ac$ being positive and not a square: in order to compare this case, the last that remains for discussion, with what has preceded, we shall write the proposed equation under the form $Az^2 - 2\alpha zy - ky^2 = P$. The roots of the equation $Ax^2 - 2\alpha x = k$ are irrational, or $t = \alpha^2 + Ak$ is positive, and not a square. Let these roots be developed in the form of continued fractions; it follows from the equation (f)

[N°. 567], that the convergent which precedes the complete fraction $\frac{\sqrt{t}+\pi}{P}$ is $\frac{n}{n'}$, with this condition

$$An^2 - 2\alpha\, nn' - kn'^2 = \pm P,$$

the sign of P being dependent on the even or odd rank of the convergent. And this equation being now compared with the one proposed, it will be seen that, if the sign of the second sides is the same, we have this solution

$$z = n,\ y = n'.$$

Hence, to find y and z, develop the roots x in continued fractions; if then, among the convergents $\frac{\sqrt{t}+\alpha}{A}, \frac{\sqrt{t}+\beta}{B}\ldots$, there be found one, the denominator of which is the second side P of the proposed equation, we must limit the continued fraction to the integer given by the preceding complete fraction, investigate the corresponding convergent $\frac{n}{n'}$, and we shall have $z = n$, $y = n'$; only this convergent must be of an even rank when the 2nd side P is positive, and of an odd one when P is negative, if the development be that of the greatest root; and the contrary for the least root. Each complete fraction that occurs in an available rank gives a solution; so that if it form part of the period, we have an infinity of values for z and y.

For example, let $2z^2 - 14yz + 17y^2 = 5$; it has been found [p. 127] that $2x^2 - 14x + 17 = 0$ has for its least root $x = 1, 1, 1, (3, 2)$, and that the 2nd complete fraction has 5 for the denominator; thus the convergent $\frac{1}{1}$, which is of an odd rank, gives this unique solution $z = 1$, $y = 1$; the period has nothing to do with the question.

Had the 2nd side been $+3$, instead of 5, there would have been no integral solution, because the complete fractions which have 3 for their denominator, being all in even ranks for the greatest of the roots x, and in odd ones for the least, are not available.

But if the 2nd side be -3, developing the greatest root $x = 5\ (2, 3)$, we shall put a stop to it at the ranks 1, 3, 5, 7... successively, because the following complete fractions have the denominator 3; and hence we get the convergents $\frac{5}{1}, \frac{38}{7}, \frac{299}{55}\ldots$, which give as many solutions. The least root $x = 1, 1, 1\ (3, 2)$, terminated at the 2nd, 4th, 6th... terms, in like manner gives $\frac{2}{1}, \frac{11}{7}, \frac{86}{55}\ldots$; and consequently with $\pm y = 1, 7, 55\ldots$, we shall take $\pm z = 5, 38, 299\ldots$ or $2, 11, 86\ldots$.

Lastly, when the second side is 2, we similarly find $\pm y = 0, 2, 16, 126, 992\ldots$ with $\pm z = 1, 11, 87, 685, 5393\ldots$, or $1, 3, 25, 197, 1551\ldots$

Since the convergents are always irreducible, this method will give us

no solutions but what are prime to each other. Suppose that the proposed equation admit others which have a common factor θ, $z = \theta z'$, $y = \theta y'$; we shall have then

$$\theta^2 (a z'^2 + 2b z' y' + c y'^2) = P;$$

so that P is a multiple of θ; let P' be the quotient, and it will remain to deduce z' and y' from an equation similar to the one proposed, the second side being P'. Thus, taking all the square factors, θ^2 others than 1, of P, we shall have as many values of θ and as many equations to be solved, the only difference between them being in the 2nd term, which will be $P' = P : \theta^2$.

Let the equation, for instance, be $z^2 + 2zy - 5y^2 = 9$; it does not admit of any solutions prime to each other; but since $9 = 3^2$, we must solve $z'^2 + 2z'y' - 5y'^2 = 1$. For this purpose, the equation $x^2 + 2x = 5$ gives

$$x = \frac{\sqrt{6}-1}{1} = 1, \frac{{}^*\sqrt{6}+2}{2} = 2, \frac{\sqrt{6}+2}{1} = 4, \frac{{}^*\sqrt{6}+2}{2}, \text{\&c.};$$

whence we have $x = 1$, $(2, 4)$, and the convergents $\frac{1}{0}, \frac{3}{2}, \frac{29}{20}, \frac{287}{198}$.... The terms of these fractions are the values of z' and y'; and multiplying above and below by 3, we finally obtain

$$\pm z = 3, 9, 87, 861 \ldots \quad \pm y = 0, 6, 60, 594 \ldots$$

The second root of x does not lead to any new solution.

The denominators of the complete fractions being all $< 2\sqrt{t}$ [p. 125], when the second side P exceeds this limit, we cannot expect to meet with it among these denominators, and our method can no longer determine the solutions. But, f being a factor of P, or $P = fP'$, and n some integer, assume $y = nz + fy'$; whence

$$\left(\frac{a + 2bn + cn^2}{f}\right) z^2 + 2y'z(b + cn) + cfy'^2 = P'.$$

Let n be now so determined [p 132] that the first coefficient shall be integral; as many times then as $b^2 - ac$ enters into the semi-period of f, so many values shall we have of $\pm n$, and so many equations to be solved, of the form $Az^2 + 2By'z + Cy'^2 = P'$, in which C and P' will be the same, as also $B^2 - AC$. Thus, P may be reduced to a value $P' < 2\sqrt{t}$, and in fact to $P' = \pm 1$.

For instance, the equation $66z^2 - 18yz + y^2 = 34$, taking $f = 17$, requires us to render $\dfrac{66 - 18n + n^2}{17}$ = an integer; whence $n = 2$

and 16, and consequently

$$y = 17y' + 2z \text{ or } +16z,\ 2z^2 \pm 14y'z + 17y'^2 = 2.$$

One of these transformed equations has been already solved [p. 134], whilst the other differs only in respect to the sign of y'; and we find, therefore,

$$\pm z = 1, 11, 87 \ldots 3, 25 \ldots \text{ with } \pm y = 2, 56, 446 \ldots 40, 322 \ldots$$
$$\text{or with } \pm y = 16, 142, 1120 \ldots 14, 128 \ldots$$

We shall suppress the demonstration, by which it might be proved that this process will serve to determine all the integral solutions.

571. These calculations may be applied with great facility to the equation $z^2 - ty^2 = \pm 1$; we must develop $\sqrt{t}$ in a continued fraction $x = \sqrt{t} = u\,(u', u'' \ldots u'' u', 2u)$, and stop it at the several complete fractions of which the denominator is 1, the rank being odd for $+1$, and even for -1. But it is easily proved that the only complete fractions which have 1 for their denominator (except the 1st $\sqrt{t}$), are those that give the last integer $2u$ of the period, and which have the form $\frac{\sqrt{t}+u}{1}$. Thus the convergents $\frac{n}{n'}$, corresponding to the several recurrences of the term u' which precedes $2u$, if they are in available ranks, give $\pm z = n$, $\pm y = n'$, the signs being independent of each other. When the period consists of an even number of terms, each period gives a solution in the case of $+1$, whilst there is not one in the case of -1. When the period has an odd number of terms, the recurrences in the 1st, 3rd, 5th... periods answer to the conditions of the question, if the second side be -1; but if it be $+1$, we must take the 2nd, 4th... periods.

For the equations $z^2 - 14y^2 = \pm 1$, we have [p. 130] $\sqrt{14} = 3, (1, 2, 1, 6)$; the term 1, which precedes 6, occurs only in the even ranks; and thus the question is absurd for integral solutions, in the case of -1. In that of $+1$, we take the convergents $\frac{1}{0}, \frac{15}{4}, \frac{449}{120}, \frac{13455}{3596} \ldots$, and we have, the signs being quite arbitrary,

$$\pm z = 1, 15, 449 \ldots,\ \pm y = 0, 4, 120 \ldots.$$

Let $z^2 - 13u^2 = \pm 1$: since $\sqrt{13} = 3, (1, 1, 1, 1, 6)$, the convergents corresponding to the recurrence of the term 1, which precedes 6, are $\frac{1}{0}, \frac{18}{5}, \frac{649}{180}, \frac{23382}{6485} \ldots$; whence $z = 1, 649 \ldots$, $y = 0, 180 \ldots$ for $+1$; and $= 18, 23382 \ldots$, $y = 5, 6485 \ldots$ for -1.

Let $z^2 - 3y^2 = 1$: since $\sqrt{3} = 1, (1, 2)$, all the convergents $\frac{1}{0}, \frac{2}{1}, \frac{7}{4}$,

$\frac{26}{15}, \frac{97}{56}, \frac{362}{209}\ldots$ give solutions; there would not be one, were the second side -1.

The equation $z^2 - 5y^2 = \pm 1$ has its solutions in the alternate fractions $\frac{2}{1}, \frac{9}{4}, \frac{38}{17}, \frac{161}{72}\ldots$.

572. The equation

$$az^2 + 2bzy + cy^2 + dy + ez + f = 0,$$

the most general one of the 2nd degree, comes under the form of the preceding one, when it is divested of the terms of the 1st dimension. To effect this, we must make

$$z = kz' + \alpha,\ y = ly' + \beta,$$

α and β being such that

$$2a\alpha + 2b\beta + d = 0,\ 2c\beta + 2b\alpha + e = 0 \ldots (1),$$

or, assuming $b^2 - ac = D$,

$$\alpha = \frac{cd - be}{2D},\ \beta = \frac{ae - bd}{2D}.$$

All our coefficients are supposed to be integral. But, it is obvious that this transformation can be of no use unless z' and y' be integral at the same time with z and y. To ensure this, let the indeterminate quantities k and l be each made $= \frac{1}{2D}$, viz.

$$z = \frac{z' + cd - be}{2D},\ y = \frac{y' + ae - bd}{2D} \ldots (2).$$

Then the values that result for y' and z' will answer to integral values of y and z; but the converse does not necessarily hold, and we must reject those of the integral solutions of z' and y', which do not render z and y also integral. Thus we shall arrive at all the values required, retaining for z' and y' those only of the solutions obtained, which are of the suitable form [No. 565]. Now multiply the equations (1) by α and β respectively, and add; when we shall have

$$-(a\alpha^2 + 2b\alpha\beta + c\beta^2) = \frac{d\alpha + e\beta}{2} = \frac{ae^2 - 2bed + cd^2}{4D};$$

and denoting the numerator of this by N, the transformed equation will be

$$az'^2 + 2bz'y' + cy'^2 + 4D^2f + ND = 0,.. (3),$$

which we have already seen how to resolve.

When $b^2 - ac = 0$, this calculation cannot be carried into effect; but multiplying by a, the three first terms form the square of $az + by$; and assuming this binomial $=z'$, the rest of the calculation is easy.

Let the equation be

$$7z^2 - 2zy + 3y^2 - 30z + 10y + 8 = 0;$$

the equations (2) become

$$z = \frac{z' - 80}{-40},\ y = \frac{y' + 40}{-40};$$

so that y and z are not integral unless 40, the common factor of the constants, be so also of z' and y': change these letters into $40z'$ and $40y'$; when this factor 40 will disappear, and we shall have

$$z = z' + 2,\ y = y' - 1,\ 7z'^2 - 2z'y' + 3y'^2 = 27.$$

This equation has been discussed [p. 133], and has given

$\pm z' = 0$ and 2, $\pm y' = 3$ and 1; and we consequently have

$$z = 4, 0, 2 \text{ and } 2, \text{ with } y = 0, -2, 2 \text{ and } -4.$$

RESOLUTION OF NUMERICAL EQUATIONS.

573. Let $X = 0$ be an equation which has been so prepared that none of its roots are commensurable, or equal, or more than one of them comprised between two successive integers [Nos. 518, 524, 527]; suppose also that for each irrational root we know the integer y which is immediately less [No. 528], and let us proceed to the ulterior approximation.

According to the rule given [No. 562], make $x = y + \frac{1}{x'}$; when $X = 0$ will become $X' = 0$, the unknown quantity of the equation being x'. By supposition then, there is one, and only one, of the values of x' which is > 1; and this root corresponds to the value of x, of which y is the integral part, and to which we wish to approximate. Following the same course in regard to $X' = 0$, let y' be the integer nearest to x'; we know that there is but one value of x' which is positive and > 1; and

assuming $x' = y' + \frac{1}{x''}$, we shall get a transformed equation $X'' = 0$, having x'' for its unknown quantity, and in which x'' has one value > 1, and only one. And thus we see that, carrying on the process, the root x will be developed in a continued fraction $x = y, y', y'', y'''...$; whence we shall deduce convergents approaching more and more nearly to the true value by excess and by defect, alternately; and for each of which, the error that results has a known limit. As to the construction of the equations $X', X''...$, it presents no difficulty; for let $X = kx^i + px^{i-1} + qx^{i-2}... + u = 0$; if we assume $x = y + t$, the transformed equation [N°. 503] is

$$X + X't + \tfrac{1}{2}X''t^2... + kt^i = 0;$$

but, in the present case, $t = \frac{1}{x'}$; and consequently, multiplying the whole by x'^i, the result is

$$Xx'^i + X'x'^{i-1} + \tfrac{1}{2}X''x'^{i-2} + ... + k = 0.$$

The coefficients $X, X', X''...$ are the values of X and its derivatives, when we make $x = y$; and these we must calculate [see note, p. 37], and substitute in the above equation.

Let the equation proposed be $x^3 - 2x - 5 = 0$, which has but one real root, and that comprised between 2 and 3 [N°. 529]. Applying our method, and making $x = 2$ in $x^3 - 2x - 5$, $3x^2 - 2$, $3x$ and 1, we find -1, 10, 6 and 1 for the coefficients of the equation in x'. The value of x' is between 10 and 11, whence we similarly find for the coefficients of the equation in x'', 61, -94, &c....; and we thus obtain these successive results, in which we have dispensed with writing the different powers of x, as being sufficiently indicated by the ranks of the terms:

(0)...... $x^3 + 0x^2 - 2x - 5 = 0$, integer 2,
(1)... $-1 + 10 + 6 + 1 = 0$, ... 10,
(2)...... $61 - 94 - 20 - 1 = 0$, ... 1,
(3)... $-54 - 25 + 89 + 61 = 0$, ... 1,
(4)...... $71 - 123 - 187 - 54 = 0$, ... 2,
(5)... $-352 + 173 + 303 + 71 = 0$, ... 1,
(6)...... $195 - 407 - 883 - 352 = 0$, ... 3,
&c.

And hence $x = 2, 10, 1, 1, 2, 1, 3... = \frac{576}{275} = 2{\cdot}09455...$, a value which is exact to five places of decimals, since its error does not amount to $(\frac{1}{275})^2$.

The equation $x^3 - x^2 - 2x + 1 = 0$ has its three roots real, and they are comprised between 1 and 2, 0 and 1, -1 and -2. Approximating to the first, we find

$$\begin{array}{lrrrrl}
(0)\ldots\ldots & x^3 - & x^2 - & 2x + & 1 = 0, & \text{integer } 1, \\
(1)\ldots - & 1 - & 1 + & 2 + & 1 = 0, & \ldots\ldots\ldots 1, \\
(2)\ldots\ldots & 1 - & 3 - & 4 - & 1 = 0, & \ldots\ldots\ldots 4, \\
(3)\ldots - & 1 + & 20 + & 9 + & 1 = 0, & \ldots\ldots\ldots 20, \\
(4)\ldots\ldots & 181 - & 391 - & 40 - & 1 = 0, & \ldots\ldots\ldots 2, \\
(5)\ldots - & 197 + & 568 + & 695 + & 181 = 0, & \ldots\ldots\ldots 3, \\
(6)\ldots\ldots & 2059 - & 1216 - & 1205 - & 197 = 0, & \ldots\ldots\ldots 1, \\
\&c. & & & & &
\end{array}$$

and consequently

$$x = 1, 1, 4, 20, 2, 3, 1, 6, 10, 5, 2 = \tfrac{1289054}{715371} = 1{\cdot}8019377358.$$

The root comprised between 0 and 1 is found in the same manner; and since after the 2nd operation we meet again with the transformed equation (2), we must have a recurrence of the following equations (3), (4), (5)...; whence

$$x = 0, 2, 4, 20, 2, 3, 1, 6, 10, 5, 2 = \tfrac{573683}{1289054} = 0{\cdot}4450418679.$$

Lastly, for the negative root, x must be changed into $-x$; and since this gives us the equation (1), we may forthwith assume

$$-x = 1, 4, 20, 2, 3, 1, 6, 10, 5, 2 = \tfrac{715371}{573683} = 1{\cdot}2469796037.$$

We meet with a peculiarity in the example proposed, that of the three roots being composed of the same terms, owing to which it becomes unnecessary to proceed with the calculation for the two last.

5;4. We shall now show how these different calculations may be abridged.

The continued fraction having been carried down to the integer y^i, $x = y, y', y'' \ldots y^i$, let $\frac{m}{m'}$ and $\frac{n}{n'}$ be the two last convergents; it follows then from the equation F [No. 562], and as we have also seen [p. 129], that if z represent the value of the remaining part of the continued fraction, we have

$$z = -\frac{m'x - m}{n'x - n}$$

whence, commencing the division, and observing that $m'n - mn' = \pm 1$,

$$z = -\frac{m'}{n'} \mp \frac{1}{n'(n'x - n)}.$$

Let δ be the difference between x and the convergent $\frac{n}{n'}$, or $\delta = \frac{n}{n'} - x$; we have then $n'x - n = -n'\delta$, and

$$z = -\frac{m'}{n'} \pm \frac{1}{\delta . n'^2}.$$

It is true that, in this expression, x denotes the root to which we are attempting to approximate, and the value of z therefore is dependent upon it; but each of the other roots x', x''... gives a similar equation; so that, z', z''... being the corresponding values of z, we have

$$z' = -\frac{m'}{n'} \pm \frac{1}{\delta'.n'^2},\ z'' = -\frac{m'}{n'} \pm \frac{1}{\delta''.n'^2},\ \&c.;$$

and taking the sum of these $(i-1)$ equations, and making $\frac{1}{\delta'}\Delta = +\frac{1}{\delta''} + \ldots$, there results

$$z' + z'' + z'''\ldots = -\frac{m'}{n'}(i-1) \pm \frac{\Delta}{n'^2}.$$

On the other hand, the transformed equation in z being represented by $Az^i + Bz^{i-1} + \ldots = 0$, the sum of the roots is $z + z' + z'' = -\frac{B}{A}$; and consequently, subtracting the preceding equation, we find

$$z = \frac{m'}{n'}(i-1) - \frac{B}{A} \mp \frac{\Delta}{n'^2}.$$

Now $\frac{n}{n'}$ can speedily be rendered so close an approximation to x that δ shall be very small; in which case δ', δ''..., which are the differences between our convergents and the other roots x', x''..., will be very nearly equal to the differences between these roots and x itself; the greater, therefore, these differences be, the less will Δ be, whilst at the same time n' is continually increasing; thus, the last term of our equation may at length be neglected; and we shall then have

$$z = \frac{m'}{n'}(i-1) - \frac{B}{A}.$$

Not only will this equation give the integer ϖ, contained in z; but having resolved it into a continued fraction, by the method of the common divisor, we may also take several successive terms, as composing

the value of z, and continuing that of x; viz. $z = \pi, \varrho, \sigma \ldots$ and $x = y, y', y'' \ldots y^i, \pi, \varrho, \sigma \ldots$ This fraction z being discontinued at one of its terms u, let $\frac{p}{p'}, \frac{q}{q'}$, be the two last convergents; then [equation F, p. 116]

$$z = \frac{qu + p}{q'u + p'};$$

and if this value be now substituted in the transformed equation in z, we shall pass at once to that corresponding to the term u, and shall in fact be supposing the value of x to be referred to this term.

Since $z = -\frac{m'x - m}{n'x - n}$, we have only to take two approximate limits between which x is comprised, and to substitute them in the preceding fraction, to obtain the limits of z; and these being then resolved into continued fractions, the terms common to both will be so also to z, and will consequently continue x.

For the first root of the last example, commencing from the transformed equation (4), the convergents are $\frac{9}{5} = 1, 1, 4$; $\frac{182}{101} = 1, 1, 4, 20$; whence we deduce $z =$ [illegible] $+$ [illegible] $=$ [illegible] $= 2, 3, 1, 6 \ldots$; and these four terms continue the value of x, which thus acquires, on the whole, eight terms. We now find the convergents $\frac{9}{4}$ and $\frac{61}{27}$; whence $z = \frac{61u + 9}{27u + 4}$, and substituting in (4), we arrive at the transformed equation (8); and so on.

When the root x is commensurable, the continued fraction comes to a termination; otherwise it extends to infinity. If the proposed equation admit of a rational factor of the 2nd degree, we obtain a period, and the recurrence of the same terms intimates this circumstance. Thus the equation $x^4 - 2x^3 - 9x^2 + 22x - 22 = 0$, when we are in search of the root which lies between 3 and 4, gives

(1)... $-10 \quad +22 \quad +27 \quad +10 \quad +1 = 0$, integer 3,
(2)...... $58 \quad -314 \quad -315 \quad -98 \quad -10 = 0$, ... 6,
(3)... $-4594 + 12322 + 6561 + 1078 + 58 = 0$, ... 3,
&c.

The last of these equations gives us $z =$ [illegible] $+$ [illegible] $=$ [illegible] $= 3, 6 \ldots$; and from this recurrence of the figures (3, 6) we are led to expect a period. Supposing it to exist, it appears that $x^2 - 11$ should be a divisor of the equation proposed [N°. 568]; and making trial of this division, it gives the exact quotient $x^2 - 2x + 2$.

The solution of the equation $x^i = A$, or the extraction of roots, comes within the scope of this method. Thus $x^3 = 17$ gives

$$x = 2,\ 1,\ 1,\ 3,\ 138 = \tfrac{2489}{968};$$

and forming the value of z, we find $z = 1,\ 3,\ 2 \ldots$; whence

$$x = \tfrac{22527}{8761} = 2{\cdot}5712818.$$

575. The equation $10^x = 29$ may be treated in the same manner. We at once see that x is between 1 and 2; viz.

$$x = 1 + \frac{1}{x'},\ 10^{1+\frac{1}{x'}} = 29;\ 10 \times 10^{\frac{1}{x'}} = 29;\ 10 = (2{\cdot}9)^{x'}.$$

We then find that x' is between 2 and 3;

$$x' = 2 + \frac{1}{x''},\ 10 = (2{\cdot}9)^2 \times (2{\cdot}9)^{\frac{1}{x''}},\ (\tfrac{1000}{841})^{x''} = 2{\cdot}9;$$

and so on. And hence

$$x = 1,\ 2,\ 6,\ 6,\ 1,\ 2,\ 1,\ 2 \ldots = \tfrac{1439}{984} = 1{\cdot}4623980.$$

This value, $> x$, approximates to it within less than $(\frac{1}{984})^2$, and thus has six exact decimal figures.

$$10^x = 23 \text{ gives } x = 1,\ 2,\ 1,\ 3,\ 4,\ 17,\ 2 = \tfrac{2270}{1667} = 1{\cdot}3617278.$$

Thus, we can obtain the approximate solution of the equation $10^x = b$; and since, instead of 10, any other base may be taken, *we can calculate the logarithm of a number in any system.*

VI. METHOD OF INDETERMINATE COEFFICIENTS.

DECOMPOSITION OF RATIONAL FRACTIONS.

576. F and ϕ being *identical* functions of x, *i. e.* functions the dissimilarity of which consists simply in the mode in which they are algebraically expressed, the equation $F = \phi$ will not require for its verification that any particular values should be assigned to x, but must subsist, whatever be the number that we think fit to substitute for x. Suppose that, by means of some analytical artifice, we have succeeded in arranging F and ϕ in respect to x, under the same form,

$$a + bx + cx^2 + dx^3 \ldots = A + Bx + Cx^2 + Dx^3 \ldots;$$

since then between F and ϕ there was only an apparent difference owing to the forms under which these functions were expressed, and

this dissimilarity of forms no longer exists, we must find every thing on one side precisely the same with what appears on the other; consequently,

$$a = A,\ b = B,\ c = C\ldots$$

And, in effect, since the equation must subsist for every value of x, if we assume $x = 0$, we have at once $a = A$. These two constants however have not been rendered equal by this supposition; they were so independently of it, and the hypothesis has been no more than a medium for putting this fact in evidence. And now, whatever x be, we have still $bx + cx^2\ldots = Bx + Cx^2\ldots$; whence, dividing by x,

$$b + cx + \ldots = B + Cx + \ldots;$$

and the same reasoning proves that $b = B$, then $c = C\ldots$

Thus, a function F being given, if we have direct assurance that it is susceptible of being expréssed under a specified form φ, containing constant coefficients $A, B, C\ldots$, it will be easy to find these numbers: 1°. We must express the identity $F = \varphi$, F being the function proposed, and φ its value put under some other known and suitable form, and containing the *indeterminate coefficients* $A, B, C\ldots$; 2°. by means of the appropriate calculations, we must *arrange* the two sides F and φ according to the powers of x; 3°. *we must equate to each other the terms affected with the same exponents of* x; 4°. and lastly, we must *eliminate* between these equations in order to deduce the values of the unknown constants $A, B, C\ldots$

We shall now apply these principles to different examples.

577. N being the numerator of a rational fraction, and D the denominator, let it be proposed to decompose it into others of which it shall be the sum.

By division, the degree of the polynomial N, in respect to x, may always be reduced below that of D; and in this state therefore we shall take the fraction. Let $D = P \times Q$, P and Q being polynomials prime to each other, of the degrees p and q, and assume

$$\frac{N}{D} = \frac{Ax^{q-1} + Bx^{q-2}\ldots + L}{Q} + \frac{A'x^{p-1} + B'x^{p-2}\ldots + L'}{P}.$$

To reduce to the same denominator $D = P \times Q$, multiply $Ax^{q-1} + \ldots$ by P and $A'x^{p-1} + \ldots$ by Q; these products will be of the degree $p + q - 1$, *i. e.* they will form a *cómplete polynomial* of a degree one lower than that of D; and since N is at the highest of this same degree, by comparing the several terms of N with those of the products above, we shall derive $p + q$ equations between the unknown coefficients $A, A', B, B'\ldots$, the number of which is obviously $p + q$, since our

numerators have q and p terms respectively; these unknown quantities will be but of the 1st degree, and our previous rules will readily lead to the determination of them. Thus it is proved that the decomposition proposed is legitimate, and the calculation will give the actual values of all the component parts.

And if P and Q be themselves decomposable into other factors prime to each other, without proceeding first to determine $A, A', B, B'\ldots$, we shall replace each fraction by others formed according to the same principle; *i. e. to decompose the rational fraction proposed, we must find the prime factors of its denominator, and equate the fraction to a series of others having these factors for their denominators, the numerators being polynomials of a degree one lower than that of the respective denominators.*

We must therefore equate D to zero, in order to resolve it into its prime factors; when two cases will present themselves, accordingly as D has all its factors unequal, or some of them be equal. We shall examine these two cases separately.

1st Case. If $D = (x-a)(x-b)(x-c)\ldots$, we shall assume

$$\frac{N}{D} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \ldots,$$

and we must determine $A, B, C\ldots$ by the process just explained.

For example, let $D = (x-a)(x-b)$; we have

$$\frac{kx+l}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b};$$

whence

$$\begin{aligned} kx + l &= A(x-b) + B(x-a) \\ &= (A+B)x - Ab - Ba; \end{aligned}$$

thus

$$k = A + B, \; -l = Ab + Ba;$$

and, lastly,

$$A = -\frac{ka+l}{b-a}, \; B = \frac{kb+l}{b-a}.$$

For $\dfrac{2-4x}{x^2-x-2}$, we shall equate the denominator to zero in order to obtain the binomial factors; when $x^2 - x = 2$ gives $x = 2$ and -1; and these are the values corresponding to b and a. From the numerator we have $k = -4$, $l = 2$; and thus

$$\frac{2-4x}{x^2-x-2} = -\frac{2}{x+1} - \frac{2}{x-2}.$$

In like manner

$$\frac{1}{a^2 - x^2} = \frac{1}{2a(a+x)} + \frac{1}{2a(a-x)}.$$

Again, assume

$$\frac{1}{x(a^2 - x^2)} = \frac{A}{x} + \frac{B}{a+x} + \frac{C}{a-x};$$

we find

$$1 = Aa^2 + ax(B + C) + x^2(C - A - B);$$

whence

$$1 = Aa^2,\ B + C = 0,\ C - A - B = 0.$$

Eliminating, we have A, B, C, and

$$\frac{1}{x(a^2 - x^2)} = \frac{1}{a^2x} - \frac{1}{2a^2(a+x)} + \frac{1}{2a^2(a-x)}.$$

The same method will still apply, when D contains imaginary binomial factors; but we generally prefer decomposing D into real trinomials, such as $x^2 + px + q$, and the proposed expression into fractions of the form $\frac{Ax + B}{x^2 + px + q}$. Thus, for

$$\frac{x^2 - x + 1}{(x+1)(x^2+1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x+1},$$

we find

$$C = \tfrac{3}{2},\ B = A = -\tfrac{1}{2}.$$

In like manner

$$\frac{x}{x^3 - 1} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 1}$$

gives

$$-A = B = C = \tfrac{1}{3}.$$

2nd Case. Each factor of D, of the form $(x - a)^i$, gives rise to a component such as $\frac{Ax^{i-1} + Bx^{i-2} \ldots}{(x-a)^i}$; but this being itself decomposable, we shall forthwith assume, instead of it, the equivalent sum

$$\frac{A}{(x-a)^i} + \frac{B}{(x-a)^{i-1}} + \frac{C}{(x-a)^{i-2}} \ldots + \frac{L}{x-a}.$$

And it is in fact evident that, reducing to the same denominator, the numerator will have the same form as before, and an equal number of unknown constants.

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2}=\frac{A}{x}+\frac{B}{(x+1)^2}+\frac{C}{x+1}+\frac{D}{(x-1)^2}+\frac{E}{\cancel{x}-1} \text{ gives}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots = \frac{2}{x}-\frac{\frac{1}{2}}{(x+1)^2}+\frac{\frac{5}{4}}{x+1}+\frac{1}{(x-1)^2}-\frac{\frac{3}{4}}{x-1}.$$

And in like manner we shall find

$$\frac{1}{x(x+1)^2(x^2+x+1)}=\frac{1}{x}-\frac{1}{(x+1)^2}-\frac{2}{x+1}+\frac{x}{x^2+x+1}.$$

Should the equal factors of the denominator be imaginary, though the same process would still be applicable, it will be preferable to combine them in real factors of the 2nd degree, under the form $(x^2+px+q)^i$; the numerator will then be $Ax^{2i-1}+Bx^{2i-2}+\ldots$; but we shall rather take the component fractions

$$\frac{Ax+B}{(x^2+px+q)^i}+\frac{Cx+D}{(x^2+px+q)^{i-1}}+\ldots+\frac{Kx+L}{x^2+px+q}.$$

For example, we shall assume

$$\frac{1}{(x+1)\,x^2(x^2+2)\,(x^2+1)^2}$$

$$=\frac{A}{x+1}+\frac{B}{x^2}+\frac{C}{x}+\frac{Dx+E}{x^2+2}+\frac{Fx+G}{(x^2+1)^2}+\frac{Hx+I}{x^2+1};$$

and the calculation will give

$A=\frac{1}{12}$, $B=-C=\frac{1}{2}$, $D=-E=\frac{1}{6}$, $F=-G=\frac{1}{2}$, $H=-I=\frac{1}{4}$.

578. The frequent use that is made of the decomposition of rational fractions renders the following method, by which the operations are abridged, of great service.

1st Case. *Factors unequal.* Let $D=(x-a)\,S$, S being the product of the other factors, all different from $x-a$. The derivative [p. 37, and N°. 664] is $D'=S+(x-a)\,S'$; we assume

$$\frac{N}{D}=\frac{A}{x-a}+\frac{P}{S}, \text{ whence } N=AS+P\,(x-a);$$

and our present object is, to determine the constant A, without knowing the polynomial P. If now we make $x=a$, and n, s and d denote the resulting values of N, S, and D, on this hypothesis (we shall make use, in what follows, of this notation), we have $d'=s$ and $n=As$; and consequently $A=\frac{n}{s}=\frac{n}{d'}$.

Hence, *replace the denominator D of the proposed fraction by its derivative D'; then change x into a, and you will have the numerator A*

of the component fraction which has $x-a$ for its denominator. And supposing $D=(x-a)(x-b)(x-c)\ldots$, we must in like manner make $x=b, c\ldots$ in $\frac{N}{D'}$, in order to obtain the numerators of

$$\frac{B}{x-b},\ \frac{C}{x-c}\ldots$$

For $\frac{-5x^2-5x+6}{x^4-2x^3-x^2+2x}$, assume $\frac{N}{D'}=\frac{-5x^2-5x+6}{4x^3-6x^2-2x+2}$;

then, since $D=(x-1)(x+1)(x-2)x$, make $x=1, -1, 2$ and 0, successively; you will have $2, -1, -4$ and 3 for results, and the proposed equation will thus be equivalent to

$$\frac{2}{x-1}-\frac{1}{x+1}-\frac{4}{x-2}+\frac{3}{x}.$$

Let the fraction be $\frac{1}{z^6-1}$; we have $\frac{N}{D'}=\frac{1}{6z^5}$; whilst [p. 86]

$$z^6-1=(z+1)(z-1)(z^2-z+1)(z^2+z+1):$$

for the two first factors, we make $z=\pm 1$, and we have $\pm\frac{1}{6}$; the following factor gives $z=\frac{1}{2}(1\pm\sqrt{-3})$; whence we deduce

$$\frac{2^5}{6(1\pm\sqrt{-3})^5}=\frac{32}{6(16\mp 16\sqrt{-3})}=\frac{1\pm\sqrt{-3}}{12};$$

the two component fractions are easily found, and being reduced to a single one, we have $\frac{1}{6}\cdot\frac{z-2}{z^2-z+1}$. Lastly, the 4th factor of D indicates that we have only to change x into $-x$ in this last result. And hence

$$\frac{1}{z^6-1}=\frac{1}{6}\left(\frac{1}{z-1}-\frac{1}{z+1}+\frac{z-2}{z^2-z+1}-\frac{z+2}{z^2+z+1}\right).$$

2nd Case. *The factors being equal.* Let $D=(x-a)^i$; if x be changed into $a+h$ in N and D, these polynomials become [N°. 503]

$$N=n+n'h+\tfrac{1}{2}n''h^2+\frac{1}{6}n'''h^3+\ldots, \text{ and } D=h^i;$$

whence, dividing and putting $x-a$ for h, we find

$$\frac{N}{D}=\frac{n}{(x-a)^i}+\frac{n'}{(x-a)^{i-1}}+\frac{\frac{1}{2}n''}{(x-a)^{i-2}}+\ldots$$

Thus, *the proposed expression resolves itself into $i-1$ fractions, the numerators of which are the values of $N, N', \frac{1}{2}N''\ldots$, when we make $x=a$.*

Take, for example, $\dfrac{3x^2 - 7x + 6}{(x-1)^3}$: since the numerator has for derivatives $6x - 7$ and 6, making $x = 1$, we obtain $2, -1$ and 3 for numerators of the component fractions; and consequently

$$\frac{3x^2 - 7x + 6}{(x-1)^3} = \frac{2}{(x-1)^3} - \frac{1}{(x-1)^2} + \frac{3}{x-1}.$$

But if the denominator contain other factors along with $(x-a)^i$, and we have $D = (x-a)^i . S$, S being known and not divisible by $x - a$, we assume

$$\frac{N}{D} = \frac{F}{(x-a)^i} + \frac{P}{S} \ldots (1);$$

whence

$$N = P(x-a)^i + FS.$$

Change x into $a + y$ in this identical equation, and develop [N°. 503]; then, adhering to the notation employed above for n, d, s and f, there results

$$\begin{aligned} n + n'y + \tfrac{1}{2}n''y^2 \ldots &= y^i(p + p'y + \tfrac{1}{2}p''y^2 \ldots) \\ &+ (f + f'y + \tfrac{1}{2}f''y^2 \ldots)(s + s'y + \tfrac{1}{2}s''y^2 \ldots); \end{aligned}$$

whence, comparing the coefficients of the same powers of y on each side [N°. 576], we have

$$\left.\begin{aligned} n &= fs,\ n' = f's + fs',\ n'' = f''s + 2f's' + fs'', \ldots \\ n^{(l)} &= sf^{(l)} + ls'f^{(l-1)} + \tfrac{1}{2}l(l-1)s''f^{(l-2)} \ldots + fs^{(l)} \end{aligned}\right\} \ldots (2),$$

l denoting any integer $< i$. Thus, these equations give $f, f', f'' \ldots$, and consequently the development of the first part,

$$\frac{F}{(x-a)^i} = \frac{f}{(x-a)^i} + \frac{f'}{(x-a)^{i-1}} + \frac{\frac{1}{2}f''}{(x-a)^{i-2}} \ldots,$$

precisely as though the proposed fraction had had but $(x-a)^i$ in the denominator. From this last equation we deduce

$$F = f + f'.(x-a) + \tfrac{1}{2}f''.(x-a)^2 + \ldots (3);$$

F therefore is known; and we have from the equation (1)

$$\frac{P}{S} = \frac{N - FS}{D} = \frac{N - FS}{S.(x-a)^i} \ldots (4).$$

This identity implies that $(x-a)^i$ is a factor of $N - FS$; the division must be carried into effect in order to obtain the quotient P; when the 2nd part of our proposed fraction becomes known, and must in turn be submitted to decomposition.

Suppose, for example, that

$$\frac{N}{D} = \frac{5x^4 - 13x^3 + 14x^2 - 5x + 3}{(x-1)^3(x+1)\,x};$$

making $x = 1$, we have

$$S = x^2 + x = 2,\ S' = 2x + 1 = 3,\ S'' = 2,$$
$$N = 5x^4 - 13x^3 + \ldots = 4,\ N' = 20x^3 \ldots = 4,\ N'' = 10.$$

Thus,

$$4 = 2f,\ 4 = 2f' + 3f,\ 10 = 2f'' + 6f' + 2f;$$

and consequently

$$f=2, f'=-1, \tfrac{1}{2}f''=3;\ F=2-(x-1)+3\,(x-1)^2=3x^2-7x+6.$$

The product FS, subtracted from N, gives

$$2x^4 - 9x^3 + 15x^2 - 11x + 3;$$

which, divided by $(x-1)^3$, gives $P = 2x - 3$; and it remains to decompose, by the first process,

$$\frac{P}{S} = \frac{2x-3}{x^2+x}$$

From this we have $\frac{P}{S'} = \frac{2x-3}{2x+1}$; making $x = -1$ and 0, the results are 5 and -3; and consequently

$$\frac{N}{D} = \frac{2}{(x-1)^3} - \frac{1}{(x-1)^2} + \frac{3}{x-1} + \frac{5}{x+1} - \frac{3}{x}.$$

It is observable that, in this example, the shorter course would have been to have determined the two last fractions first, by making $x = -1$ and 0 in N and D'; whence

$$\frac{N}{D} = \frac{F}{(x-1)^3} + \frac{5}{x+1} - \frac{3}{x}.$$

Transposing these two last fractions and reducing, we now easily find the first $\frac{F}{(x-1)^3} = \frac{3x^2-7x+6}{(x-1)^3}$, which comes under the first part of this case, and can very readily be decomposed.

In like manner, for $\frac{N}{D} = \frac{x^3 - 6x^2 + 4x - 1}{x^4 - 3x^3 - 3x^2 + 7x + 6}$, since

$D=(x+1)^2(x-2)(x-3)$, we shall make $x=2$ and 3 in N and D'; when we shall have the fractions $\frac{1}{x-2}-\frac{1}{x-3}$; and subtracting these from the one proposed, it will remain to decompose $\frac{x}{(x+1)^2}$. But, for this, we find $f=-1$, $f'=1$; and therefore, combining these parts, there results on the whole

$$\frac{N}{D}=\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{(x+1)^2}+\frac{1}{x+1}.$$

RECURRING SERIES.

579. Every rational fraction, arranged according to the increasing powers of x, and in which the numerator N is of a lower degree than the denominator D, may be developed in an infinite series $A+Bx+Cx^2+Dx^3\ldots$. The actual division of N by D would lead to this, since the quotient could never give either a negative or a fractional power of x; and this operation would also make known the coefficients $A, B, C\ldots$; but we prefer the following process, which exhibits the law itself of the series. We assume

$$\frac{N}{D}=\frac{a+bx+cx^2\ldots+hx^{i-1}}{1+\alpha x+\beta x^2\ldots+\theta x^i}=A+Bx+Cx^2+Dx^3\ldots;$$

reduce to the same denominator, and compare the terms in which x is affected with equal exponents; and the equation thus becomes subdivided into others, which serve to determine $A, B, C\ldots$ [N°. 576]. The denominator has the constant 1 for its first term; but this does not at all detract from the generality of the case, as N and D may be divided by this first term, whatever it be.

Let $\frac{N}{D}=\frac{a}{1+\alpha x}=A+Bx+Cx^2+Dx^3\ldots$;

we have then

$$a=A+\left.\begin{matrix}B\\+A\alpha\end{matrix}\right|x+\left.\begin{matrix}C\\+B\alpha\end{matrix}\right|x^2+\left.\begin{matrix}D\\+C\alpha\end{matrix}\right|x^3\begin{matrix}+\ldots\\+\ldots\end{matrix}$$

whence

$$a=A,\ B+A\alpha=0,\ C+B\alpha=0\ldots;$$

and the 1st of these equations gives A, the 2nd B, the 3rd C....

Generally, M and N being any two successive coefficients of our

series, we have $N + M\alpha = 0$; whence $N = -M\alpha$: thus, *any term is the product of the preceding one by* $-\alpha x$, *i. e.* the series is a progression by quotient, the ratio of which is $-\alpha x$. And thus we can form the several terms one after the other, commencing from the first, $A = a$, which is obtained by making $x=0$ in the fraction proposed:

$$\frac{a}{1+\alpha x} = a\left[1 - \alpha x + \alpha^2 x^2 - \alpha^3 x^3 \ldots + (-\alpha x)^n \ldots\right].$$

The *general term* T, or that which has n terms before it, and the *term of summation* Σ, or the sum of the first n terms [N°. 144], are

$$T = a\,(-\alpha x)^n,\ \Sigma = a.\frac{1-(-\alpha x)^n}{1+\alpha x}.$$

Conversely, if the series and the law which governs it be given, we can readily deduce from it the generating fraction; for the 1st term a is the numerator, and the denominator is 1 — the ratio of the progression.

For example, $\frac{3}{6-4x}$, which, dividing above and below by 6, is $= \frac{\frac{1}{2}}{1-\frac{2}{3}x}$, gives the series $\frac{1}{2}\left(1+\frac{2}{3}x+\frac{4}{9}x^2\ldots\right)$, the first term being $\frac{1}{2}$ and the ratio $\frac{2}{3}x$; at the same time, we find

$$T = \frac{2^{n-1}x^n}{3^n},\quad \Sigma = \frac{1-(\frac{2}{3}x)^n}{2(1-\frac{2}{3})x}.$$

And if this series and its law be given, we regain the generating fraction by dividing the first term $\frac{1}{2}$ by 1 — the factor $\frac{2}{3}x$.

$$\text{For } \frac{a+bx}{1+\alpha x+\beta x^2} = A + Bx + Cx^2 + Dx^3\ldots,$$

we have

$$\begin{array}{rl|l|l|l} a + bx = & A + B & x + C & x^2 + D & x^3 \ldots \\ & + A\alpha & + B\alpha & + C\alpha & \ldots \\ & & + A\beta & + B\beta & \ldots \end{array}$$

whence

$$A = a,\ B + A\alpha = b,\ C + B\alpha + A\beta = 0\ldots;$$

and these equations give $A, B, C\ldots$ successively. The first $A = a$ may also be derived from the assumed equation by making $x = 0$.

Let M, N, P be three consecutive indeterminate coefficients of the development; it follows then from our operation that

$$P + N\alpha + M\beta = 0;\ \text{whence } P = -N\alpha - M\beta:$$

and thus, *any term of the series is deducible from the two preceding ones, by multiplying the one by* $-\alpha x$, *the other by* $-\beta x^2$, *and taking the sum.* It will be observed also that these factors, subtracted from 1, give the denominator of the fraction proposed. To develop this fraction, we must first find the leading terms $A + Bx$, either by division, or by means of the equations $A = a$, $B = b - a\alpha$; and then, by means of the factors $-\alpha x$ and $-\beta x^2$, compose the following terms, one after the other.

Conversely, if the series and its law be given, a very simple calculation leads us back to *the generating fraction, which is the sum total of this series continued to infinity:* 1 minus the two factors forms the trinomial denominator $1 + \alpha x + \beta x^2$; and as to the numerator $a + bx$, we have $a = A$, $b = B + A\alpha$.

Take, for example, $\dfrac{2-4x}{x^2-x-2}$: dividing above and below by -2, this fraction becomes $\dfrac{2x-1}{1+\frac{1}{2}x-\frac{1}{2}x^2}$; and the factors therefore are $-\frac{1}{2}x$, $+\frac{1}{2}x^2$; we also find $-1 + \frac{5}{2}x$ for the two 1st terms; whence this series

$$-1 + \tfrac{5}{2}x - \tfrac{7}{4}x^2 + \tfrac{17}{8}x^3 - \ldots$$

And if, conversely, the series be known, *i. e.* if we have the two first terms and the factors $-\frac{1}{2}x$, $+\frac{1}{2}x^2$, these last, subtracted from 1, give at once the denominator of the generating fraction; and, lastly, we have $a = -1$, $b = 2$; whence results the numerator.

Reasoning in the same manner for $\dfrac{a+bx+cx^2}{1+\alpha x+\beta x^2+\gamma x^3}$, we find that the first three terms of the series give

$$A = a,\ B + A\alpha = b,\ C + B\alpha + A\beta = c,$$

from which equations we derive the values of A, B and C.

The following terms are deducible from these, as before, and four successive coefficients are connected by the equation $Q = -P\alpha - N\beta - M\gamma$, so that any term may be derived from the three preceding ones, by multiplying them by $-\alpha x$, $-\beta x^2$, $-\gamma x^3$. And conversely, we may revert from the series to the generating fraction which expresses its entire sum. This law extends to all rational fractions.

580. The denomination of *Recurring* is applied to every series in which each term is deduced from those which precede it, by multiplying them by certain invariable quantities; these factors being called the *Scale of Relation.* Thus, the sines and cosines of equi-different arcs

[N^{os}. 361, 542], and the sums of the powers of the roots of equations [N^{o}. 553], form recurring series. And, from what has preceded, we may further pronounce every rational fraction, the denominator of which is $1 + \alpha x + \beta x^2 \ldots + \theta x^i$, to be developable in a recurring series, of which the scale of relation is composed of the n factors $-\alpha x, -\beta x^2, \ldots -\theta x^i$; we must first investigate the i leading terms, either by division, or by the method of indeterminate coefficients; and the following terms can then be deduced from them, one after the other.

For example,

$$\frac{x^3+5x^2-10x+2}{x^4-3x^3+x^2+3x-2} = -1 + \tfrac{7}{2}x + \tfrac{9}{4}x^2 + \tfrac{49}{8}x^3 + \tfrac{73}{16}x^4 \ldots$$

The first four terms are easily found; and since, after dividing the fraction above and below by -2, we obtain $\frac{3}{2}x$, $\frac{1}{2}x^2$, $-\frac{3}{2}x^3$ and $\frac{1}{2}x^4$ for the four factors, this scale of relation serves to extend the series to any length we please.

It will be almost superfluous to remind the student that, if the terms of the series go on decreasing, we make the nearer approach to the total value, as we take a greater number of terms; whilst this is not the case when the series is divergent, but it must be taken throughout its whole extent, in order that it may represent the fraction of which it is the development [N^{os}. 99 and 488.]

581. Let us now turn our attention to the general terms T and those of summation Σ.

The denominator D being decomposed into its simple factors, it will be easy to resolve the fraction proposed into others [N^{o}. 577]; and if D have no equal factors, these component fractions will be reducible to the form $\dfrac{A}{1+\alpha x}, \dfrac{B}{1+\beta x} \ldots$, each of which generates a geometric progression. Thus, *the recurring series is the sum of i progressions, the terms of the same rank being added together:* and consequently, the general term T, or that of summation Σ, is the sum of those of the progressions, which it is easy to calculate for each.

In our example of the 2nd degree, it is readily seen that the component fractions are $\dfrac{2}{2-x} - \dfrac{2}{1+x}$; whence result

$$1 + \tfrac{1}{2}x + \tfrac{1}{4}x^2 \ldots + (\tfrac{1}{2}x)^n \ldots, \text{ and } -2[1 - x + x^2 \ldots (-x)^n \ldots];$$

and adding the terms in which x has the same exponent, we again arrive at the series $-1 + \frac{5}{2}x - \frac{7}{4}x^2 \ldots$, the development of the fraction proposed: the general term therefore is

$$T = (\tfrac{1}{2}x)^n - 2(-x)^n.$$

In like manner $\frac{1}{1-4x+3x^2}$ resolves itself into

$$\frac{3}{2-6x}=\tfrac{1}{2}(1+3x+9x^2\ldots)\text{ and }\frac{1}{2x-2}=-\tfrac{1}{2}(1+x+x^2\ldots);$$

and the series required is the sum of these two; the general term therefore is $T=\frac{1}{2}x^n(3^{n+1}-1)$.

$$\frac{2+x+x^2}{2-x-2x^2+x^3}=\frac{2}{1-x}+\frac{\frac{1}{3}}{1+x}-\frac{\frac{8}{3}}{2-x}\text{ gives}$$

$$T=\tfrac{1}{3}x^n[6+(-1)^n-(\tfrac{1}{2})^{n-2}];$$

and this is the general term of the series

$$1+x+2x^2+\tfrac{3}{2}x^3+\tfrac{9}{4}x^4\ldots,$$

of which the scale of relation is $-\frac{1}{2}x^3$, x^2 and $\frac{1}{2}x$.

The term of summation Σ is found without any difficulty in these examples.

582. When the denominator has equal factors, the component fractions are of the form

$$\frac{K}{1+ax},\ \frac{K}{(1+ax)^2},\ \frac{K}{(1+ax)^3},\ \frac{K}{(1+ax)^4}\ldots,$$

the general terms of which have for coefficients [p. 20]

$$T=1,\ n+1,\ \tfrac{1}{2}(n+1)(n+2),\ \tfrac{1}{6}(n+1)(n+2)(n+3)\ldots$$

For $\frac{K}{(1+ax)^i}$, we have $T=(n+1)\frac{n+2}{2}\ldots\frac{n+i-1}{i-1}$;

and we must likewise introduce throughout the factor $K(-ax)^n$. In the example of the 4th degree [p. 154], the component fractions are

$$\frac{-\frac{5}{3}}{1-\frac{1}{2}x},\ \frac{-\frac{4}{3}}{1+x},\ +\frac{1}{(1-x)^2},\ +\frac{1}{1-x};$$

whence

$$T=[-\tfrac{5}{3}(\tfrac{1}{2})^n-\tfrac{4}{3}(-1)^n+n+2]x^n.$$

In like manner

$$\frac{1+4x+x^2}{(1-x)^4}=\frac{6}{(1-x)^4}-\frac{6}{(1-x)^3}+\frac{1}{(1-x)^2}$$

gives

$$T=(n+1)(n+2)(n+3)-3(n+1)(n+2)+(n+1)=(n+1)^3;$$

the series is

$$1^3 + 2^3 x + 3^3 x^2 + 4^3 x^3 + \dots + (n+1)^3 x^n \dots$$

583. If the denominator $1 + \alpha x + \beta x^2 \dots + \theta x^i$ have no equal factors, it has been just seen that the series is the sum of i geometric progressions. Let the unknown ratios of these progressions be represented by $y, z, t\dots$, and their initial terms by $a, b, c\dots$; the general term of the series will be

$$T = ay^n + bz^n + ct^n + \dots (1).$$

Lagrange has given this elegant mode of determining the unknown quantities $a, b, c\dots x, y, z\dots$ It follows from the nature of the recurring series and from the form of the denominator of the generating fraction, that each term depends on the i antecedent ones, and that in particular any coefficient T is deduced from the i which precede it, by multiplying them respectively by $-\alpha, -\beta, -\gamma\dots$; so that these coefficients, taken in inverse order, being represented by $S, R\dots M$, we have

$$T = -S\alpha - R\beta\dots - M\theta\dots (2).$$

If now the general term were only $T = ay^n$, the terms of the ranks $i+h, i+h-1\dots h$, would be $T = ay^{i+h}$, $S = ay^{i+h-1}$, $R = ay^{i+h-2}\dots$, $M = ay^h$; whence, substituting and suppressing the common factor ay^h, we should have

$$y^i + \alpha y^{i-1} + \beta y^{i-2} + \dots + \theta = 0\dots (3).$$

We must observe that it is not necessary to enter into any calculation in order to form this equation, since it results immediately from the given relation (2), or still more easily from the denominator of the generating fraction, by changing x into y^{-1}, and equating to zero.

If, again, we suppose $T = ay^n + bz^n$, we find in like manner

$$a(y^i + \alpha y^{i-1}\dots) + b(z^i + \alpha z^{i-1}\dots) = 0;$$

an equation which is satisfied by assuming for y and z two of the roots of the equation (3), whatever be the values of a and b. And the same reasoning proves that if the general term T be put under its form (1), the condition (2) of the mutual dependence of the i consecutive terms of our series is satisfied, by assuming for $y, z, t\dots$ the i roots of the equation (3). Thus, then, the ratios of our progressions are known; and it remains to find their initial terms $a, b, c\dots$ But we are presumed to know the first i terms of our recurring series $A + Bx + Cx^2\dots$; and making $n = 0, 1, 2\dots$ in the equation (1), and comparing the results

with the given quantities $A, B, C...$, we shall have these i equations of the 1st degree.

$$A = a + b + c..., \; B = ay + bz + ct..., \; C = ay^2 + bz^2... \; (4);$$

whence we easily determine $a, b, c...$ by elimination, and the problem is solved.

Hence we conclude that, *to find the general term T of the recurring series proposed, or to decompose it into i geometric progressions, we must form and solve the equation* (3) ; *its roots are the ratios $y, z...$ of the progressions, and we must substitute these roots for $y, z...$ in the equations* (4), *which will then give $a, b, c...$*

In the example of the 2nd degree [p. 153], the series is $-1 + \frac{5}{2}x - \frac{7}{4}x^2...$; the denominator $x^2 - x - 2$, or rather the scale of relation $-\frac{1}{2}$ and $+\frac{1}{2}$, gives $1 - y - 2y^2 = 0$; whence $y = \frac{1}{2}$ and -1 ; thus our equations (4) become

$$-1 = a + b, \; \frac{5}{2} = \frac{1}{2}a - b; \text{ whence } a = 1, \; b = -2,$$

and lastly, $T = (\frac{1}{2})^n - 2(-1)^n$, as before.

The example of the third degree [p. 155] gives this series

$$1 + x + 2x^2 + \tfrac{3}{2}x^3 + \tfrac{9}{4}x^4...;$$

we find

$$1 - 2y - y^2 + 2y^3 = 0, \; y = \tfrac{1}{2} \text{ and } \pm 1;$$

then

$$1 = a + b + c, \; 1 = \tfrac{1}{2}a - b + c, \; 2 = \tfrac{1}{4}a + b + c;$$

whence

$$a = -\tfrac{4}{3}, \; b = \tfrac{1}{3}, \; c = 2, \; T = \&c.$$

When the denominator has k factors equal to $x - h$, besides the terms $ay^n, bz^n...$, arising from the unequal factors, there will be others also of the form

$$(a' + b'n + c'n^2... + f'n^{k-1})\, a^{n-k+1};$$

the coefficients $a', b'...$ are found, as above, by assuming $n = 0, 1, 2...$, and comparing the results with the initial terms $A + Bx + Cx^2...$

EXPONENTIAL AND LOGARITHMIC SERIES.

584. We shall commence our developments of *transcendental* functions with that of the *exponential* a^x.

Making $a = 1 + y$, the formula of the binomial gives

$$(1+y)^x=1+xy+x.\frac{x-1}{2}y^2...+\frac{x(x-1)(x-2)...(x-n+1)}{1.2.3...n}y^n...;$$

this being arranged in respect to x, the only term without x is 1, and the exponents of that letter are all integral and positive; so that we may assume

$$a^x=1+kx+Ax^2+Bx^3+Cx^4+...(1).$$

To obtain kx, it is evident, from the inspection of our general term, that to derive the term of its product in which x is but of the 1st degree, we must confine ourselves to the second terms of the binomial factors, or take $\frac{x.-1.-2...-(n-1)}{1.2.3...n}y^n = \pm\frac{y^n x}{n}$, the sign being + if n is odd. The aggregate of all these products is kx, viz.

$$k=y-\tfrac{1}{2}y^2+\tfrac{1}{3}y^3-\tfrac{1}{4}y^4...(2),$$

y being $a-1$; thus k is known, and we must now proceed to find $A, B, C...$

These constants still remain the same when x is changed into z; so that

$$a^z=1+kz+Az^2+Bz^3+Cz^4...$$

Make $z=x+i$, and subtract (1); then

$$a^z-a^x+a^{x+i}-a^x=a^x.a^i-a^x=a^x(a^i-1) \text{ gives}$$

$$a^x(a^i-1)=(z-x)[k+A(z+x)+B(z^2+zx+x^2)...];$$

the general term being [N°. 99]

$$P(z^n-x^n)=P(z-x)(z^{n-1}+xz^{n-2}...+x^{n-1}).$$

But, according to equation (1), $a^i-1=ki+Ai^2...$; so that the two sides of our equation are divisible by $i=z-x$; whence

$$a^x(k+Ai...)=k+A(z+x)+B(z^2+zx+x^2)...$$

This being premised, let the arbitrary quantity $i=0$, or $z=x$, and replace a^x by its value (1); we find then

$$(1+kx+Ax^2+Bx^3...)k=k+2Ax+3Bx^2...+(n+1)Px^n...;$$

whence

$$2A=k^2, 3B=kA, 4C=kB, 5D=kC...;$$

these equations being multiplied consecutively in order to eliminate $A, B, C...$, the results are

$$2A=k^2, 2.3B=k^3, 2.3.4C=k^4, \text{\&c.};$$

and, finally,

$$a^x=1+kx+\frac{k^2x^2}{2}+\frac{k^3x^3}{2.3}...+\frac{k^nx^n}{2.3...n}...(A).$$

585. The equation (2) gives k in terms of y or a. On the other hand, to find a, when k is known, we must make $x = 1$ in (A); whence $a = 1 + k + \frac{1}{2}k^2 + \frac{1}{6}k^3 \ldots$; and this series and (2) are the developments of the equation which expresses in finite terms the connection between a and k: this equation we shall now investigate. Make $k = 1$, and let e represent the value then assumed by the base a; $e = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \ldots$

The calculation of this number is easily effected in the manner we see at the side, each term being deduced from the preceding one, by dividing by 3, 4, 5..., as follows from the nature of this series.

	2·5
3rd term ...	0·16666 66666 66
4th	0·04166 66666 66
5th	0·00833 33333 33
6th	0·00138 88888 88
7th	0·00019 84126 98
8th	0·00002 48015 87
9th	0·00000 27557 32
&c.	....
$e =$	2·71828 1828459

But, on the other hand, x being quite arbitrary, we may assume $kx = 1$ in (A); when the 2nd side becomes $= e$, and consequently $a^{\frac{1}{k}} = e$. Thus $e^k = a$, and this is the finite equation which connects k and a; *k is the logarithm of a, taken in the system of which the base is e.* This base e is the one usually adopted in algebraical calculations, in on account of the greater simplicity, which, it will be obvious, must thence ensue. The logarithms taken in this system are styled *Napierian;* we shall for the future designate them by the symbol l, continuing, as in N°. 146, to indicate by Log that the base is any arbitrary number, and by *log* that this base is 10.

Taking the logarithms on each side of the equation $a = e^k$, we have

$$(3)\ k = \frac{\text{Log}\ a}{\text{Log}\ e} \ldots \text{ the base being any number } b;$$

$$(4)\ k = la = nap.\ log.\ a \ldots \text{ the base being } e;$$

$$(5)\ k = \frac{1}{\text{Log}\ e} \ldots \text{ the base being } a;$$

and these are the different values that may be taken for k in the equation (A).

When we take the base $a = e$, k is 1, and we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} \ldots (B).$$

The base being any number b, we must take the first value of k, $\text{Log}\ a = k\ \text{Log}\ e$; and since $a = 1 + y$, the equation (2) becomes

$$\text{Log}\,(1 + y) = \text{Log}\ e\,(y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 \ldots) \ldots (C).$$

Add Log h to each side, and assume $hy = z$; we have then, h and z, as also the base of the system, being any whatever,

$$\text{Log}(h+z) = \text{Log}\, h + \text{Log}\, e\left(\frac{z}{h} - \frac{z^2}{2h^2} + \frac{z^3}{3h^3} - \dots\right) \dots (D).$$

But when the logarithms referred to are the Napierian, Log.e changes into $le = 1$, since this factor is now the log of the base itself of the system considered [N°. 146, 1°]; whence the equation (C) becomes

$$l(1+y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 \dots;$$

and consequently

$$\text{Log}\,(1+y) = \text{Log}\, e \times l\,(1+y):$$

thus we change the Napierian log into those taken in any other system base b, by multiplying the former by Log e [N°. 148]. This constant factor, Log e, is what is called the *Modulus; it is the log of the Napierian base e taken in the system b*, or it may otherwise be defined to be *Unity divided by the Napierian log of the base b*, since, from equations (4) and (5), we have Log $e = \frac{1}{k} = \frac{1}{lb}$. We shall shortly proceed to calculate this factor for any given system of logarithms.

586. Before the equation (C) can be applied to the calculation of the log. of a given number, the series must be rendered convergent. Now, making the modulus Log $e = M$, the equation (C), when y is changed into $-y$, gives

$$\text{Log}\,(1-y) = -M\,(y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 + \dots);$$

and this being subtracted from (C), there results

$$\text{Log}\left(\frac{1+y}{1-y}\right) = 2M\,(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 \dots) \dots (E).$$

It is at once evident that if this fraction $\frac{1+y}{1-y}$ be equated to any positive number N, y will be < 1, and the series consequently will be convergent; but this effect will be enhanced, by assuming

$$\frac{1+y}{1-y} = \frac{z}{z1}, \text{ whence } y = \frac{1}{2z-}.$$

The first side becomes, on this supposition, $\Delta = \text{Log}\, z - \text{Log}\,(z-1)$, *i. e.* it is the difference of the log of the consecutive numbers z and $z-1$; and we have

$$\Delta = 2M\left[\frac{1}{2z-1} + \frac{1}{3(2z-1)^3} + \frac{1}{5(2z-1)^5} + \dots\right] \dots (F)$$

Thus, the modulus M having been determined, we shall easily calculate, one after the other, the logs of the integers 2, 3, 4...; since the value Δ of the difference between these logs is highly convergent, and becomes the more so as the number z increases. And if the system to be formed be the Napierian, M or Log e becomes $le = 1$, so that Δ is very easily calculated, and a table of Napierian logs can thus be composed without any difficulty.

As to the value of M, it depends on the system for which the logs are to be calculated, since $M = \frac{1}{la}$, and we have therefore to investigate the Napierian log of the base a. If, for example, $a = 10$, we shall, in the equation (F), make $M=1$, $z = 2$; and, $l1$ being $= 0$, we shall have $\Delta = l2$; the double of $l2$ is $l4$; $z = 5$ will then give $l5$, and we shall finally obtain $l10$. The calculation itself is annexed. We must now take the quotient of 1 by $l10$; and we thus find

$$\begin{aligned} l2 &= 0{\cdot}69314718 \\ l4 &= 1{\cdot}38629436 \\ \Delta \text{ for } z = 5 &= 0{\cdot}22314355 \\ l5 &= 1{\cdot}60943791 \\ l2 &= 0{\cdot}69314718 \\ l10 &= 2{\cdot}30258509 \end{aligned}$$

$$\begin{aligned} M &= 0{\cdot}43429\ 44819\ 03251\ 82765, \\ \text{Log } M &= \bar{1}{\cdot}63778\ 43113\ 00536\ 77817, \\ \text{Compl.} &= 0{\cdot}36221\ 56886\ 99463\ 22183, \\ e &= 2{\cdot}71828\ 18284\ 59045\ 23536. \end{aligned}$$

Had 3 been the base of the system, after having obtained $l2$, we should have made $z = 3$, when we should have had $l3 = 1{\cdot}09861229$; and, lastly,

$$M = 1 : l3 = 0{\cdot}9102392.$$

And similarly, for the base 5,

$$M = 1 : l5 = 0{\cdot}6213349.$$

It will be easy now to form the tables of logarithms of Briggs and Callet. The base $a = 10$; the value of M enhances the convergence of the series (F); and when z exceeds 100, the 2nd term may be neglected, the 1st sufficing to give Δ with 8 decimals. It will be necessary to calculate 2 or 3 decimal figures beyond those which we propose to retain, in order to avoid the accumulation of errors. At the same time, we need commence only from $z = 10000$, since the inferior logs are easily deduced from the others; and when z exceeds 12000, we may neglect the 1 in $2z - 1$, and assume $\Delta = \frac{M}{z}$.

For example, $z = 10001$ gives $\Delta = 0{\cdot}000043425$; whence log $10001 = 4{\cdot}000043425$. For $z = 99857$ we have $\Delta = 0{\cdot}000004349$, and this must be added to log 99856, to obtain

$$\log 99857 = 4{\cdot}9993785.$$

It is to be observed also that, for the consecutive logs, the difference continues the same for a certain extent of the table [Vol. I. p. 100]; and it will only be necessary therefore to calculate the values of Δ from one distance to another. Thus $z = 99840$ gives the same number Δ (the value above) as for $z = 99860$; and consequently, limiting ourselves to 9 decimals, Δ is constant for the interval of these two values of z.

Having given the logs of two numbers B and C, to find that of $B \pm C$. Of this problem, the following solution is given by M. Legendre, in the *Conn. des Temps.* for 1819.

C being $< B$, if we assume $\varphi = \frac{C}{B}$, $\log\varphi$ is known; and we have

$$\log(B \pm C) = \log B + \log(1 \pm \varphi) = \log B \pm M\varphi - \tfrac{1}{2} M\varphi^2 \ldots,$$

where M is the modulus, and its log therefore supposed to be known. When φ is very small, this series very easily resolves the question. But if the degree of convergence be only slight, we look out for a number a near to φ; let δ be the difference between their logs, we have then

$$\log \frac{\varphi}{a} = \delta, \frac{\varphi}{a} = 10^{\delta}, \varphi = a . 10^{\delta} = a(1 + k\delta + \ldots).$$

And δ may be supposed so small that we are at liberty to neglect δ^2, which gives $1 \pm \varphi = (1 \pm a)\left(1 \pm \frac{ak\delta}{1 \pm a}\right)$; and taking the log and developing, we find, since $Mk = 1$,

$$\log(B \pm C) = \log B + \log(1 \pm a) \pm \frac{a\delta}{1 \pm a},$$

observing that the $\pm$ must correspond on the two sides.

For example, $\log \sin\theta$ being $= \bar{1}{\cdot}2216164$, let $\log\cos\theta$ be required. Since $\cos\theta = \sqrt{(1 - \sin^2\theta)}$, we have $\log\cos\theta = \frac{1}{2}\log(1 - \sin^2\theta)$; and $1 - \sin^2\theta$ gives $B = 1$, $\log B = 0$, $C = \sin^2\theta$.

$\theta = 9^\circ 35' 20''$ $\quad$ $\log C \ldots \bar{2}\cdot4482328 = \log \varphi$

Assume $a = 0\cdot02774\ldots$ $\quad \log a \ldots \bar{2}\cdot4431065$

$1 - a = 0\cdot97226$ $\quad \delta = 0\cdot0001263$

$\log a \ldots\ldots \bar{2}\cdot4431065$

$\log \delta \ldots\ldots \bar{4}\cdot1014034$

$\log (1-a) \ldots - \bar{1}\cdot9877824 \ldots\ldots\ldots\ldots\ldots \bar{1}\cdot9877824$

$\bar{6}\cdot5567275$ $\quad$ 3rd term...$0\cdot0000036$

$\log(1-\sin^2\theta)\ldots\bar{1}\cdot9877788$

half...$\bar{1}\cdot9938894 = \log\cos\theta$

This process is especially useful, when we wish to carry on the calculations with a great number of decimals.

CIRCULAR SERIES.

587. Let it be proposed to develop $\sin x$ and $\cos x$ in series of ascending powers of x,

$$\sin x = Mx^m + M'x^{m'}\ldots, \cos x = Nx^n + N'x^{n'}\ldots$$

In the first place, since $x = 0$ gives $\sin x = 0$ and $\cos x = 1$, it follows that, making $x = 0$, these series must reduce themselves, the one to zero, the other to 1; and no negative exponents therefore can be admitted for x, since a term such as Kx^{-k} would become infinite for $x=0$.

Moreover, $\cos x$ has the constant term 1, so that $N = 1$, $n = 0$: $\sin x$ has no such term; but it has been seen [N°. 362] that unity is the limit of the ratio of the sine to the arc; whence it follows that the series

$$\frac{\sin x}{x} = Mx^{m-1} + M'x^{m'-1}\ldots$$

must be of the form $1-\varphi$, φ decreasing indefinitely with x. It is clear therefore, since the constants parts must destroy each other independently of the others, that we must have $Mx^{m-1} = 1$, viz. $M=1, m=1$; and consequently

$$\sin x = x + M'x^{m'} + \ldots, \cos x = 1 + N'x'\ldots$$

Substituting these values in the equation $\sin^2 x + \cos^2 x = 1$, we further find that $n' = 2$, and $2N' = -1$; thus, making $\alpha = 2$, we may assume these series, in which the coefficients and the exponents are to be determined:

$$\cos x = 1 + Ax^\alpha + Cx^\gamma + Ex^\epsilon + \ldots,$$
$$\sin x = x + Bx^\beta + Dx^\delta + Fx^\varphi + \ldots$$

Let x be now changed into $x+h$, in $P\sin x + Q\cos x$, and the expression developed according to the powers of x. This operation may be carried into effect either by developing the binomial according to x, and then replacing x by $x+h$; or by first changing x into $x+h$ in this binomial and developing; and the two results $a+bh+ch^2\ldots$, $a'+b'h+c'h^2\ldots$ must be identical, whence $b=b'$. Let this double calculation be executed, attending only to the 1st powers of h, which will be sufficient for our purpose.

1°. $P\sin x + Q\cos x = P(x+Bx^\beta\ldots) + Q(1+Ax^\alpha+Cx^\gamma\ldots)$, when $x+h$ is put for x, gives for the coefficient of h (each term is a derivative), [N°. 503],

$$b = P(1+\beta Bx^{\beta-1}+\delta Dx^{\delta-1}\ldots) + Q(\alpha Ax^{\alpha-1}+\gamma Cx^{\gamma-1}\ldots).$$

2°. $x+h$ being first put for x in the binomial, we have

$$\begin{aligned} &P(\sin x.\cos h + \sin h\cos x) + Q(\cos x.\cos h - \sin x.\sin h) \\ &= (P\sin x + Q\cos x)\cos h + (P\cos x - Q\sin x)\sin h. \end{aligned}$$

But $\cos h = 1 + Ah^2\ldots$, $\sin h = h + Bh^\beta\ldots$; whence, substituting and retaining the terms in h alone, the first part will not give any thing, and from the other there will remain for the coefficient only $b' = P\cos x - Q\sin x$.

Equating these values of the coefficients, since P and Q are arbitrary, the equation $b=b'$ divides itself into two:

$$\begin{aligned} \cos x &= 1 + \beta Bx^{\beta-1} + \delta Dx^{\delta-1} + \phi Fx^{\phi-1}\ldots \\ &= 1 + Ax^2 + Cx^\gamma + Ex^\epsilon\ldots \\ \sin x. &= -2Ax - \gamma Cx^{\gamma-1} - \epsilon Ex^{\epsilon-1}\ldots \\ &= x + Bx^\beta + Dx^\delta + Fx^\phi\ldots \end{aligned}$$

Now, in these identical equations, the same terms must be found on each side: the comparison of the exponents gives

$$\beta-1=2,\ \gamma-1=\beta,\ \delta-1=\gamma,\ \epsilon-1=\delta,\ \phi-1=\epsilon,\ldots;$$

whence

$$\beta=3,\ \gamma=4,\ \delta=5,\ \epsilon-6,\ \phi=7\ldots$$

the exponents of the series for $\sin x$ following the order of the odd numbers 1, 3, 5..., those of $\cos x$ the order of the even numbers 0, 2, 4... And it will be easily seen that this ought to be the case; since, changing x into $-x$, the values of $\sin x$ and $\cos x$ must remain the same, except that the sign of $\sin x$ must be changed. The coefficients being next compared, we have

$$-2A = 1, \beta B = A, -\gamma C = B, \delta D = C, -\varepsilon E = D\ldots;$$

whence

$$A = -\tfrac{1}{2}, B = \frac{-1}{2.3}, C = \frac{1}{2.3.4}, D = \frac{1}{2.3.4.5}, E = \frac{-1}{2\ldots6}\ldots$$

and, finally,

$$\sin.\ x = x - \frac{x^3}{2.3} + \frac{x^5}{2\ldots5} - \frac{x^7}{2\ldots7} + \ldots\ (G),$$

$$\cos.\ x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2\ldots6} + \ldots\ (H).$$

588. These series give the sines and cosines of an arc, the length of which is x, the radius of the circle being 1. Suppose the circumference 2π to be known [N°. 591]; then

$$\pi : x :: 180^\circ : \text{number of degrees of arc } x;$$

and substituting in our series $\frac{\pi x}{180}$ instead of x, *the letter x will denote the number of degrees of the arc x*, and our series will become

$$\sin x = Ax - Bx^3 + Cx^5\ldots, \cos x = 1 - A'x^2 + B'x^4\ldots$$

The calculation of the coefficients gives

$\log A = \bar{2}{\cdot}24187736759$	$\log C = \overline{11}{\cdot}13020559$	$\log E = \overline{22}.61713$
$\log B = \bar{7}{\cdot}947480852-$	$\log D = \overline{17}{\cdot}990711-$	$\log F = \overline{27}{\cdot}05950-$
$\log A' = \bar{4}{\cdot}1827247395$	$\log C' = \overline{14}{\cdot}593932-$	$\log E' = \overline{25}{\cdot}85901-$
$\log B' = \bar{9}{\cdot}58729823$	$\log D' = \overline{19}{\cdot}329498$	$\log F' = \overline{30}{\cdot}22219$

589. But the calculation of the sines and cosines is by no means so important as that of their logarithms. Let δ be the constant difference of the arcs of the table to be constructed; then any arc x will be of the form $n\delta$, whence

$$\sin.\ x = n\delta\,(1 - \tfrac{1}{6}n^2\delta^2\ldots), \cos.\ x = 1 - \tfrac{1}{2}n^2\delta^2 + \ldots;$$

and making

$$y = \frac{n^2\delta^2}{2.3} - \frac{n^4\delta^4}{2\ldots5}\ldots, \quad z = \frac{n^2\delta^2}{2} - \frac{n^4\delta^4}{2.3.4}\ldots,$$

we have

$$\sin.\ x = n\delta\,(1-y), \cos.\ x = 1 - z.$$

Taking the logs of these expressions in any system, the modulus of which is M [N°. 586], we find

$$\text{Log}\sin x = \text{Log}\, n\delta \pm M\left(y + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 \ldots\right),$$

$$\text{Log}\cos x = -\ M\left(z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 \ldots\right);$$

and, finally, replacing y and z by their values, we have

$$\text{Log}\sin x = \text{Log}\,(n\delta) - \frac{M\delta^2}{2.3}n^2 - \frac{M\delta^4}{4.5.9}n^4 - \frac{M\delta^6}{9^2.5.7}n^6 \ldots$$

$$\text{Log}\cos x = \qquad - \frac{M\delta^2}{2}n^2 - \frac{M\delta^4}{3.4}n^4 - \frac{M\delta^6}{9.5}n^6 \ldots$$

If the base of the logs be 10, and the arcs of the table proceed from 10″ to 10′, as is the case in the tables of Callet, δ is the length of the arc of 10″, or the 64800th of the semi-circumference π; and, from the values of π and of M given in N^{os}. 591, 586, we find, after going through the calculation, that

$$\log\sin x = \log\delta + \log n - An^2 - Bn^4 \ldots,\ \log\cos x = -A'n^2 - B'n^4 \ldots;$$

where

$$\begin{array}{ll}
\log\delta\ = \bar{5}\cdot 68557\ 48668\ 23541 & \log B\ = \overline{20}\cdot 12481\ 12735 \\
\log A\ = \overline{10}\cdot 23078\ 27994\ 564 & \log B' = \overline{19}\cdot 30090\ 25326 \\
\log A' = \overline{10}\cdot 70790\ 40492\ 84 & \log D\ = \overline{40}\cdot 54439\ 2 \\
\log C\ = \overline{30}\cdot 29868\ 045 & \log D' = \overline{38}\cdot 95143\ 2 \\
\log C' = 28\cdot 09802\ 100 &
\end{array}$$

Thus, for the arc of $4^\circ\frac{1}{2}$ or 16200″, we have $n = 1620$.

$$\begin{array}{rrr}
\log\delta = \bar{5}\cdot 68557487 & \log A = \overline{10}\cdot 2307828 & \log B = \overline{20}\cdot 1248113 \\
\log n = 3\cdot 20951501 & \log n^2 = \underline{6\cdot 4190300} & \log n^4 = \underline{12\cdot 8380600} \\
-0\cdot 00044649 & \bar{4}\cdot 6498128 & \bar{8}\cdot 9628713 \\
\underline{-0\cdot 00000009} & \multicolumn{2}{l}{\text{The corresponding numbers are subtracted.}} \\
\log\sin 4^\circ\,30' = \bar{2}\cdot 89464330 & &
\end{array}$$

$$\begin{array}{rrr}
\log A' = \overline{10}\cdot 7079041 & \log B = \overline{19}\cdot 3009025 & -0\cdot 00133947 \\
\log n^2 = \underline{6\cdot 4190300} & \log n^4 = \underline{12\cdot 8380600} & \underline{-0\cdot 00000138} \\
\bar{5}\cdot 1269341 & \bar{6}\cdot 1389625 & -0\cdot 00134085
\end{array}$$

$$\text{Complement} = \log\cos 4^\circ\,30' = \bar{1}\cdot 99865915$$

If we wish to have $\log R = 10$, we must add 10 to the characteristics [see N°. 362]. The logarithms of the tangents and cotangents are obtained by simple subtractions.

Since n is continually increasing, our series will scarcely answer the purpose beyond 12°, as they become too slightly convergent. We, in fact, employ them only as far as 5°; beyond which, we have recourse to the following process.

We have

$$\frac{\sin(x+\delta)}{\sin x} = \frac{\sin x \cos\delta + \sin\delta \cos x}{\sin x}$$
$$= \cos\delta + \sin\delta . \cot x = \cos\delta\,(1 + \tan\delta . \cot x);$$

and taking the logarithms, the 1st side is the difference Δ between the logs of the sines of the arcs $x+\delta$ and x, viz.

$$\Delta = \log\cos\delta + M(\tan\delta . \cot x - \tfrac{1}{2}\tan^2\delta . \cot^2 x + \tfrac{1}{3}...).$$

Reasoning in the same manner also for $\cos(x+\delta)$, we find that the difference between the logs of the consecutive cosines is

$$\Delta' = \log\cos\delta - M(\tan\delta . \tan x + \tfrac{1}{2}\tan^2\delta . \tan^2 x + \tfrac{1}{3}...).$$

If now we limit ourselves to 9 decimal figures, and take $\delta = 10''$, the only term of these series that gives significant figures is the 1st, whence

$$\Delta = M\tan\delta . \cot x, \quad \Delta' = -M\tan\delta . \tan x,$$

and we have

$$\log(M\tan\delta) = \bar{5}\cdot3233591788.$$

When δ is $1'$, we have

$$\log(M\tan\delta) = \bar{4}\cdot10151043.$$

Thus, commencing from the arc $x = 5°$, of which we are presumed to know the sine, cosine, tangent and cotangent, we shall be able to calculate all the sines and cosines, successively, by means of their consecutive differences Δ and Δ', either from $10''$ to $10''$, or from $1'$ to $1'$; and thence to deduce the tangents and cotangents. Suppose, for example, that $x = 10°\,10'\,30''$; then

$\log\cot x =$	$0\cdot7459888$	$\log\tan x =$	$\bar{1}\cdot2540112$
constant $=$	$\bar{5}\cdot3233592$		$\bar{5}\cdot3233592$
	$\bar{4}\cdot0693480$		$\bar{6}\cdot5773704$
$\Delta =$	$0\cdot00011731$	$\Delta' =$	$-0\cdot000003779$

The remark will apply here, as in p. 162, that the quantities Δ and Δ' are the same for a certain extent of the table.

To avoid the accumulation of errors, we take the precaution of calculating terms at different distances, making them points whence to commence our operations. The equation $\sin 2x = 2\sin x\cos x$, which gives

$$\log\sin 2x = \log 2 + \log\sin x + \log\cos x,$$

will serve for this purpose.

Since we have sin. $45° = \frac{1}{2}\sqrt{2}$, and tan. $45° =$ cot. $45° = 1$, we may commence from this arc and calculate sin. $45° \pm 10''$; these two complemental arcs will reciprocally have the sine of the one for the cosine of the other; from them we shall deduce their tangents and cotangents; and then pass on to

$$45° \pm 20'', \; 45° \pm 30'', \; \&c.$$

590. The series (G) and (H) being compared with the equation (B), it will be seen that their sum is e^x, except as to the sign of every two alternate terms. But, let x be changed into $\pm x\sqrt{-1}$ in the development (B) of e^x; since then $\sqrt{-1}$ has for its powers $\sqrt{-1}, -1, -\sqrt{-1}, +1$, and these values recur periodically on to infinity, the signs of the terms will on this supposition be found to be the same as those in the series G and H; and consequently

$$e^{\pm x\sqrt{-1}} = \cos x \pm \sqrt{-1}.\sin. x \ldots (I).$$

Let these two equations be first added and then subtracted; and we have

$$\cos. x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}, \; \sin. x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} \; (K);$$

whence

$$\tan x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{(e^{x\sqrt{-1}} + e^{-x\sqrt{-1}})\sqrt{-1}},$$

or, multiplying above and below by $e^{x\sqrt{-1}}$,

$$\tan x = \frac{e^{2x\sqrt{-1}} - 1}{(e^{2x\sqrt{-1}} + 1)\sqrt{-1}}.$$

These expressions we must regard simply as analytical results, in which the imaginary quantities have but an apparent existence, and must disappear immediately on the calculation being carried into effect.

In conclusion, changing x into nx in (I), we have

$$e^{\pm nx\sqrt{-1}} = \cos nx \pm \sqrt{-1}.\sin nx \ldots (L);$$

but the 1st side of this is the n^{th} power of the equation (I); and we therefore have, whatever be the value of n,

$$\cos nx \pm \sqrt{-1}.\sin nx = (\cos x \pm \sqrt{-1}.\sin x)^n \ldots (M).$$

These formulæ are of continual use; but we shall at present confine the application of them to the solution of triangles. Making

$$z = e^{C\sqrt{-1}}, \; z' = e^{-C\sqrt{-1}},$$

we find

$$\cos C = \tfrac{1}{2}(z + z'), \ \sin C . \sqrt{-1} = \tfrac{1}{2}(z - z').$$

If now A, B, C be the three angles of a triangle, and a, b, c the sides respectively opposite, we have

$$a \sin B = b \sin A = b \sin (B + C);$$

whence

$$\frac{\sin B}{\cos B} = \tan B = \frac{b \sin C}{a - b \cos C};$$

the latter of these equations leads to

$$\frac{e^{2B\sqrt{-1}} - 1}{e^{2B\sqrt{-1}} + 1} = \frac{b(z - z')}{2a - b(z + z')}, \text{ whence } e^{2B\sqrt{-1}} = \frac{a - bz'}{a - bz};$$

from this we have

$$2B\sqrt{-1} = l(a - bz') - l(a - bz),$$

and, finally, from equation (D),

$$2B\sqrt{-1} = \frac{b}{a}(z - z') + \frac{b^2}{2a^2}(z^2 - z'^2) + \frac{b^3}{3a^3}(z^3 - z'^3) \ldots$$

But the formula (L) gives

$$z^m = \cos mC + \sqrt{-1} \sin mC, \ z'^m = \cos mC - \sqrt{-1} \sin mC;$$

whence

$$z^m - z'^m = 2\sqrt{-1} . \sin mC;$$

and substituting and suppressing the common factor $2\sqrt{-1}$, there results

$$B = \frac{b}{a} \sin C + \frac{b^2}{2a^2} \sin 2C + \frac{b^3}{3a^3} \sin 3C + \ldots$$

Again, the equation

$$c^2 = a^2 - 2ab \cos C + b^2 = a^2 - ab(z + z') + b^2$$

reduces itself, zz' being $= 1$, to $c^2 = (a - bz)(a - bz')$. Taking the logarithms, we obtain

$$2 \log c = 2 \log a - M\left[\frac{b}{a}(z + z') + \frac{b^2}{2a^2}(z^2 + z'^2) \ldots\right];$$

and since $z^m + z'^m = 2 \cos mC$, we have

$$\log c = \log a - M\left(\frac{b}{a} \cos C + \frac{b^2}{2a^2} \cos 2C + \frac{b^3}{3a^3} \cos 3C \ldots\right).$$

These two series serve for the solution of a triangle, in which b is very small in respect to a, *the two sides a and b and the included angle C* being known.

591. Taking the Napierian logarithms, the equation (I) gives

$$\pm x \sqrt{-1} = l(\cos x \pm \sqrt{-1}.\sin x);$$

whence, subtracting these two equations one from the other, and observing that $\sin x = \cos x.\tan x$, we get

$$2x\sqrt{-1} = l\,\frac{\cos x + \sqrt{-1}.\sin x}{\cos x - \sqrt{-1}.\sin x} = l\left(\frac{1+\sqrt{-1}.\tan x}{1-\sqrt{-1}.\tan x}\right).$$

But the development of this logarithm is given by the formula (E) p. 160; and suppressing the common factor $2\sqrt{-1}$, we have this expression for the arc x, when we know its tangent,

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x \ldots (N).$$

This formula will serve for finding *the ratio π of the circumference to the diameter.* Two arcs x and x', the tan of which are $\frac{1}{2}$ and $\frac{1}{3}$ respectively, have for the tan of their sum

$$\tan.(x+x') = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}} = 1;$$

and this sum consequently is $x + x' = 45°$. If, therefore, in (N), we make $\tan x = \frac{1}{2}$, then $\tan x' = \frac{1}{3}$, and add the results, we shall have the length of the arc of 45°, which is the quarter of the semi-circumference π of the circle, the radius of which is 1:

$$\tfrac{1}{4}\pi = \tfrac{1}{2} - \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{5}(\tfrac{1}{2})^5 \ldots + \tfrac{1}{3} - \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 - \ldots$$

Series still more convergent may be obtained by Machin's method. Take the arc x, the tan of which is $\frac{1}{5}$; then [L, N°. 359]

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \tfrac{5}{12},\ \tan 4x = \frac{2.\frac{5}{12}}{1-(\frac{5}{12})^2} = \tfrac{120}{119};$$

this arc therefore differs very slightly from 45°; and A being the excess of it above 45°, or $A = 4x - 45°$, we have

$$\tan A = \frac{\tan 4x - 1}{1+\tan 4x} = \tfrac{1}{239}.$$

Consequently, if we make $\tan x = \frac{1}{5}$ and repeat the series (N) four times, we shall have the arc $4x$; in like manner $\tan. A = \frac{1}{239}$ will give the arc A; and subtracting, we shall obtain the arc of 45°, or

$$\tfrac{1}{4}\pi = 4\left[\tfrac{1}{5} - \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5 \ldots\right] - \tfrac{1}{239} + \tfrac{1}{3}(\tfrac{1}{239})^3 - \ldots$$

We have given [N°. 248] the result of these calculations with 20 decimal figures: $\pi = 3{\cdot}14159\ 26535\ 89793$,

$$\log \pi = 0{\cdot}497149872694,\ l\pi = 1{\cdot}1447298858494.$$

592. Make $x = k\pi$ in equation (I), k denoting any integer; then $\sin. x = 0$, $\cos. x = \pm 1$, as k is even or odd, and

$$e^{\pm k\pi\sqrt{-1}} = \pm 1, \text{ whence } l(\pm 1) = \pm k\pi\sqrt{-1};$$

and multiplying by the modulus M, and adding the numerical value A of $\log a$, we find from this

$$\text{Log}\,(\pm a) = A \pm kM\pi\sqrt{-1},$$

k being any even number, if the expression be that for $\text{Log}\,(+a)$, and any odd one for $\text{Log}\,(-a)$. Hence, *every number has an infinite number of logs in the same system; these logs are all imaginary if the number be negative; when it is positive, a single one is real.**

593. Let us now develop $\sin^u z$ and $\cos^u z$ in terms of the sines and cosines of the arcs z, $2z$, $3z$... Assume

$$\cos z + \sqrt{-1}.\sin z = y,\ \cos z - \sqrt{-1}.\sin z = v;$$

then $yv = 1$, $2\cos. z = y + v$; and $1, u, A', A''$... being the coefficients of the power u, we have, whatever u be,

$$2^u \cos^u z = y^u + uy^{u-2} + A'y^{u-4} + A''y^{u-6}\ldots$$

But the equation (M) gives $y^k = \cos kz + \sqrt{-1}.\sin kz$, and hence

---

* From $a^2 = (-a)^2$, we deduce $2\,\text{Log}\,a = 2\,\text{Log}\,(-a)$; but we must not hence conclude with D'Alembert, that $+a$ and $-a$ have the same logarithms. For, k and l being even, we have

$$\text{Log}\,a = A \pm kM\pi\sqrt{-1} \text{ and } = A \pm lM\pi\sqrt{-1},$$

whence, by addition, $2\,\text{Log}\,a = 2A \pm (k+l)M\pi\sqrt{-1}$; and in like manner k' and l' being odd, we find

$$2\,\text{Log}\,(-a) = 2A \pm (k'+l')M\pi\sqrt{-1}.$$

But it is evident that this latter expression is comprised in the former, from $k'+l'$ being an even number, without our having at the same time $2\,\text{Log}\,(-a)$ generally $= 2\,\text{Log}\,a$: that $\text{Log}\,a$ may be real, it is incumbent that $k = l = 0$; whereas k', l', being odd, cannot be $= 0$; and we consequently can never have $\text{Log}\,a = \text{Log} - a$ in real numbers.

D'Alembert therefore ought to have inferred nothing more than that, among the imaginary log of $+a$ and $-a$, there are some which, added two and two, give equal sums.

$$2^u \cos^u z = \cos uz + u \cos(u-2)z + A' \cos(u-4)z \dots (P)$$
$$\pm \sqrt{-1}\,[\sin uz + u \sin(u-2)z + A' \sin(u-4)z \dots].$$

The $\pm$ arises from $\sqrt{-1}$, which always admits of this double sign. When u is integral, $\cos^u z$ can have but one value; and these two expressions must therefore be equal, and the series will consequently reduce itself to the 1st line (P). But if u be fractional, as $u = \frac{p}{n}$, there will be a root to be extracted, which admits of n values; in this case if we take for z the values z, $z + 2\pi$, $z + 4\pi \dots$ $z + (n-1)\pi$, $\cos z$ and the 1st side will continue unaltered, whilst the developments will be different, the $\sqrt{-1}$ remaining where it is proper; and these will be the n values required.

To obtain $\sin^u z$, let z be changed in the preceding series into $90^\circ - z$. Denoting $u, u-2, u-4 \dots$ by h, so that h shall represent the factors $u - 2x$ of the arc z on the 2nd side, the $\cos hz$ will on this supposition become

$$\cos(\tfrac{1}{2}h\pi - hz) = \cos \tfrac{1}{2}h\pi . \cos hz + \sin \tfrac{1}{2}h\pi . \sin hz.$$

Let $u - 2x$ be now reinstated for h; and the arc $\frac{1}{2}h\pi$ will become $\frac{1}{2}\pi u - \pi x$, or $\frac{1}{2}\pi u$, since we may add to the arc any number of semicircumferences, observing only to take the sine or the cosine with the proper sign. Reasoning in the same manner for the sines of uz, $(u-2)z \dots$, and denoting the cosine and the sine of the arc $\frac{1}{2}\pi u$ by C and S, we find that the change of z into $90^\circ - z$ corresponds to that of $\cos z$ into $\sin z$, and

$$\text{of } \cos(u-2x)z \text{ into } \pm (C \cos hz + S \sin hz),$$
$$\text{of } \sin(u-2x)z \text{ into } \pm (S \cos hz - C \sin hz),$$

taking the sign $+$ if x is even, and $-$ when x is odd. We must, therefore, make $h = u, u-2, u-4 \dots$, and take the successive results with the signs $+$ and $-$ alternately; and hence we shall have

$$2^u \sin^u z = (C \pm \sqrt{-1}.S)\,[\cos uz - u \cos(u-2)z + A' \cos(u-4)z \dots]$$
$$+ (S \mp \sqrt{-1}.C)\,[\sin uz - u \sin(u-2)z + A' \sin(u-4)z \dots]$$

When u is an integer, since $\sin^u z$ has but one value, these two developments are equal, and the two imaginary terms must consequently destroy each other. There will, however, be two cases, as C and S, the cos and sin of the multiples of the quadrants $\frac{1}{2}\pi u$, reduce themselves to zero, to $+1$ or to -1, accordingly as u is even or odd.

1°. If u is *even*, $S = 0$; C is $+1$ for $u = 4n$, and -1 if $u = 4n + 2$; whence

$$\pm 2^u \sin^u z = \cos uz - u \cos (u-2) z \ldots$$

But the coefficients of the binomial formula are the same at equal distances from the extremes. Moreover, the arc which has x terms before it is $u - 2x$; whilst that which has x after it, has $u - x$ before it, and is consequently $u - 2(u - x) = -(u - 2x)$; the cosines of these arcs therefore are also the same two and two, and thus the terms of the series, added together two and two, become divisible by 2, with the exception of the middle term; consequently, $u, A', A''\ldots$ denoting the coefficients of p. 6, we have

$$\pm 2^{u-1} \sin^u z = \cos uz - u \cos (u-2) z + A' \cos (u-4) z \ldots \quad (Q);$$

where the development must be continued only as far as the middle term, which is constant, and *of which we must take the half* [See p. 5 for the value of this coefficient]. The sign $+$ is to be adopted when u is of the form $4n$, and the sign $-$, if $u = 4n + 2$.

2°. If u is *odd*, $C = 0$, $S = \pm 1$, and we have

$$\pm 2^u \sin^u z \mp \sin uz - u \sin (u-2) z \ldots;$$

the arcs equi-distant from the extremes are still equal, only with contrary signs; and since the sign of the coefficient also results different, the terms can again be added two and two; and we have

$$\pm 2^{u-1} \sin^u z = \sin uz - u \sin (u-2) z + A' \sin (u-4) z \ldots \quad (R).$$

The development must be continued to the middle term, which contains $\sin z$, and of which we no longer take the half; the sign $+$ is to be used when $u = 4n + 1$ and the $-$ when $u = 4n + 3$.

3° and lastly: the series (P) also, when u is integral and positive, presents terms which admit of being added together two and two, and we have

$$2^{u-1} \cos^u z = \cos uz + u \cos (u-2) z + A' \cos (u-4) z + \ldots \quad (S);$$

confining the series to positive arcs, and *taking the half of the middle term* (which is constant), when u is even.

Hence we easily deduce the following equations:

$$\begin{aligned}
2 \cos^2 z &= \cos 2z + 1, \\
4 \cos^3 z &= \cos 3z + 3 \cos z, \\
8 \cos^4 z &= \cos 4z + 4 \cos 2z + 3, \\
16 \cos^5 z &= \cos 5z + 5 \cos 3z + 10 \cos z, \\
32 \cos^6 z &= \cos 6z + 6 \cos 4z + 15 \cos 2z + 10, \text{ \&c.} \\
-2 \sin^2 z &= \cos 2z - 1 \\
-4 \sin^3 z &= \sin 3z - 3 \sin z,
\end{aligned}$$

$$8 \sin^4 z = \cos 4z - 4 \cos 2z + 3,$$
$$16 \sin^5 z = \sin 5z - 5 \sin 3z + 10 \sin z,$$
$$-32 \sin^6 z = \cos 6z - 6 \cos 4z + 15 \cos 2z - 10,$$
$$\&c. = \&c.$$

594. Conversely, to develop the sines and cosines of the multiple arcs according to the powers of $\sin z = s$, $\cos z = c$, we have, for the 2nd side of the equation M [p. 168], $(c + \sqrt{-1}.s)^n$; which being developed by the binomial formula, we arrive at an equation of the form

$$\cos nz + \sqrt{-1}.\sin nz = P + Q\sqrt{-1};$$

and since the imaginary parts must mutually destroy each other, this equation separates itself into two; $\cos nz = P$, $\sin nz = Q$, the first containing all the terms in which $s\sqrt{-1}$ is affected with even exponents: thus, n being integral or fractional, positive or negative, we have

$$\cos nz = c^n - n.\frac{n-1}{2} c^{n-2} s^2 + \frac{n.(n-1)(n-2)(n-3)}{2.3.4} c^{n-4} s^4 - \ldots$$

$$\sin nz = nc^{n-1} s - \frac{n(n-1)(n-2)}{2.3} c^{n-3} s^3 + \frac{n(n-1)\ldots(n-4)}{2.3.4.5} c^{n-5} s^5 - \ldots$$

Hence, s being $= \sin z$ and $c = \cos z$, we have

$\cos 2z = c^2 - s^2,$	$\sin 2z = 2cs,$
$\cos 3z = c^3 - 3cs^2,$	$\sin 3z = 3c^2 s - s^3,$
$\cos 4z = c^4 - 6c^2 s^2 + s^4,$	$\sin 4z = 4c^3 s - 4cs^3,$
$\cos 5z = c^5 - 10c^3 s^2 + 5cs^4,$	$\sin 5z = 5c^4 s - 10c^2 s^3 + s^5,$
$\cos 6z = c^6 - 15c^4 s^2 + 15c^2 s^4 - s^6,$	$\sin 6z = 6c^5 s - 20c^3 s^3 + 6cs^5$
&c. = &c.	&c. = &c.

595. In these formulæ, the sines are interspersed indifferently with the cosines; but others may be found in functions of the sine alone, or the cosine. Since the arcs z, $2z$, $3z$..., form an equi-difference, the sines and cosines constitute a recurring series [N°. 361], the factors of which are $2 \cos z$ and -1. And in like manner, if the arcs proceed from 2 to 2, viz. z, $3z$, $5z$..., or $0z$, $2z$, $4z$...; the factors are $2 \cos 2z$ and -1, of which the former $2 \cos 2z = 2(c^2 - s^2) = 2 - 4s^2$. Thus, commencing from $\cos 0z = 1$, $\sin 0z = 0$, $\cos z = c$, $\sin z = s$, we shall have no difficulty in forming the recurring series that follow, of which we know the two 1st terms and the law [N°. 580].

$\sin 2z = s\,(2c),$ $\cos 2z = 2c^2 - 1,$
$\sin 3z = s\,(4c^2 - 1),$ $\cos 3z = 4c^3 - 3c,$
$\sin 4z = s\,(8c^3 - 4c),$ $\cos 4z = 8c^4 - 8c^2 + 1,$
$\sin 5z = s\,(16c^4 - 12c^2 + 1),$ $\cos 5z = 16c^5 - 20c^3 + 5c,$
$\sin 6z = s\,(32c^5 - 32c^3 + 6c),$ $\cos 6z = 32c^6 - 48c^4 + 18c^2 - 1,$
$\sin 7z = s\,(64c^6 - 80c^4 + 24c^2 - 1),$ &c.

$\sin 2z = c\,(2s),$ $\cos 2z = 1 - 2s^2,$
$\sin 4z = c\,(4s - 8s^3),$ $\cos 4z = 1 - 8s^2 + 8s^4,$
$\sin 6z = c\,(6s - 32s^3 + 32s^5),$ $\cos 6z = 1 - 18s^2 + 48s^4 - 32s^6,$
$\sin 8z = c(8s - 80s^3 + 192s^5 - 121s^7),$ &c. &c.

$\sin 3z = 3s - 4s^3,$ $\cos 3z = c\,(1 - 4s^2),$
$\sin 5z = 5s - 20s^3 + 16s^5,$ $\cos 5z = c\,(1 - 12s^2 + 16s^4),$
$\sin 7z = 7s - 56s^3 + 112\,s^5 - 64s^7,$ $\cos 7z = c\,(1 - 24s^2 + 80s^4 - 64s^6),$
&c. = &c. &c. = &c.

As to the law of these equations, it is demonstrated in the eleventh section of the *Calcul des Fonctions*, where Lagrange finds these general formulæ:

$$\sin nz = s\,[(2c)^{n-1} - (n-2)\,(2c)^{n-3} + (n-3)\,\frac{n-4}{2}\,(2c)^{n-5}$$
$$- \frac{(n-4)\,(n-5)\,(n-6)}{2\,.\,3}\,(2c)^{n-7} + \frac{(n-5)\ldots(n-8)}{2\,.\,3\,.\,4}\,(2c)^{n-9}\ldots];$$

$$2\cos nz = (2c)^n - n\,(2c)^{n-2} + n.\frac{n-3}{2}(2c)^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}\,(2c)^{n-6}$$
$$+ n\,\frac{(n-5)\ldots(n-7)}{2.\,3.\,4}\,(2c)^{n-8}\ldots \pm 1, \text{ or } nc.$$

When n is even, we may assume

$$\sin nz = c\,[ns - n.\frac{n^2-2^2}{2\,.\,3}\,s^3 + n.\frac{n^2-2^2}{2\,.\,3}.\frac{n^2-4^2}{4\,.\,5}\,s^5\ldots\,(2s)^{n-1}],$$

$$\cos nz = 1 - \frac{n^2}{2}\,s^2 + \frac{n^2}{2}.\frac{n^2-2^2}{3\,.\,4}\,s^4 - \frac{n^2}{2}.\frac{n^2-2^2}{3\,.\,4}.\frac{n^2-4^2}{5\,.\,6}\,s^6\ldots\tfrac{1}{2}\,(2s)^n;$$

and when n is odd,

$$\sin nz = ns - n.\frac{n^2-1}{2\,.\,3}\,s^3 + n.\frac{n^2-1^2}{2\,.\,3}.\frac{n^2-3^2}{4\,.\,5}\,s^5\ldots\tfrac{1}{2}\,(2s)^n,$$

$$\cos nz = c\,[1 - \frac{n^2-1^2}{1\,.\,2}\,s^2 + \frac{n^2-1^2}{1\,.\,2}.\frac{n^2-3^2}{3\,.\,4}\,s^4\ldots\,(2s)^{n-1}].$$

INVERSE METHOD OF SERIES.

596. Having given the equation $y = \phi x$, in which ϕx is a series, our object now is to find $x = Fy$ in a series arranged according to y. If this latter series have a known form, as, for instance,

$$x = Ay + By^2 + Cy^3 + Dy^4 \ldots,$$

nothing more will be requisite than to determine the coefficients $A, B, C \ldots$ And for this purpose, the series and its powers must be substituted in the proposed equation $y = \phi x$ for x, x^2, $x^3 \ldots$, when we shall have an identical equation, which we shall sub-divide into others, by comparison of the terms in which y has the same powers; and these equations will make known $A, B, C \ldots$

Let

$$y = M\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right) \ldots;$$

there can be no doubt as to the preceding series in y being suitable for x; for y is the log of $1 + x$, or $a^y = 1 + x$ [see N°. 585]; and we shall therefore substitute for x the series $Ay + By^2 \ldots$, when there results

$$\begin{array}{rlr}
\dfrac{y}{M} = Ay + By^2 + Cy^3 + & Dy^4 \ldots\ldots & \text{for } x, \\
- \tfrac{1}{2}A^2y^2 - ABy^3 - & \left(\tfrac{1}{2}B^2 + AC\right)y^4 \ldots\ldots & \text{for } - \tfrac{1}{2}x^2, \\
+ \tfrac{1}{3}A^3y^3 + & A^2By^4 \ldots\ldots & \text{for } + \tfrac{1}{3}x^3, \\
- & \tfrac{1}{4}A^4y^4 \ldots\ldots & \text{for } - \tfrac{1}{4}x^4, \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & ;
\end{array}$$

whence

$$AM = 1,\ B = \tfrac{1}{2}A^2,\ C = AB - \tfrac{1}{3}A^3,\ D = \ldots;$$

consequently

$$A = \frac{1}{M},\ B = \frac{A^2}{2},\ C = \frac{A^3}{2.3},\ D = \frac{A^4}{2.3.4} \ldots;$$

and, finally,

$$x = Ay + \frac{A^2y^2}{2} + \frac{A^3y^3}{2.3} + \frac{A^4y^4}{2.3.4} \ldots$$

In like manner, $y = x - x^2 + x^3 - x^4 \ldots$ gives, by reversion,

$$x = y + y^2 + y^3 + y^4 \ldots$$

It seldom happens, however, that we are thus acquainted *à priori* with the form of the required series $x = Fy$; when that is not the case, the powers of y must be indicated by letters, as $x = Ay^\alpha + By^\beta + Cx^\gamma \ldots$, and we shall then have to determine both the coefficients and the

exponents, from the consideration that, after the substitution in $y = \varphi x$, each term must be destroyed by others in which y is of the same power. This is the plan that we have followed in N°. 587.

Let

$$y = \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 + \tfrac{1}{4} x^4 + \ldots;$$

and assume

$$x = Ay^{\alpha} + By^{\beta} + Cy^{\gamma} \ldots\ldots,$$

$\alpha, \beta, \gamma \ldots$ being increasing numbers. We have not introduced a term devoid of y, because $x = 0$ corresponds to $y = 0$. This value being now substituted for x, it will be seen that

1°. The exponents 2, 3, 4... that x had, forming an equi-difference, those $\alpha, \beta, \gamma \ldots$ of y must equally form one; since, the development being effected, the powers, $x^2, x^3 \ldots$ will evidently possess the same property.

2°. When α and β are found, $\gamma, \delta \ldots$ will follow from them.

3°. The term in which y will have the least exponent is $\frac{1}{2} A^2 y^{2\alpha}$; and this term must correspond to the 1st side y, whence $2\alpha = 1$, $\frac{1}{2} A^2 = 1$; and consequently, $\alpha = \frac{1}{2}$, $A = \sqrt{2}$.

4°. The terms which have the next lowest exponent being $ABy^{\alpha+\beta}$ and $\frac{1}{3} A^3 y^{3\alpha}$, in order that they may correspond to each other, we must have $\alpha + \beta = 3\alpha$, or $\beta = 1$, and consequently, $\gamma = \frac{3}{2}$, $\delta = \frac{4}{2} \ldots$; viz.

$$x = Ay^{\frac{1}{2}} + By^{\frac{2}{2}} + Cy^{\frac{3}{2}} + \&c.;$$

by a renewal of the calculation, we shall easily find $A, B, C \ldots$; and their values give

$$x = y^{\frac{1}{2}} . \sqrt{2} - \tfrac{2}{3} y + \tfrac{1}{18} \sqrt{2} \cdot y^{\frac{3}{2}} - \tfrac{2}{135} y^2 + \ldots$$

In this manner

$$y = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2\ldots5} - \frac{x^7}{1.2\ldots7} \ldots$$

appears, by reversion, under the form

$$x = Ay + By^3 + Cy^5 \ldots;$$

and, the calculation being completed, we find [see N°. 800]

$$x = y + \frac{1 . y^3}{2.3} + \frac{1.3\, y^5}{2.4.5} + \frac{1.3.5\, y^7}{2.4.6.7} + \frac{1.3.5.7\, y^9}{2.4.6.8.9} \ldots$$

From $x = ay + by^2 + cy^3 + \ldots$, we obtain

$$y = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{2b^2 - ac}{a^5} x^3 + \frac{5abc - a^2 d - 5b^3}{a^7} x^4 \ldots$$

The series $x = ay + by^3 + cy^5 + dy^7 \ldots$ gives

$$y = \frac{x}{a} - \frac{bx^3}{a^4} + \frac{3b^2 - ac}{a^7}x^5 + \frac{8\,abc - a^2 d - 12\,b^3}{a^{10}}x^7 \ldots$$

Lastly, $y = x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}} - \frac{5}{128}x^{\frac{7}{2}} \ldots$

gives
$$x = Ay^{-2} + By^{-4} + Cy^{-6} \ldots;$$
and subsequently,
$$x = y^{-2} - y^{-4} + y^{-6} - y^{-8} \ldots$$

Should the proposed equation be of the form $y = a + bx + cx^2 \ldots$, it will be advisable, for the convenience of calculation, to transpose a, and to make

$$\frac{y-a}{b} = z, \text{ whence } z = x + \frac{c}{b}x^2 + \frac{d}{b}x^3 \ldots;$$

and then to develop x in terms of z. We shall conclude with referring to N°. 711, where we have discussed the subject of the inversion of series in the most general manner.

EQUATIONS OF CONDITION.

597. Whilst the law that governs a physical phenomenon is known and expressed by means of an equation $\phi\,(x, y \ldots, a, b \ldots) = 0$, it frequently happens that the constants $a, b, c \ldots$ are unknown, $x, y \ldots$ being magnitudes variable with the circumstances of the phenomenon. In this case recourse is had to experiment for the determination of $a, b, c \ldots$; simultaneous values of $x, y, z \ldots$ are measured, and substituted in the equation $\phi = 0$; the experiment is then repeated, and other values observed for $x, y, z \ldots$, which gives fresh *equations of condition* between the unknown constants $a, b, c \ldots$, and these constants then become known by elimination.

But the values gained by observation never being quite accurate, the numbers $a', b', c' \ldots$, which we thus obtain for $a, b, c \ldots$, can only be considered as approximate; so that in $\phi = 0$, we must assume $a = a' + A$, $b = b' + B \ldots$, and proceed to determine the errors $A, B \ldots$, with which $a, b \ldots$ are affected. At the same time, $A, B \ldots$ being very small quantities, we shall be authorized to neglect their higher powers; and thus the equation $\phi = 0$ will contain the unknown quantities $A, B \ldots$ only in the 1st degree, and will appear, for instance, under the form

$$0 = x + Ay' + Bz + Ct \ldots (1).$$

We then compensate for the inaccuracy in the measures of x, y, by the number of the observations. The experiment being repe several times, we obtain as many equations (1), in which $x, y, z...$ known; these equations we compare, and combine one with anothe as to arrive at a mean equation, in which one of the constants has greatest factor possible, whilst on the contrary the other factors are small; when the resulting error in the determination of the coeffici will be very considerably reduced. And as many of these equatio condition being derived as there are unknown quantities, elimination readily give the values of $A, B...$

This method is practised in astronomy; but it is inferior in poi accuracy to that of the *least squares*, proposed by M. Lege which compensates for the length of the calculations by the precisio the results. Suppose that observation have given values of no very g exactness for $x, y, z...$; these being substituted in the equation (1) 1st side will not be zero, but a very small and unknown quanti Other experiments will in like manner give the errors, $e', e''...$ c sponding to the values $x', y'..., x'', y''...$, viz.

$$e' = x' + Ay' + Bz'..., e'' = x'' + Ay'' + Bz''..., \&c.;$$

and forming the sum of the squares of these equations, we find, wr down only the terms in A, since the others have the same form, $e^2 + e'^2 + e''^2...$ is

$$= A^2 (y^2 + y'^2...) + 2A (xy + x'y'...) + 2AB (yz...) + 2AC$$

This 2nd side has the form $A^2m + 2An + k$; and will be the possible when A is so taken that the derivative shall be not $Am + n = 0$ [see N^{os}. 140, II., and 717]: thus, considering only constant and unknown factor A, we have

$$xy + x'y'... + A(y^2 + y'^2...) + B(yz + y'z'...) + C(yt...) \&c. =$$

whence it appears that *each of the equations* (1) *of condition mu multiplied by the factor y of A, and the sum equated to zero.* The f y must have its proper sign continued to it. Proceeding in the manner for $B, C...$, we shall obtain as many equations, similar to as there are unknown constants; these equations will all be of th degree, and the process of elimination will be attended wit difficulty.

For example, it appears from the principles of Mechanics that, in latitude y, the length of the simple seconds pendulum is $x = A + B$ s A and B being certain invariable numbers, which it is required t termine. For this purpose, it might suffice carefully to measure lengths x in two different latitudes y, when we should obtain two e

N 2

tions of condition that would serve to determine A and B. But the accuracy of the results will be much greater, if, as M. M. Mathieu and Biot have done, we measure x in six different latitudes, and treat the six equations of condition according to the preceding method. The quantities $A+B \sin^2 y - x$, expressed in metres, give these six errors,

$$A+B.0{\cdot}3903417-0{\cdot}9929750, \quad A+B.0{\cdot}4932370-0{\cdot}9934740$$
$$A+B.0{\cdot}4972122-0{\cdot}9934620, \quad A+B.0{\cdot}5136117-0{\cdot}9935967$$
$$A+B.0{\cdot}5667721-0{\cdot}9938784, \quad A+B.0{\cdot}6045628-0{\cdot}9940932.$$

Since the coefficient of A is 1, the equation which corresponds to it is formed of the sum of the six errors: for B, we must multiply each trinomial by the factor which affects B, and add the six products; whence

$$6A + B \times 3{\cdot}0657375 - 5{\cdot}9614793 = 0,$$
$$A \times 3{\cdot}0657375 + B \times 1{\cdot}5933894 - 3{\cdot}0461977 = 0.$$

Elimination then gives A and B; and we finally have

$$x=0{\cdot}9908755 + B \sin.^2 y, \log B=\bar{3}{\cdot}7238509, B=0{\cdot}0052941816.$$

See the *Conn. des Temps* for 1816, where M. Mathieu has discussed, by this method, the observations on the pendulum made in different places by the Spaniards.

BOOK VI.

ANALYSIS APPLIED TO THREE DIMENSION

I.—SPHERICAL TRIGONOMETRY.

FUNDAMENTAL PRINCIPLES.

598. Three planes MON, NOP, MOP [fig. 7], which pass th the centre of a sphere, determine a trihedral angle O, and cut the s in great circles, of which the arcs CA, CB, AB form a spherical tr ABC; the plane angles of the trihedral O are respectively measu the sides or arcs of this triangle, viz. NOP by AB, MON by AC, by BC. The angle C of the triangle is measured by that wh formed by two tangents at C to the contiguous arcs AC, BC these tangents being situated in the planes of the arcs, form the di angle $NOMP$ of the planes themselves, *i. e.* measure the inclinat the face NOM to POM. Hence, *the plane angles of the trihe are measured by the sides of the spherical triangle ABC, and the nations of the faces are the angles of this triangle.*

The problems in which certain parts of a spherical triangle being it is proposed to find the other parts, are precisely the same with th which, knowing some of the elements of a trihedral, we wish to the others. *There are six elements: three angles A, B, C, an three opposite sides a, b, c, of the spherical triangle;* or, otherwise, *plane angles a, b, c, and the three opposite dihedral angles A, B, the trihedral under consideration.* And *any three of these six parts given, the question is to determine the three others.*

Now, let visual radii be directed to any three distinct points M, in space, as to three stars, for instance; these lines then will b edges of a trihedral O, the constituent elements of which will be th a spherical triangle ABC, formed by the arcs of great circles the

the points in which these edges pass through the surface of a sphere of arbitrary radius, but having its centre in O.

On these grounds the following theorems may be demonstrated:

1°. Every plane angle of a trihedral being less than two right angles, *each side of any spherical triangle is* $< 180°$. *Each angle also is less than two right angles;* as follows again from the polar triangle [see N°. 599 below].

Whenever, therefore, the result of a calculation, for finding the value of an angle or side of a triangle, shall be an arc $> 180°$, this solution must be rejected as impossible.

2°. Since the sum of the plane angles of any polyhedral angle is less than 4 right angles [N°. 280], *the sum of the three sides of any spherical triangle is less than* 360°. The trihedral angle of a cube, composed of three right angles, shows that each side of a spherical triangle may equal and even exceed 90°.

3°. *Two spherical triangles are equal when the three angles, or the three sides, or two sides and the included angle, or two angles and the adjacent side, are respectively equal each to each.* These theorems, as also the two following, may be demonstrated in the same manner as for the rectilinear triangles [N°. 198].

4°. *In a spherical isosceles triangle, the equal angles are opposite to the equal sides, and the converse; and the arc let fall from the vertex perpendicularly on the base, bisects this base and the angle at the vertex.*

5°. *In any spherical triangle, the greatest angle is always opposite to the greatest side, the mean angle to the mean side, and the least to the least.*

6°. *Any one side is always less than the sum of the two others, and greater than their difference:* for the sum of two plane angles of a trihedral exceeds the third; whence $a < b + c$, $b < a + c$, and consequently $a > b - c$. Hence also, *the semi-sum of the three sides of a triangle is always greater than any one side.*

599. Let our trihedral O be cut by three planes respectively perpendicular to the edges; these planes will determine a second trihedral O' opposite to the first; and *the plane angles of the one will be the supplements of the dihedral angles of the other, and the converse.*

For, AOB being [fig. 8] one of the faces of the proposed trihedral O, if AA', BA' be the lines in which it is cut by the two planes perpendi-

cular to the edges OA, OB, and therefore to the face OAB, the angles A, B of the quadrilateral $AOBA'$ will be right angles; and consequently the angle A' will be the supplement of O. But our two cutting planes are faces of the new trihedral O', the straight line $O'A'$, in which they intersect each other, being an edge of this body; and the dihedral angle formed by these planes is evidently measured by the angle $AA'B$, since the plane AOB is perpendicular to them. Thus, the plane angle O of the first trihedral is the supplement of the dihedral angle A' of the second; and the same may be said of the other faces; consequently, the plane angles of the one O are the supplements of the dihedral angles of the other O'.

And the converse is true, since the trihedral O' may be considered as the one proposed, and O as the one constructed.

Hence we conclude that

1°. *The sum of the three angles of any spherical triangle is always comprised between* 2 *and* 6 *right angles.* For, each of the dihedral angles A, B, C being less than 2 right angles, their sum $A+B+C$ is less than 6 right angles. On the other hand, since each of the dihedral angles A, B, C is the supplement of the plane angle corresponding to it in O', we have $A+B+C=6$ right angles $-$ the sum of the three plane angles in O'; and since this latter sum is <4 right angles, we have $A+B+C>2$ right angles.

2°. The two trihedrals O and O' determine two spherical triangles, such that the angles of the one are the supplements of the sides of the other, and the converse.

Having given a spherical triangle ABC of which the sides are a, b, c, we may always construct another $A'B'C'$, with the sides a', b', c', such that the angles A, B, C of the first shall be the supplements of the respective sides a', b', c' of the second, and the converse, viz.

$$a' = 180^\circ - A,\ b' = 180^\circ - B,\ c' = 180^\circ - C \ldots (1),$$
$$A' = 180^\circ - a,\ B' = 180^\circ - b,\ C' = 180^\circ - c \ldots (2).$$

The triangle thus formed is called the *polar* or *supplemental* triangle of the first.

These equations are of great importance, for they reduce the six problems of Spherical Trigonometry to three. Are the given quantities, for instance, the three angles of a triangle ABC? To find a side a, take instead of this triangle the supplemental one $A'B'C'$, of which we shall know the three sides a', b', c' from equ. (1); and having found one of its angles A', we shall thence be able to deduce the side a of the proposed triangle, it being [equ. 2] $= 180^\circ - A'$. Thus, when able to solve a

triangle of which we have the three sides, we shall be able also to solve that in which we know the three angles; and similarly for the other cases. This will appear more clearly from what follows.

Hence we again arrive at the consequence 1°. p. 182; for the sum of the three last equations is $A'+B'+C'=6$ right angles $-(a+b+c)$; and since the sum $a+b+c$ of the sides is less than 4 right angles, that of the angles, or $A'+B'+C'$, is $>$ 2 right angles, *i. e.* the sum of the three angles of every triangle is $> 180°$.

600. If the trihedral O [fig. 9] be cut by a plane pmn perpendicular to an edge OA, in a point m, such that $Om=1$, we have

$$mn = \tan c,\ nO = \sec c,\ mp = \tan b,\ Op = \sec b.$$

But the rectangular triangles mnp, npO give [N°. 355]

$$np^2 = mn^2 + pm^2 - 2mn.pm.\cos A,$$
$$np^2 = nO^2 + pO^2 - 2nO.pO.\cos a;$$

the first of which being subtracted from the second, we have, in consequence of the triangles being rectangular and $Om=1$,

$$0 = 1 + 1 - 2\sec c \sec b \cos a + 2\tan c \tan b \cos A.$$

From this, substituting $\dfrac{1}{\cos}$ for sec, $\dfrac{\sin}{\cos}$ for tan, &c. there results

$$0 = 1 - \frac{\cos a}{\cos c \cos b} + \frac{\sin c \sin b \cos A}{\cos c \cos b};$$

and this gives the *fundamental equation*

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \ldots (3).$$

Of course, we may here change a and A into b and B, or into c and C, so that this equation in fact represents three.

We hence deduce

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

whence

$$1 - \cos^2 A \text{ or } \sin^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c};$$

and reducing to a common denominator, and making $\sin^2 = 1 - \cos^2$, there results

$$\sin^2 A = \frac{1 - \cos^2 b - \cos^2 c - \cos^2 a + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Taking the root of this and dividing the two sides by $\sin a$, the 2nd

will be a *symmetrical* function of a, b, c, which we shall call M, viz. $\frac{\sin A}{\sin a} = M$.

But if A and a be now changed into B and b, or into C and c, the 2nd side will remain the same; thus the 1st must continue constant, and we shall have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \dots (4).$$

In every spherical triangle, the sines of the angles are proportional to the sines of the opposite sides.

According to the property of the supplemental triangle, change, in equ. (3), a into $180 - A$, &c.; and we shall have

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a \dots (5)$$

To eliminate b from the equ. (3), first put for $\sin b$ its value $= \frac{\sin B \sin a}{\sin A}$: let a and A be then replaced in (3) by b and B, whence

$$\cos b = \cos a \cos c + \sin a \sin c \cos B;$$

and these values of $\sin. b$ and $\cos b$ being now substituted in (3), $1 - \sin^2 c$ put for $\cos^2 c$, and the whole divided by the common factor, $\sin a. \sin c$, we have

$$\sin c \cot a = \cos c \cos B + \sin B \cot A \dots (6).$$

The equations 3, 4, 5 and 6 are the foundation of the whole of Trigonometry, and serve for the solution of all triangles, under the condition of the letters $A, B \dots$ being changed, one into the other.

SOLUTION OF SPHERICAL RECTANGULAR TRIANGLES.

601. Let A be the right angle and a the hypotenuse [fig. 7]; if now we make $A = 90°$ in the equations 3, 4, 5, 6, we shall have

$$\begin{aligned} \cos a &= \cos b \cos c \dots (m), \\ \sin b &= \sin a \sin B \dots (n), \\ \cos a &= \cot B \cot C \dots (p), \\ \tan c &= \tan a \cos B \dots (q); \end{aligned}$$

but, we are also at liberty to change B into A, and b into a, and the

converse, in the equ. 5 and 6, otherwise they would cease to be true for any triangle whatever; and after this, making $A = 90°$, there results

$$\cos B = \sin C \cos b \ldots (r),$$
$$\cot B = \cot b \sin c \ldots (s).$$

These six equations are sufficient for the treatment of all the cases in the solution of rectangular triangles: *of the five elements a, b, c, B, C, two are given, and we have to investigate some one of the three others.* Questions of this sort contain three elements, one of them unknown; the angles of the triangle are denoted by A, B, C, A being the right angle, and we must look out, among the above equations, for one that contains the three elements proposed. Only it will sometimes be necessary, in order to arrive at this equation, to transpose the letters B and C. Thus, every problem relative to the solution of rectangular triangles contains three elements, two given, and one unknown. The following are all the different cases that can present themselves:

$$\text{The hypot. } a \text{ and } \begin{cases} \text{two angles } B, C \ldots\ldots\ldots\ldots \text{take equ. } (p), \\ \text{an angle } B \text{ and } \begin{cases} \text{the side } b \text{ opp} \ldots\ldots\ldots\ldots (n), \\ \text{the side } c \text{ adj} \ldots\ldots\ldots\ldots (q), \end{cases} \\ \text{two sides } b, c \ldots\ldots\ldots\ldots\ldots\ldots (m), \end{cases}$$

A side b of the right angle and the angles B, C ....(r),
Two sides b, c of this angle and an angle B opp......................(s).

The frequent use that is made of these formulæ renders it indispensable to have them constantly within our recollection, a task of some difficulty, as several of them are not symmetrical. Mauduit has given a method which, though empirical, is convenient as an aid to the memory. The three elements which enter into a problem being compared in the order in which they present themselves in the figure, as we make the tour of the triangle, it will be found that, if the right angle be left out of consideration, these three arcs are either *successive* or *alternate*. And, provided that the sides of the right angle be changed into their complements, we have, in all the cases, these two equations:

$$\textit{cos of the intermediate arc} = \textit{prod. of the} \begin{cases} \textit{sines of the } \textsc{alternate} \textit{ arcs}, \\ \textit{cot of the } \textsc{contiguous} \textit{ arcs}. \end{cases}$$

It will, in fact, be easily seen that these two conditions lead us again to the six preceding equations.

Our equations demonstrate various general properties which it will be of use to observe in all rectangular triangles.

1°. From the equ. (m) we conclude that any one of the three sides is $<$ or $>$ 90°, accordingly as the other two sides are of similar or different

species. This follows from the circumstance that the cosines of arcs $> 90°$ are negative.

2°. The equ. (p) shows that if the hypotenuse be compared with the two adjacent angles B and C, any one of these three arcs is $<$ or $>$ $90°$, accordingly as the two others are of similar or different species.

3°. The equ. (r or s) prove that each of the angles B and C is always of the same species with the side opposite to it.

4°. From the equ. (q) we conclude that the hypotenuse and a side are of the same species when the included angle is acute, and of different species when that angle is obtuse.

5°. And lastly, if the side b of the right angle $= 90°$, and consequently $\cos b = 0$, the equ. (m and r) gives $\cos a = 0$, $\cos B = 0$; so that the sides CA, CB are also equal to $90°$ and perpendicular to AB: the triangle is isosceles and bi-rectangular; and C is the pole of the arc AB [fig. 7.]

These theorems will come to be applied, when any given triangle is decomposed into two rectangular ones by an arc perpendicular to its base.

602. These formulæ, though they give solutions of all the cases, will be found deficient in point of accuracy, should the unknown quantity be very small and be given by a cosine, or be very near to $90°$ and be given by a sine. In these cases we must proceed as follows:

The equation $\tan^2 \frac{1}{2} x = \dfrac{1-\cos x}{1+\cos x}$ [Vol. i. p. 309] serves to change $\cos x$ into $\tan \frac{1}{2} x$. Thus:

1°. To find the hypotenuse a, the angles B and C being given, the equ. (p) becomes

$$\tan^2 \tfrac{1}{2} a = \frac{1 - \cot B \cot C}{1 + \cot B \cot C} = \frac{\sin B \sin C - \cos B \cos C}{\sin B \sin C + \cos B \cos C},$$

$$\tan^2 \tfrac{1}{2} a = \frac{-\cos (B+C)}{\cos (B-C)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From this equation we conclude that *the sum of the two angles B and C is always* $> 90°$, otherwise the 2nd side could not be positive.

2°. Similarly, to obtain a side b of the right angle, knowing the

angles B and C, the equ. (r) gives $\cos b = \frac{\cos B}{\sin C} = \frac{\sin x}{\sin C}$, assuming $x = 90° - B$; whence [equ. cited and N°. 360]

$$\tan^2 \tfrac{1}{2} b = \frac{\sin C - \sin x}{\sin C + \sin x} = \frac{\tan \tfrac{1}{2}(C-x)}{\tan \tfrac{1}{2}(C+x)},$$

$$\tan \tfrac{1}{2} b = \sqrt{\tan\left(45°+\frac{B-C}{2}\right)\tan\left(\frac{B+C}{2}-45°\right)} \ldots (9).$$

3°. Knowing the hypotenuse a and a side c, to obtain the adjacent angle B, the equ. (q) gives

$$\tan^2 \tfrac{1}{2} B = \frac{1 - \tan c \cot a}{1 + \tan c \cot a} = \frac{\cos c \sin a - \sin c \cos a}{\cos c \sin a + \sin c \cos a},$$

$$\tan^2 \tfrac{1}{2} B = \frac{\sin (a-c)}{\sin (a+c)} \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

We shall observe that the sines of $a-c$ and $a+c$ must have the same sign, in order that the result of this equation may not be imaginary; thus, when $a+c > 180°$, the hypotenuse a is $< c$.

The hypotenuse ceases to be the greatest side, when the triangle has obtuse angles; as will be farther evident from the fig. 10 and N°. 607.

4°. The equ. (m) gives $\cos c = \frac{\cos a}{\cos b}$, whence

$$\tan^2 \tfrac{1}{2} c = \tan \tfrac{1}{2}(a+b) \cdot \tan \tfrac{1}{2}(a-b) \ldots (11).$$

5°. And lastly, if, knowing the opposite angle B and the hypotenuse a, we have to investigate a side b, when b is very near to 90°, instead of at once employing the equ. (n), we must make the following assumptions:

$$b = 90 - 2z, \ \tan x = \sin a \sin B.$$

The equ. (n) then becomes $\cos 2z = \tan x$; whence

$$\tan^2 z = \frac{1-\tan x}{1+\tan x} = \tan (45-x);$$

and thus

$$\tan (45° - \tfrac{1}{2} b) = \sqrt{\tan (45° - x)} \ldots (12).$$

The arc x will be given by the equation above, and we shall then have b from this.

We shall subjoin a statement of a triangle, which will serve for practice in these calculations.

Elements.	Log. sin.	Log. cos.	Log. tan.
$a = 71°\ 24'\ 30''$	$\bar{1}$·9767235	$\bar{1}$·5035475 +	0·4731759 +
$b = 140.\ 52.\ 40$	$\bar{1}$·8000134	$\bar{1}$·8897507 –	$\bar{1}$·9102626 –
$c = 114.\ 15.\ 54$	$\bar{1}$·9598303	$\bar{1}$·6137969 –	0·3460333 –
$B = 138.\ 16.\ 45$	$\bar{1}$·8232900	$\bar{1}$·8728570 –	$\bar{1}$·9504341 –
$C = 105.\ 52.\ 39$	$\bar{1}$·9831068	$\bar{1}$·4370867 –	0·5460201 –

OBLIQUE-ANGLED TRIANGLES.

603. We shall go through the several cases that can present themselves in this theory.

1st Case. *The three sides a, b, c being given, to find an angle A.*

The equ. (3), p. 184, when for cos A we substitute its value $1 - 2 \sin^2 \frac{1}{2} A$, becomes

$$\cos a = \cos (b-c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2} A \ldots (7).$$

This equation is of frequent use; from it, and the equation in the note to N°. 360, which expresses $\cos B - \cos A$, we derive

$$\begin{aligned} 2 \sin b \sin c \sin^2 \tfrac{1}{2} A &= \cos (b-c) - \cos a \\ &= 2 \sin \tfrac{1}{2} (a+b-c) \cdot \sin \tfrac{1}{2} (a+c-b).^{*}, \end{aligned}$$

an equation adapted for logarithmic computation, and which makes known A by means of a, b and c. Assuming, for conciseness, that

$$2p = a + b + c,$$

we have

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (p-b) \cdot \sin (p-c)}{\sin b \sin c};$$

and in like manner, substituting $2 \cos^2 \frac{1}{2} A - 1$ for $\cos A$, in the equ. (3), we find

$$\cos^2 \tfrac{1}{2} A = \frac{\sin p \sin (p-a)}{\sin b \sin c}.$$

* The 1st side being essentially positive, we must on the 2nd have $c < a + b$ along with $c > b - a$, since it would be absurd to suppose that the contrary of these relations existed. We arrive a second time therefore at the theorem 6°, p. 182.

2nd Case. *The three angles A, B, C being given, to find a side a.*

The property of the supplemental triangle N°. 599 being applied to the formulæ just found, the substitution of the values (1) p. 138 gives, making $2P = A + B + C$,

$$\sin^2 \tfrac{1}{2} a = \frac{-\cos P. \cos (P - A)}{\sin B. \sin C},$$

$$\cos^2 \tfrac{1}{2} a = \frac{\cos (P - B) \cos (P - C)}{\sin B. \sin C}.$$

3rd Case. *Two sides a and b, and the included angle C, being given, to find the 3rd side c.*

The question will be satisfied by the equation (3), p. 184, put under the form

$$\cos. c = \cos. a. \cos. b. (1 + \tan. a \tan. b. \cos. C).$$

Neither this formula, however, nor the following one, are adapted to logarithmic computation; we shall shortly resume the consideration of these problems.

4th Case. *Having given two angles C and B, and the adjacent side a to find the third angle A.*

The equ. (5), N°. 600, gives

$$\cos A = \cos B. \cos C (\tan B. \tan C. \cos a - 1).$$

5th Case. *Of two sides and the angles opposite, knowing three of the parts, to find the fourth.* We must make use of the rule (4) [N°. 600] in respect to the four sines.

604. Except when the three sides, or the three angles, are given, every problem in Spherical Trigonometry contains among the given quantities an angle and a side adjacent to it (which in what follows we shall denote by A and b), besides a third element. If, now, from one of the angles, as C [fig. 2β], an arc CD be let fall perpendicularly to the opposite side c, this side will be divided into two segments φ and φ', and the angle C into two angles θ and θ', viz.

$$c = \varphi + \varphi', \; C = \theta + \theta';$$

one of these parts being of course negative in each equation, whenever *the perpendicular falls without the triangle.* But this we know will be the case, when *one of the angles A and B at the base is acute, and the other obtuse; whilst, on the contrary, the perpendicular falls within, when these angles are of the same species.* And, in fact, the triangles ACD, BCD give [equ. p. 185]

$$\tan CD = \tan A. \sin \varphi = \tan B. \sin \varphi';$$

where the sines are essentially positive; and consequently tan A and tan B have the same signs, or A and B are of the same species. But, if the perpendicular fall within the triangle, A and B are the angles at the base; whilst if it fall without, as for the triangle ACB, B is then the supplement of the angle $CB'A$ of the proposed triangle; and the latter angle therefore is of a different species to A, which establishes our theorem.

This exposition being given, we see that the proposed triangle is decomposed into two others, which may be treated separately in order to arrive at the elements required.

605. The several cases are comprised in the formulæ below; of which (1) and (2) are deduced from the equation (q and p); 5, 6, 7 and 8 are found by deriving from each rectangular triangle [fig. 2, β], by means of the equations (m, r, s and q), the values of the perpendicular arc CD, and equating them two and two; viz.

$$\tan\varphi = \tan b\cos A \ldots (1) \qquad \cot\theta = \tan A\cos B \ldots (2)$$

$$c = \varphi + \varphi' \ldots\ldots\ldots (3) \qquad C = \theta + \theta' \ldots\ldots\ldots (4)$$

$$\frac{\cos a}{\cos b} = \frac{\cos\varphi'}{\cos\varphi} \ldots\ldots\ldots (5) \qquad \frac{\cos A}{\cos B} = \frac{\sin\theta}{\sin\theta'} \ldots\ldots\ldots (6)$$

$$\frac{\tan A}{\tan B} = \frac{\sin\varphi'}{\sin\varphi} \ldots\ldots\ldots (7) \qquad \frac{\tan A}{\tan B} = \frac{\cos\theta}{\cos\theta'} \ldots\ldots\ldots (8)$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \ldots\ldots\ldots (9).$$

According to the case, φ or θ must be deduced from the equation (1) and (2); regard being had to the signs of the trigonometrical lines, these arcs φ and θ will receive a determinate sign; and we must introduce their values with the appropriate sign in the subsequent equations.

The following is the detail of the different cases:

Besides *the given elements b and A*,

1°. If we know c *(two sides, and the included angle, b, c, A)*, the equation (1) gives φ, (3) φ', which may be negative, (5) a, (7) b, (9) C, the species of which is otherwise known [N°. 604].

2°. If we have C *(two angles and the adjacent sides A, C, b)*: the equation (2) gives θ, (4) θ', which may be negative, (6) B, (8) a, (9) c, known in species.

3°. When we have a *(two sides and an opposite angle, b, a, A)*, equ. (1) gives φ, (5) φ', (3) c, (7) and (9) B and C.

Or, otherwise, (2) gives θ, (8) θ', (4) C, (6) and (9) B and c.

The problem will, in general, have two solutions; for φ' or θ' being

determined by means of its cosine, the arc has the double sign $\pm$; and c and C will therefore have two values, unless we find reason to reject one as negative: ϕ' and θ' enter into (7) and (6) in terms of their sines; which correspond to two values of B; and similarly for C and c.

4°. If we have B (*two angles and an opposite side, A, B, b*), the equ. (1) gives ϕ, (7) ϕ', (3) c, (5) and (9) a and C.

Or, otherwise, (2) gives θ, (6) θ', (4) C, (8) and (9) a and c.

There are again two solutions; for ϕ' or θ' is given by a sine, the arc of which has two values supplemental to each other; thus c in (3), or a in (8), receives two values; and similarly for a or C in (5) and (4), &c....

It will be observed that, in each of these cases, it is requisite to employ only the equations of the even, or of the odd orders, respectively; when we have the choice of these systems, we must of course adopt the one which leads to the most simple results.

606. The following are some important consequences from what has preceded:

1°. The equation (5) gives $\dfrac{\cos b - \cos a}{\cos b + \cos a} = \dfrac{\cos \phi - \cos \phi'}{\cos \phi + \cos \phi'}$;

and since $c = \phi + \phi'$, we have, by virtue of the equations of the note to N°. 360,

$$\tan \tfrac{1}{2}(\phi' - \phi) = \tan \tfrac{1}{2}(a + b) . \tan \tfrac{1}{2}(a - b) \cot \tfrac{1}{2} c \ldots (10).$$

The three sides being known, this equation gives the difference of the segments, whence we have the segments themselves; and then the angles A and B, by the solution of the rectangular triangles ACD, BCD, which give

$$\cos A = \tan \phi . \cot b, \ \cos B = \tan \phi' . \cot a \ldots (11).$$

2°. The equation (6), treated in the same manner, gives

$$\tan \tfrac{1}{2}(\theta' - \theta) = \tan \tfrac{1}{2}(A + B) . \tan \tfrac{1}{2}(A - B) . \tan \tfrac{1}{2} C \ldots (12).$$

When the three angles are given, this equation makes known θ and θ'; and we have thence the sides a and b in the triangles BCD, ACD,

$$\cos b = \cot \theta . \cot A, \ \cos a = \cot \theta' . \cot B \ldots (13).$$

3°. The equation (8) in like manner gives

$$\tan \tfrac{1}{2}(\theta' - \theta) = \frac{\sin (a - b)}{\sin (a + b)} \cot \tfrac{1}{2} C \ldots (14).$$

Knowing two sides a, b, and the included angle C, we shall have θ and θ'; and then A and B from the equation (13).

4°. The equ. (7) gives

$$\tan\tfrac{1}{2}(\varphi'-\varphi)=\frac{\sin(A-B)}{\sin(A+B)}.\tan\tfrac{1}{2}c\ldots\ldots(15).$$

When therefore we have two angles A, B, and the adjacent side c, this equation gives φ and φ'; and the equ. (11) make known a and b.

One of the chief uses of the equations just found is the demonstration they afford of the *analogies of Napier.* Equating the values 10 and 15, and also 12 and 14 (which comes to the same thing with eliminating φ and φ' between 5 and 7, or θ and θ' between 6 and 8); we shall have, considering that $\sin 2\alpha=2\sin\alpha.\cos\alpha$,

$$\tan\tfrac{1}{2}(a+b)\cdot\tan\tfrac{1}{2}(a-b)=\tan^2\tfrac{1}{2}c.\frac{\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B).\cos\frac{1}{2}(A+B)},$$

$$\tan\tfrac{1}{2}(A+B).\tan\tfrac{1}{2}(A-B)=\cot^2\tfrac{1}{2}C.\frac{\sin\frac{1}{2}(a-b).\cos\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b).\cos\frac{1}{2}(a+b)}.$$

But the equation (9) gives

$$\frac{\sin a+\sin b}{\sin a-\sin b}=\frac{\sin A+\sin B}{\sin A-\sin B};$$

whence

$$\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}(a-b)}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

Now let each of the two equations above have their sides multiplied and divided by the corresponding sides of this last; then all those factors which are not destroyed will appear in the square; and taking the root, there will result the following equations, as though each of the first had been decomposed into two factors:

$$\tan\tfrac{1}{2}(a+b)=\tan\tfrac{1}{2}c.\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)},^{*}$$

$$\tan\tfrac{1}{2}(a-b)=\tan\tfrac{1}{2}c.\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)},$$

$$\tan\tfrac{1}{2}(A+B)=\cot\tfrac{1}{2}C.\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}$$

$$\tan\tfrac{1}{2}(A-B)=\cot\tfrac{1}{2}C.\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$$

Such are the celebrated *Analogies of Napier.* They are chiefly of use for obtaining at the same time two sides or two angles of a triangle, of

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* Since $\tan\frac{1}{2}c.\cos\frac{1}{2}(A-B)$ is a positive quantity, it follows that $\tan\frac{1}{2}(a+b)$ and $\cos\frac{1}{2}(A+B)$ have the same signs. Hence, *the semi-sum of any two angles is always of the same species with that of the opposite sides, and the converse.*

which we know two angles and the adjacent side, or two sides and the included angle.

Isosceles triangles. Let C and B be the equal angles, c and b the equal sides, A the angle at the vertex, and a the base; the arc AD, which bisects BC, gives two equal rectangular triangles [N°. 598, 5°.], in which we have

$$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2} A . \sin b \ldots\ldots (n),$$
$$\tan \tfrac{1}{2} a = \tan b . \cos B \ \ldots\ldots (q),$$
$$\cos b = \cot B . \cot \tfrac{1}{2} A \ldots\ldots (p),$$
$$\cos \tfrac{1}{2} A = \cos \tfrac{1}{2} a . \sin B \ldots\ldots (r).$$

These relations are formed of the combinations 3 and 3 of the four elements A, B, a, b.; and they will make known any one of these arcs, when two others are given. Thus, *of these four elements of an isosceles triangle, the angle A at the vertex, the base a, the arc b of the equal sides, the arc B of the equal angles, any two being given, we can always find the two others.*

AMBIGUOUS CASE OF SPHERICAL TRIANGLES.

607. Spherical triangles result from the section of a sphere by three planes which pass through the centre. The fig. 10 has the great circle $MKmf$ for its *base*, and is intended to represent the hemisphere produced by one of these planes; the two others are $AC\alpha$ and BC, which cut each other in the radius CO, and determine the triangle ABC.

Every plane, such as $AC\alpha$, cuts our hemisphere in a semi-circumference; and the arcs CA, $C\alpha$ are supplemental to each other: the angle $A = \alpha$ is the inclination of this plane to the base. Let the plane MCm be drawn through the radius CO, and perpendicular to the base MKm; then taking $MA' = MA$ on the other side of this plane, we obtain a 2nd plane $A'C\alpha'$ symmetrical to $AC\alpha$; and we have

$$m\alpha = m\alpha',\ AC = A'C,\ C\alpha = C\alpha',\ A = A' = \alpha = \alpha'.$$

If now the plane $AC\alpha$ be turned about the radius CO, this plane will be perpendicular to the base in the position MCm, and will then assume different inclinations to it in B, A, $K\ldots$, forming angles supplemental to each other on its opposite sides. The arcs CB, CA, $CK\ldots$ go on increasing as they separate from the perpendicular arc $CM = \psi$, which is the least of all, till we arrive at Cm, which is the greatest; for the rectangular triangle ACM, in which $CA = b$, gives $\cos ACM = \cot b . \tan \psi$, and the factor $\tan \psi$ is constant. When the angle ACM becomes a right angle, as for the arc CK, the plane of which is perpendicular to MCm, we have $\cot b = 0$, and the arc CK is one of 90°. The plane still continuing to turn, $\cos ACM$ becomes negative, and

increases as also cot b; and the arcs CA, CK, Ca' still go on increasing. In the course of this revolution, the cutting plane becomes more and more inclined to the base, as it assumes the positions CB, CA, CK; for the rectangular triangle ACM also gives

$$\sin \psi. = \sin b. \sin A... \; (16),$$

an equation in which the 1st side is constant whilst, as we have just seen, $\sin b$ at first goes on increasing; and $\sin A$ therefore at the same time decreases. But, after b has reached 90°, the arc then having the position CK perpendicular to CM, $\sin b$ diminishes; consequently $\sin A$ increases, the angle A, acute to the base, has passed through its least value k, and again begins to increase. This point K is the pole of the arc MCm; we have $CK = MK = MK' = 90°$ [N°. 601, 5°]; the angle K is measured by CM, or $K = \psi$ on one side, and by CM or $K = 180° - \psi$ on the other side of the plane CK perpendicular to CM.

It appears, therefore, that all the arcs starting from C and terminating in some point of the semi-circle $K'MK$ are $< 90°$; whilst the others from K through m to K' are $> 90°$; the least of all is $CM = \psi$, the greatest $Cm = 180° - \psi$; and every arc CA, CK, Ca'... is comprised between these limits. The nearer an arc approaches to CM, the less it is; and the contrary is the case for Cm; the intermediate arc CK is of 90°.

The inclination of the planes to the base diminishes, from 90° its value in CM, as it assumes the successive positions CB, CA... onwards to CK, where it becomes $K = \psi$; and then increases for the positions beyond K, till it again becomes 90° in Cm. The obtuse angles, on the other side of the plane, are supplemental to those we have been speaking of, and have $180° - \psi$ for their limit. The angle is acute when it opens towards the smaller perpendicular arc M, and obtuse when its direction is towards Cm.

From this, it will be easy to perceive whether, for any given triangle BCA, the arc perpendicular to the base AB falls within or without the triangle [see N°. 604]; and we also have a verification of the corollaries of N°. 601, as to the relations in point of magnitude of the sides and the angles of rectangular triangles.

608. The problems which give rise to double solutions are those in which an angle and the side opposite appear among the given elements.

Suppose that *two sides a and b and the opposite angle A are given.* The hemisphere $MKmf$ being cut by a plane $AC\alpha$ passing through the centre O, and inclined to the base at the angle A, if we take $AC = b$, C will be the vertex of the triangle, which will be closed by an arc $CB = a$. Let us examine into the circumstances of this triangle.

Suppose first that b is $< 90°$: if then A be acute and also $a < b$, CA

being the side b, the other side a may fall any where within the space $A'CA$ (making $MA' = MA$); thus there will be *two solutions*, such as ACB and ACB': and if the angle A be obtuse, b being throughout $< 90°$, there will still be two solutions, provided that $a > C\alpha'$ or $C\alpha = 180° - b$, since two equal arcs may be placed between $C\alpha$ and $C\alpha'$. But there will be only *one solution*, if the arc a fall within the space $\alpha CA'$ or $\alpha' CA'$, whether A be acute or obtuse; and this will be the case when the length of the arc a is intermediate to $C\alpha$ and CA', *i. e.* to b and $180° - b$.

Let us now assume $b > 90°$, as the arc $C\alpha$, and we shall in like manner see that there are *two solutions* when the bounding side a falls within either of the spaces $A'CA$ or $\alpha C\alpha'$, and but *one* if it fall within $\alpha CA'$ or $\alpha' CA$; the latter of which cases requires that a be again between b and $180° - b$. It will moreover be seen that when the side a falls in either of these latter spaces, the angle B is of the same species with the side b which is opposite to it.

From this we shall conclude that *when two sides b and a are given, with an opposite angle A, there are in general two solutions; but that one only is admissible when a is comprised between b and* $180° - b$; in which case B and b are of the same species. But, we have seen [N°. 604] that the perpendicular let fall from the vertex C on the base c is situated within the triangle when the angles A and B are of the same species, and without it in the contrary case. Hence therefore we shall know whether the base c is $\phi + \phi'$ or $\phi - \phi'$, and whether C is $\theta + \theta'$ or $\theta - \theta'$; and the analysis of the 3rd case of N°. 605 is rendered complete by this consideration, which enables us to decide on the solution that ought to be preferred.

Thus, when it is proposed to solve a triangle, of which a, b and A are known, there will in general be two solutions; but the side a must be compared with the arcs b and $180° - b$; and if a be found to be intermediate to these values, we shall have but one solution; B and b are of the same species, and we shall be able to determine what is the position of the arc perpendicular to the base.

It may also happen that the elements given are such that they cannot be brought to form any triangle. If the side CA or $C\alpha$ make an acute angle A with the base, the other side must necessarily exceed $CM = \psi$, whilst at the same time it must not be greater than $C\alpha$, since we should thus pass to the other side of the plane $AC\alpha$, and the acute angle A would no longer form part of the triangle; and if A be obtuse, the side a cannot be $< b$, nor exceed $CM = 180° - \psi$, for the same reason: ψ has been already determined by the equation (16). Hence *the triangle is impossible*, when we have

$A < 90°$ *with* $a < \psi$ *or* $>$ *that of the arcs b and $180° - b$ which is obtuse,*

or $A > 90°$ *with* $a > 180° - \psi$ *or* $<$ *that of the arcs* b *and* $180-b$ *which is acute.*

These circumstances do not require any special calculation, the impossibility becoming manifest of itself from the operations.

609. Let us now proceed to the case in which are given *two angles* A *and* B *with the opposite side* b. The hemisphere [fig. 10] having been cut by a plane ACa inclined to the base at the angle A, we take $AC = aC = b$; and then through the point C draw another plane BC inclined at the angle B to the base, which completes the triangle.

We shall examine the different cases.

When the bounding side a falls within the angle aCA', it may be situated on either side of the plane $K'CK$, as Cf and Cf', there being two planes which make the same angle B with the base; whilst, on the contrary, if the side a fall within the angle ACA', there is but one solution. Reasoning as before we shall see that A being acute or obtuse, and whether b be $<$ or $>$ 90°, it is when the side a falls within the angle $a'CA$ or aCA that there are two solutions; whilst in the angles aCa' or ACA', there is but one; which is the reverse of what took place in the preceding case. It is at the same time evident that, when there is but one solution, the angle B is intermediate to A and $180° - A$.

Hence, *when two angles* B *and* A *are given, with the opposite side* b, *there are in general two solutions; but one only is admissible when* B *is comprised between* A *and* $180° - A$. In this case the unknown side a is of the same species as A; also the perpendicular arc let fall from the vertex C on the base lies within or without the triangle [N°. 604], accordingly as B and A are of the same or different species; so that we shall be able to make a proper choice of signs in $c = \varphi \pm \varphi'$, $C = \theta \pm \theta'$.

Thus, when we wish to solve a triangle in which A, B and b are known, we must compare B with A and $180° - A$, since, if B be intermediate to those values, there will be but one solution: A and a will be of the same species, which will inform us as to the position of the arc perpendicular to the base.

There are also cases in which *the triangle is impossible;* and we shall see, as before, that this happens when we have

$b < 90°$ *with* $B < \psi$, *or* $>$ *the greatest of the arcs* A *and* $180° - A$,
or $b > 90°$ *with* $B > 180° - \psi$, *or* $<$ *the least of the arcs* A *and* $180 - A$.

The calculation itself, however, evidences the absurdity, by giving sines or cosines > 1, and it is not necessary to go through any special process for the purpose of recognising the existence of the present case.

610. *When the triangle proposed is rectangular*, one of the sides has the position CM or Cm; and if an angle and the side opposite to it be given, there are two solutions, reducible in certain cases to a single one.

1°. *The hypotenuse a and a side b being given, to find the opposite angle B, and the converse.* The equ. (n) [N°. 601] gives b or B in the value of a sine, which corresponds to two angles supplementary to each other. Nevertheless there will be but one solution, since the two arcs CA, or CA', which complete the triangle CMA or CMA', are symmetrical. Thus B and b are of the same species [N°. 601, 3°], and there is no longer any ambiguity.

2°. A side b of the right angle and the opposite angle B being given, the 3rd part sought admits of two values; for if the hypotenuse a be required, the equ. (n) gives $\sin a$; if we are in search of the third side c, the equ. (s) gives $\sin c$; and lastly, to obtain the angle C adjacent to the given side b, we have $\sin C$ from equ. (r). Thus the unknown element has two supplemental values for the angle corresponding to this sine.

611. We subjoin some numerical applications.

I. Let $a = 133° 19'$, $b = 57° 28'$, $A = 45° 23'$: the triangle is impossible, since a exceeds $180° - b = 122° 32'$, and A is an acute angle.

II. The same conclusion must be drawn if we have $A = 120°$, $B = 51°$, $b = 101°$; for we find $B < 180° - A$ or $60°$, at the same time that $b > 90°$.

III. Let $b = 40° 0' 10''$, $a = 50° 10' 30''$, $A = 42° 15' 14''$: there is but one solution, since A, B and b are $< 90°$; the perpendicular falls within the triangle, ϕ and ϕ' are positive, and c is the sum of these arcs:

$\tan b$...	$\bar{1}{\cdot}9238563$	$\cos a$...	$\bar{1}{\cdot}8064817$	$\phi =$	$31° 50' 46''$
$\cos A$...	$\bar{1}{\cdot}8693330$	$\cos \phi$...	$\bar{1}{\cdot}9291471$	$\phi' =$	$44.44.50$
$\tan \phi$...	$\bar{1}{\cdot}7931893$ —	$\cos b$...	$\bar{1}{\cdot}8842363$	$c =$	$76.35.36$
		$\cos \phi'$...	$\bar{1}{\cdot}8513925$		

To find the angle C at the vertex

$\cos b$...	$\bar{1}{\cdot}8842363$	$\tan b$...	$\bar{1}{\cdot}9238563$	$\theta =$	$55° 9' 59''$
$\tan A$...	$\bar{1}{\cdot}9583058$	$\cot a$...	$\bar{1}{\cdot}9211182$	$\theta' =$	$66.26.21$
$\cot \theta$...	$\bar{1}{\cdot}8425421$	$\cos \theta$...	$\bar{1}{\cdot}7567851$	$C =$	$121.36.20$
		$\cos \theta'$...	$\bar{1}{\cdot}6017596$		

Lastly, the rule of the four sines gives $B = 34°15'3''$.

IV. For $B = 42°15'14''$, $A = 121°36'20''$, $b = 50°10'30''$, there are two solutions, B being $< 180° - A$ or $58°23'40''$, and $b < 90°$.

$\cos b \ldots \bar{1}\cdot8064817$
$\tan A \ldots 0\cdot2108864 -$
$\cos \theta \ldots 0\cdot0173681 -$
$\theta = -43°51'16''$
$\theta' = \quad 78.6.19$ or $101.53.41$
$C = \quad 34.15.3$ or $58.2.25$

$\cos B \ldots \bar{1}\cdot8693330$
$\sin \theta \ldots \bar{1}\cdot8406262 -$
$\cos A \ldots \bar{1}\cdot7193880 -$
$\sin \theta' \ldots \bar{1}\cdot9905712 -$

$\sin b \ldots \bar{1}\cdot8853636$
$\sin A \ldots \bar{1}\cdot9302745$
$\sin B \ldots \bar{1}\cdot8276379$
$\sin a \ldots \bar{1}\cdot9880002$
$a = 76°\,35'\,36''$
or $= 103.24.24$

$\tan b \ldots 0\cdot0788818$
$\cos A \ldots \bar{1}\cdot7193874 -$
$\tan \phi \ldots \bar{1}\cdot7982692 -$
$\phi = -32°\,8'\,50''$
$\phi = \quad 72.9.0$ or $107°\,51'$
$c = \quad 40.0.10$ or $75.42.10$

$\cot B \ldots 0\cdot0416956$
$\tan A \ldots 0\cdot2108873 -$
$\sin \phi \ldots \bar{1}\cdot7259905 -$
$\sin \phi' \ldots \bar{1}\cdot9785734 +$

One of these two solutions gives us the triangle of Ex. III; for one we have the triangle fCA' [fig. 10], for the other $f'CA'$.

For practice in this species of calculation, we shall give the several elements of a spherical triangle, its angles, sides, and segments.

The questions may be varied by taking for the given elements the arcs which correspond to the several cases that have been successively discussed.

Arcs.		Log sin.	Log cos.	Log tan.
$A =$	$121°36'19''81$	$\bar{1}\cdot9302747$	$\bar{1}\cdot7193874 -$	$0\cdot2108873 -$
$B =$	$42.15.13\cdot66$	$\bar{1}\cdot8276379$	$\bar{1}\cdot8693336$	$\bar{1}\cdot9583044$
$C =$	$34.15.\ 2\cdot76$	$\bar{1}\cdot7503664$	$\bar{1}\cdot9172860$	$\bar{1}\cdot8330804$
$a =$	$76.35.36$	$\bar{1}\cdot9880008$	$\bar{1}\cdot3652279$	$\bar{0}\cdot6227729$
$b =$	$50.10.30$	$\bar{1}\cdot8853636$	$\bar{1}\cdot8064817$	$0\cdot0788819$
$c =$	$40.\ 0.10$	$\bar{1}\cdot8080926$	$\bar{1}\cdot8842363$	$\bar{1}\cdot9238563$
$\phi =$	$-\ 32.\ 8.50$	$\bar{1}\cdot7259905 -$	$\bar{1}\cdot9277212$	$\bar{1}\cdot7982692 -$
$\phi' =$	$72.\ 9.\ 0$	$\bar{1}\cdot9785741$	$\bar{1}\cdot4864674$	$0\cdot4921067$
$\theta =$	$-\ 43.51.16\cdot2$	$\bar{1}\cdot8406262 -$	$\bar{1}\cdot8580013$	$\bar{1}\cdot9826249 -$
$\theta' =$	$78.\ 6.19$	$\bar{1}\cdot9905733$	$\bar{1}\cdot3141056$	$0\cdot6764677$
We have for the perpendicular arc				
$P =$	$40\cdot51\cdot3$	$\bar{1}\cdot8156385$	$\bar{1}\cdot8787600$	$\bar{1}\cdot9368784$

II. SURFACES AND CURVES OF DOUBLE CURVATURE.

GENERAL PRINCIPLES.

612. To fix the position of a point M [fig. 11] in space, we take three axes Ax, Ay, Az (which for greater simplicity we shall suppose to be rectangular), and through these lines draw the planes zAx, zAy, xAy; we then give the distance PM, or $z = c$, from the point to its projection P on one of these planes, as also the projection itself, and consequently the co-ordinates AN, AS of the point P, or $x = a$, $y = b$: the given quantities a, b, c, are in fact the distances MQ, MR, MP, from the point to the three planes; and these straight lines complete the parallelipiped QN.

Considering that besides the trihedral angle $zAxy$, the three co-ordinate planes form seven other such angles, we shall readily see that the absolute position of M in space is not fixed by the lengths of a, b, c, unless we also introduce the principle of signs [N°. 340]. Thus, below the plane xAy, conceived to be indefinite in extent, the values of z are negative; if the point be on the left of the plane zAy, towards x', x is negative; whilst y is so behind the plane zAx.

613. Suppose that between the three co-ordinates x, y, z, we have an equation, such as $f(x, y, z) = 0$. This equation will be indeterminate; but if, for two of the variables, we assume certain values $x = a = AN$, $y = b = PN$ [fig. 11], it will then give, for z, at least one root $z = c$. If this root c be real, we must erect at P the perpendicular $PM = c$ to the plane yAx, and the point M in space will thus be determined.

Now let the values assumed for the arbitrary quantities x and y be changed, *i. e.* let points P be taken *ad libitum* in the plane xAy; we shall derive from the equation as many corresponding values of z, and as many points M; and these points will all lie in a surface, which we may imagine to be formed by their being united together, and a continuity established between them. This surface may, for instance, be a cone, a cylinder, a sphere, &c.; and $f(x, y, z) = 0$ will be *the equation of the surface*, since it distinguishes the several points in it from the rest of space. If z have several real roots, the surface will have several sheets; whilst, if z be imaginary, the indefinite perpen-

dicular erected at P on the plane xy will not meet the surface. If, having taken a fixed value of y, as $y = b = AS$, we make x vary, the ordinate $PM = z$ will move along SP parallel to the plane xz, and the variations that it undergoes will be determined by $f(x, b, z,) = 0$, which is consequently the equation of the intersection of the surface by the plane SM, the variables being the two co-ordinates x and z, measured in the plane $QMPS$. Similarly, making $x = a$, or $z = c$, we have the intersections of the surface by the planes MN, or QR, parallel to yz or to xy: $z = 0$ is evidently the equation of the plane xy, $z = c$ that of a plane which is parallel to it, and at a distance c from it; $x = 0$ is the equation of the plane yz, $x = a$ that of the plane drawn parallel to it at the distance a.

614. The rectangular triangle AMP gives $z^2 + AP^2 = AM^2$; and since from APN we find $AP^2 = x^2 + y^2$, we have, making $AM = R$,

$$x^2 + y^2 + z^2 = R^2.$$

Hence,

1°. The distance of a point from the origin is the root of the squares of the three co-ordinates of this point.

2°. If x, y, z are variable, this equation will characterise all the points in space, the distance of which from the origin is the same and $= R$; and it consequently is *the equation of the sphere* which has R for its radius and the origin for its centre.

Taking two points, the one $N(x, y, z)$, the other $M(x', y', z')$, [fig. 12], if n and m be their projections on the plane xy, mn will be that of the line $MN = R$. But [N°. 373], we have

$$mn^2 = (x-x')^2 + (y-y')^2:$$

also, MP being parallel to mn, the triangle MNP will be rectangular at P; whence $MN^2 = MP^2 + PN^2 = mn^2 + PN^2$; and since $PN = Nn - Mm = z - z'$, there results

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = R^2.$$

R is the distance between the points (x, y, z), (x', y', z')*; and if we

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* Since mB, nC [fig. 12], parallels to Ay, give $BC = x - x'$ = the projection of MN on the axis of x, it appears that *the length of a line in space is the square root of the sum of the squares of its projections on the three axes.*

We also have $MP = MN. \cos NMP$; so that the projection mn is the product of the line projected by the cosine of its inclination; and conversely a line in space

consider x, y, z as variable, *this equation is that of a sphere of radius R, and having its centre situated in the point M (x', y', z').*

615. Let our surface be a right cylinder with any base [N°. 287]; this base will be a curve given on the plane xy by its equation $f(x, y) = 0$. Assigning to x and y values which satisfy this equation, the point of the plane xy, that these co-ordinates determine, is one of those of the curve which serves for the base of the cylinder; and the indefinite perpendicular z, erected at this point on the plane xy, is a generating line of the body: thus, whatever value we assign to z, and at whatever point we terminate it, the extremity of this perpendicular will be on the surface of the cylinder. Hence, *the equation of the surface of a right cylinder is that of its base,* or $f(x, y) = 0$.

If the generating line of the right cylinder be perpendicular to the plane of xz, the equation of its surface is that of the base traced on this plane, &c.

The same reasoning proves that the equation of a plane perpendicular to one of the co-ordinate planes is that of its *Trace* on the latter, *i. e.* of the line of intersection of the two planes. Thus, let $AB = \alpha$ [fig. 13] and $a = \tan CBI$; then $x = az + \alpha$, which is the equation of the line BC in the plane zAx, is also that of the plane $FEBC$, perpendicular to zAx, and passing through BC.

616. Let $M = 0$, $N = 0$ be the equations of two surfaces; these equations respectively indicate those points of space which belong to one or other of the surfaces; so that their simultaneous existence corresponds to the line in which these surfaces cut one another. Hence, *a point is determined by three equations between the co-ordinates x, y, z of the point; a surface by a single equation; a curve by two, which are those of the surfaces that, by their intersection, determine this line.* Since an infinite number of surfaces may be made to pass through a given line, it will be seen that the same curve in space has an infinite number of equations.

If z be eliminated between $M = 0$ and $N = 0$, we shall arrive at a third equation $P = 0$ in x and y; and this will be the equation of a right cylinder which cuts our two surfaces in the curve in question, and also the equation of the projection [N°. 272] of this curve on the plane of xy. Similarly, eliminating y, we shall have the equation $Q = 0$ of the

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is the quotient of its projection on a plane divided by the cosine of the angle which it makes with this plane. These theorems may also be extended to plane areas situated in space.

projection on the plane xz, or that of the projecting cylinder. $P=0$, $Q=0$ are the equations of our two cylinders, which may be substituted for the given surfaces; they are also the equations of the projections of the curve, and those of the curve itself; and consequently, *we may take for the equations of a curve those of its projections on two of the co-ordinate planes.*

617. Applying these principles to the straight line; we may take for its equations those of any two planes, in each of which it is contained; at the same time, it will be proper to select those which furnish the most simple results. The axis of z has for its equations $x=0$, $y=0$, which are those of the planes yz and xz. Similarly, $x=\alpha$, $y=\beta$ are the equations of a straight line PM [fig. 11] parallel to z, and of which the foot P, in the plane xy, has $x=\alpha$, $y=\beta$ for its co-ordinates. The same reasoning may be applied to the other axes; thus $x=0$, $z=0$ are the equations of that of y, &c...

EF being any straight line in space [fig. 13], draw through it a plane $FEBC$ perpendicular to the plane xz; BC will be its projection on the latter plane [N°. 272]. Let EF be similarly projected into HG on the plane yz; then the equations of these projections, or of the projecting planes, are those of the straight line EF:

$$x=az+\alpha,$$

$$y=bz+\beta.$$

It will be easily seen that α and β are the co-ordinates AB, AG, of the point E in which the straight line EF meets the plane xy, and that a and b are the tangents of the angles which its projections BC, HG, make with the axis Az. Eliminating z, we obtain the equation of the projection on the plane xy,

$$ay=bx+a\beta-b\alpha.$$

618. If the straight line EF [fig. 13] pass through a given point $F(x',y',z')$, the projections C and H of this point are situated in those of the line; and hence the equations are [N°. 369]

$$x-x'=a(z-z'),$$

$$y-y'=b(z-z').$$

The values of a and b will readily be determined when the straight line is to pass through a second point (x'',y'',z'').

When the straight line passes through the origin A, its equations are

$$x = az,\ y = bz.$$

It will be easily seen that the projections of two parallel straight lines on the same plane are themselves parallel [N°. 268]; whence the equations of these lines must have the same coefficients a and b for z, and differ only as to the values of the constants α and β.

EQUATIONS OF THE PLANE, CYLINDER, CONE, &c.

619. Whatever be the conditions which determine the nature of a surface, they always reduce themselves, in the higher analysis, to giving the law of its generation; which consists in a *generating* curve, variable or constant in form, moving along one or more given lines, called the *Directrices*. The equation of the surface generated is obtained by the same course of reasoning as that of N°. 426; we shall proceed to give some examples, commencing with the plane.

A plane DC [fig. 14] may be conceived to be generated by a straight line, which slides along two others that cross each other. Let the traces of this plane on those of xz, yz, be BC, BD, meeting in B on the axis of z; making $AB = C$, these lines have for equations,

$$\begin{aligned} &BC \ldots\ y = 0,\ z = Ax + C, \\ &BD \ldots\ x = 0,\ z = By + C \ldots\ (1). \end{aligned}$$

But this plane BDC might also be generated by the trace BC moving parallel to itself along BD; a fact which must now be expressed in terms of analysis.

Let EF be any straight line in space parallel to BC; the projecting plane $EHIF$ will be parallel to zx, and HI to Ax: also, the projection of EF on the plane zx will be parallel to BC; so that the equations of EF will be

$$y = \alpha,\ z = Ax + \beta \ldots\ (2).$$

To obtain the locus E of intersection of EF with the directrix BD, let x, y, z be eliminated between the four equations (1) and (2); and there will result the equation of *condition*

$$\beta = B\alpha + C \ldots\ (3),$$

which expresses that the lines BD and EF cut each other. And if, therefore, we assign to α and β values that satisfy this condition, we may rest assured that the equ. (2) are those of the generating line in one of its positions.

Thus, supposing that in (2) we substitute for β its value $B\alpha + C$, these equations will be those of some one of the generating lines, the position of which will depend on the value assigned to the arbitrary quantity α. And hence we conclude that if α be eliminated between these, *i. e.* α and β eliminated between the three equations (2) and (3), the resulting equation

$$z = Ax + By + C$$

will be that of the plane; since x, y, z represent the co-ordinates of the several points of any generating line whatever. C is the value of z at the origin, or AB; A and B are the tangents of the angles that the traces BC, BD of the plane, on those of xz and yz, make with the axes of x and y.

If C alone be made to vary, the plane moves parallel to itself, its traces still continuing parallel [N°. 268]. Hence

1°. Every equation of the 1st degree is that of a plane.

2°. Any two equations of the 1st degree are those of a straight line.

3°. When the equation of a plane is given, the equations of its traces on the planes xz, yz, and xy are obtained, by successively making $y=0$, $x=0$, $z=0$; which are the equations of those planes. Thus, $Ax + By + C = 0$ is the equation of the trace of the plane on that of xy.

Any other straight line in space might have been taken for the generatrix, and made to move along the traces; and the calculation, though more complicated, would have led to the same result. The investigation under this form will serve for additional practice.

620. Similar reasoning will serve for finding the equation of the *Cylinder*. Let $M = 0$, $N = 0$ be the equations of the given curve in space, on which the generating line is required to move, always continuing parallel to itself [N°. 287]; also, let

$$x = az + \alpha,\quad y = bz + \beta \ldots (1)$$

denote the equations of a parallel to the generatrix, a and b being known, α and β dependent on the position of this straight line. In order, then, that this parallel may cut the directrix, our four equations must be coexistent; *i. e.* if x, y, z be eliminated between them, α and β must be such, that the final equation $\beta = F\alpha$ shall be satisfied. Thus, if in (1) we put $F\alpha$ for β, these two equations will be those of some one of the generating lines, the position depending on the value of α; and if α be next eliminated between them, we shall have an equation between x, y and z, which will hold good for any generatrix whatever; and this consequently will be the equation required.

Hence we conclude that to find the equation of a cylindrical surface, we must eliminate x, y and z between the equations (1) and those $M=0$, $N=0$ of the directrix; and then, in the resulting equation of condition $\beta = F\alpha$, substitute $x - az$ for α and $y - bz$ for β; so that the equation of the cylinder is of the form $y - \beta z = F(x - az)$, the form* of the function F depending on the nature of the directrix [see N^{os}. 705, 879].

If, for example, the base be a circle of radius r, described in the plane xy, and placed similarly to that of fig. 18, the diameter AE being on the axis of x, and the origin in A, the equations of the directrix are $z = 0$, $y^2 + x^2 = 2rx$; and eliminating x, y, z between these and the equations (1), there results $\beta^2 + \alpha^2 = 2r\alpha$, for the equation of condition.†

Thus,

$$(y - bz)^2 + (x - az)^2 = 2r(x - az)$$

is the *equation of the oblique cylinder with a circular base;* the direction of the axis giving the values of a and b. If this axis be in the plane xz, we have $b = 0$; and consequently,

$$y^2 + (x - az)^2 = 2r(x - az).$$

Lastly, if the centre of the circle be situated at the origin, we have only to replace the 2nd side by r^2.

621. Let

$$M = 0,\ N = 0 \ \ldots\ldots\ldots\ (1)$$

be the equations of the directrix of a *conical* surface [N°. 289]. The co-ordinates of the vertex being a, b, c, every straight line that passes through this point has for its equation [N°. 618],

$$x - a = \alpha(z - c),\ y - b = \beta(z - c) \ \ldots\ldots\ (2).$$

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* The symbols $Fx, fx, \phi x \ldots$ serve to denote different functions of x; and indicate formulæ into which the same quantity x enters, but combined in different ways with the given quantities. On the other hand, fx, fz are the same function of two different quantities x and z; so that if in the latter z should be changed into x, the result would be identically the first: $f(\sqrt{z}+a), f\left(\frac{a}{b+\log z}\right)$ denote that if we made $\sqrt{z} + a = x$, and $\frac{a}{b+\log z} = x$, these functions would become identical, and $= fx$.

† This is evident of itself, since α and β are the co-ordinates of the foot of the generatrix. The same remark applies to the cone.

The equation of the plane may be found, by considering it as a cylinder of which the base is a straight line.

If now this line meet the curve, it will be a generatrix; so that, eliminating x, y, z between these four equations; and then, by means of the equations (2), eliminating α and β from the final equation $\beta = F\alpha$, we shall have for the cone an equation of the form

$$\frac{y-b}{z-c} = F\left(\frac{x-a}{z-c}\right).$$

The form of the function F, dependent on the curve which serves for the directrix, is given by the calculation itself that we have been explaining [see N^{os}. 705 and 879].

Thus, if the base be the circle AE [fig. 18], its equations are $z = 0$, $y^2 + x^2 = 2rx$, the origin being at the extremity A of the diameter, which lies on the axis of x; and the equation of condition $\beta = F\alpha$ is in this case

$$(a - \alpha c)^2 + (b - \beta c)^2 = 2r(a - \alpha c);$$

whence

$$(az - cx)^2 + (bz - cy)^2 = 2r(z - c)(az - cx),$$

the *equation of the oblique cone with a circular base*, the vertex S being in the point (a, b, c). If the axis SC be supposed to be in the plane xz, as it appears in fig. 18, we must make $b = 0$, and there will result

$$c^2(x^2 + y^2) + 2c(r - a)xz + a(a - 2r)z^2 + 2acrz + 2c^2rx = 0.$$

Lastly, for a right cone, $a = r$. In this case it is more convenient to take the axis of z for that of the cone, and we find, for the equation of the surface thus disposed,

$$c^2(x^2 + y^2) = r^2(z - c)^2, \text{ or } x^2 + y^2 = m^2(z - c)^2,$$

m being the tangent of the angle formed by the axis and the generatrix, or $mc = r$.

If the circular base do not lie in the plane xy, but in a plane inclined to xy, and perpendicular to xz, A being the tangent of the angle that this base makes with the plane xy, the equation (1) must be replaced by $z = Ax$, and $x^2 + y^2 + z^2 = r^2$.

622. Every *surface of revolution* may be conceived to be generated [N°. 286] by the motion of a circle BDC [fig. 15], the plane of which is perpendicular to an axis Az, the centre I being in this axis, and the radius IC so varied, that this circle shall always cut some given curve CAB. At present we shall consider only the case in which the axis is taken for that of z.

Every circle BDC, the plane of which is parallel to xy, has for its

equations those of its plane and its projecting cylinder; or, making $AI = \beta$, and the radius $IC = \alpha$,

$$z = \beta, \text{ and } x^2 + y^2 = \alpha^2 \text{ ....} (1).$$

The equations of the given directrix CAB being

$$M = 0, \; N = 0 \text{ ....} (2),$$

in order that these curves may meet, it is requisite that the relation $F\alpha = \beta$, at which we arrive from the elimination of x, y, z between our four equations, be satisfied. If then $F\alpha$ be substituted for β in (1), these equations will become those of the generating circle in one or other of its positions according to the value of α; and α being next eliminated, we shall have the equation required.

Thus, we shall have to eliminate x, y, and z between the four equations (1) and (2); then, in the final equation $F\alpha = \beta$, to substitute z for β, and $\sqrt{(x^2 + y^2)}$ for α; and the equation of the surface of revolution will therefore be of the form $z = F(x^2 + y^2)$: that of the function F depends on the nature of the directrix, and the calculation itself serves to determine it.

I. In the first place, let us take for the directrix a circle in the plane xz, and the centre of which is at the origin; we have, for the equations (2), $y = 0$, $x^2 + z^2 = r^2$, and for the equation of condition $\alpha^2 + \beta^2 = r^2$, as is in fact self-evident; consequently, substituting $x^2 + y^2$ for α^2, and z for β, we have $x^2 + y^2 + z^2 = r^2$ for the equation of the sphere [N°. 614].

II. In the plane xz describe a parabola in a position similar to that of fig. 15; its equations will be $y = 0$, $x^2 = 2pz$; whence $\alpha^2 = 2p\beta$, and

$$x^2 + y^2 = 2pz,$$

the equation of the *Paraboloid of revolution* about the axis of z.

III. Similarly, the equation of the *Ellipsoid* or *Hyperboloid of revolution*, in which the major axis A coincides with that of z, is

$$A^2(x^2 + y^2) \pm B^2z^2 = \pm A^2B^2,$$

the upper sign corresponding to the ellipsoid.

IV. Supposing that any straight line revolve about the axis of z, let us investigate the surface of revolution which it generates. The equations of this revolving straight line, which is the directrix, are

$$x = az + A, \; y = bz + B;$$

whence we have

$$(a\beta + A)^2 + (b\beta + B)^2 = a^2$$

for the equation of condition; and that of the surface therefore is

$$x^2 + y^2 = (a^2 + b^2) z^2 + 2(Aa + Bb) z + A^2 + B^2.$$

Making $x = 0$, we find [N°. 450] that the intersection with the plane yz is an hyperbola; and since x and y only enter into this expression combined in the binomial $x^2 + y^2$, z is a function of $x^2 + y^2$, and the surface generated is an hyperboloid of revolution.

If, however, the generating line cut the axis of z, its two equations must be satisfied when we make $x = y = 0$ and $z = c$; whence $A = -ac$, $B = -bc$; and we consequently have

$$(a^2 + b^2)(z - c)^2 = x^2 + y^2,$$

which belongs to a right cone [N°. 621].

To find the equation of a surface of revolution, the axis of which has any position whatever, we must either have recourse to a transformation of co-ordinates [N°. 636], or treat the problem directly in a manner analogous to the preceding [see N°. 629].

PROBLEMS ON THE PLANE AND STRAIGHT LINE.

623. Here, as in N°. 375, the remark will apply, that the problems on surfaces which may be proposed are of two sorts. At one time the object may be to determine the points which possess certain properties; at another, to give to the surface such a position and dimensions, that it shall fulfil certain proposed conditions. In the 1st case, x, y and z are the unknown quantities; in the 2nd, certain constants of the equation will have to be determined accordingly as circumstances may require. The conditions given must, in all the cases, lead to as many equations as there are unknown quantities; otherwise, the problem will be indeterminate or absurd. We shall now proceed to apply these general principles to the plane.

624. *To find the projections of the intersection of two planes given by their equations*

$$z = Ax + By + C, \quad z = A'x + B'y + C'.$$

Eliminating z, we have, for the projection on the plane xy,

$$(A - A')x + (B - B')y + C - C' = 0;$$

and similarly, getting quit of x or y,

$$(A' - A)z + (AB' - A'B)y + AC' - A'C = 0$$
$$(B' - B)z + (A'B - AB')x + BC' - B'C = 0$$

are the equations of the projections on the planes of yz and xz.

625. *To make a plane pass through one, two, or three given points.* The equation of this plane being $z = Ax + By + C$, if it pass through the point (x', y', z'), we have

$z' = Ax' + By' + C$; and subtracting, there results

$$z - z' = A(x - x') + B(y - y'),$$

the equation of the plane which passes through this point (x', y', z').

The problem will continue indeterminate if the constants A and B be not given, unless at least they be connected by two equations whence they may be deduced. If, for example, the plane be required to be parallel to some other, the equation of which is $z = A'x + B'y + C'$, we shall have

$$A = A', \ B = B'.$$

If the plane is to pass through a 2nd point (x'', y'', z''), we have $z'' = Ax'' + By'' + C$, which leaves one arbitrary constant, and allows of the plane being made to pass through a 3rd point, &c. [see N°. 369].

626. *To find the point of intersection of two straight lines.*

Let the equations

of the 1st line be $x = az + \alpha, \quad y = bz + \beta,$
of the 2nd ...... $x = a'z + \alpha', \quad y = b'z + \beta'.$

For the point in question, x, y and z must satisfy these four equations; and eliminating, we find the equation of condition

$$(\alpha - \alpha')(b - b') = (\beta - \beta')(a - a').$$

If it be not satisfied, the lines do not cut each other; if it be, the point of intersection has for its co-ordinates

$$z = \frac{\alpha - \alpha'}{a' - a} = \frac{\beta - \beta'}{b' - b}, \quad x = \frac{a'\alpha - a\alpha'}{a' - a}, \quad y = \frac{b'\beta - b\beta'}{b' - b}.$$

627. *To find the conditions that a straight line and a plane may be coincident or parallel.* Let the equations of the plane and the line be

$$z = Ax + By + C,$$
$$x = az + \alpha, \ y = bz + \beta;$$

substituting $az + \alpha$ and $bz + \beta$ for x and y in the 1st of these, we have

$$z(Aa + Bb - 1) + A\alpha + B\beta + C = 0;$$

and if the straight line and the plane were required to have but one point common, we should thus arrive at the co-ordinates of that point. But, that the line may lie entirely in the plane, this equation must be satisfied whatever z be; whence [N°. 576],

$$Aa + Bb = 1, \quad A\alpha + B\beta + C = 0;$$

and these are the equations of condition required.

If the straight line is simply parallel to the plane, it follows that, on being transferred in parallel directions to the origin, they will become coincident; thus the above equations must be satisfied when α, β and C are each supposed to be nothing; whence

$$Aa + Bb - 1 = 0.$$

628. *To express that a straight line is perpendicular to a plane.* The plane by which the straight line is projected on xy is perpendicular both to the given plane and to that of xy; and these two latter planes therefore cut each other in a perpendicular to the projecting plane [N°s. 272 and 273]; *i. e.* the trace of the given plane on xy is perpendicular to every straight line in the projecting plane, and consequently to the projection on the plane xy of the given line. Hence, *when a line is perpendicular to a plane, the traces of this plane and the projections of the line are at right angles to each other.* Accordingly, the equations of the plane and the straight line being the same as in the preceding problem, those of the traces of the plane on xz and yz are

$$z = Ax + C, \quad z = By + C,$$

or

$$x = \frac{1}{A}z - \frac{C}{A}, \quad y = \frac{1}{B}z - \frac{C}{B};$$

and the known relation [N°. 370, equation 4] gives

$$A + a = 0, \quad B + b = 0.$$

These equations determine two of the constants of the plane, or of the straight line which is perpendicular to it. The other constants must be given, or be subject to some other conditions.

629. When therefore a plane is to be drawn perpendicular to a given straight line, the equation of this plane is

$$z + ax + by = C.$$

The co-ordinates of the foot of the straight line on the plane xy are

α and β; and the sphere, the centre of which is in this point, has for its equation [N°. 614],

$$(x-\alpha)^2+(y-\beta)^2+z^2=r^2.$$

These two last equations therefore belong to a circle, the plane of which is perpendicular to the given straight line; the radius and the absolute position of the circle depending on r and C.

Let $M=0$, $N=0$ be the equations of a curve; that it may cut our circle, these four equations must be co-existent. Eliminating x, y, and z, we shall obtain an equation of condition $r=F(C)$, and substituting

$$z+ax+by \text{ for } C, \text{ and } \sqrt{[(x-\alpha)^2+(y-\beta)^2+z^2]} \text{ for } r,$$

we shall have the equation of the surface generated by the revolution of the given curve about any axis.

630. If, on the other hand, the plane be given, and it be required that the straight line shall be perpendicular to it, and pass through a given point (x', y', z'), we have for the equations of the straight line,

$$x-x'+A(z-z')=0, \quad y-y'+B(z-z')=0.$$

631. Hence we may deduce the distance from the point to the plane; for, making $L=C-z'+Ax'+By'$, and so putting the equation of the plane under the form

$$z-z'=A(x-x')+B(y-y')+L;$$

then eliminating the co-ordinates x, y, z of the foot of the perpendicular, there results

$$z-z'=\frac{L}{1+A^2+B^2},\quad x-x'=\frac{-AL}{1+A^2+B^2},\quad y-y'=\frac{-BL}{1+A^2+B^2};$$

and consequently [N°. 614] the distance δ between the extremities is

$$\delta=\frac{L}{\sqrt{(1+A^2+B^2)}}.$$

632. *To find the distance from a point to a straight line.*

The equations of the straight line being the same with those of N°. 627, the perpendicular plane, drawn through the given point (x', y', z'), has for its equation

$$a(x-x')+b(y-y')+z-z'=0;$$

and eliminating x, y, z by means of the equations of the straight line, we find, for the co-ordinates of the point of intersection,

$$x = \frac{aM}{1+a^2+b^2} + \alpha,\ y = \frac{bM}{1+a^2+b^2} + \beta,\ z = \frac{M}{1+a^2+b^2},$$

where $M = a(x'-\alpha) + b(y'-\beta) + z'$.

The distance P between the points (x, y, z), (x', y', z'), the calculation being gone through, is given by

$$P^2 = (x'-\alpha)^2 + (y'-\beta)^2 + z'^2 - \frac{M^2}{1+a^2+b^2}.$$

633. *To find the angle A formed by two straight lines.* Draw, through the origin, parallels to these lines; the angle of these two parallels is that which is called the angle of the straight lines, whether they cut each other or not. Let the equations of these parallels be

$$(1) \ldots\ x = az,\ y = bz,\quad (2) \ldots\ x = a'z,\ y = b'z;$$

A then must be found in a function of a, b, a', b'. Now, suppose a sphere to have the origin for its centre, and unity for its radius; the co-ordinates of the points in which it cuts our straight lines will be had by eliminating x, y and z between their respective equations and that of the sphere, which is $x^2+y^2+z^2=1$. The result is $(a^2+b^2+1)z^2=1$; whence we derive z, and then x and y from the equ. (1); a and b must then be accented for the values of z', x' and y':

$$z = \frac{1}{\sqrt{(1+a^2+b^2)}},\ x = \frac{a}{\sqrt{(1+a^2+b^2)}},\ y = \frac{b}{\sqrt{(1+a^2+b^2)}},$$

$$z' = \frac{1}{\sqrt{(1+a'^2+b'^2)}},\ x' = \frac{a'}{\sqrt{(1+a'^2+b'^2)}},\ y' = \frac{b'}{\sqrt{(1+a'^2+b'^2)}}.$$

The distance D between these points is given by

$$D^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 = 2 - 2\,(xx'+yy'+zz'),$$

$x^2+y^2+z^2$ and $x'^2+y'^2+z'^2$ being each $= 1$. Thus we have, in space, an isosceles triangle, the three sides of which are 1, 1 and D; and the angle A is opposite to the last of these; so that the equation D [N°. 355] gives for this angle $\cos A = 1 - \frac{1}{2}D^2 = xx' + yy' + zz'$, or

$$\cos A = \frac{1+aa'+bb'}{\sqrt{(1+a^2+b^2)}\,\sqrt{(1+a'^2+b'^2)}}.$$

1°. From this to deduce the angles X, Y, Z, that a straight line makes with the axes of x, y and z, we must give to the 2nd line the position of each of these axes successively, and substitute in the above

expression the corresponding values of a' and b'. For example, $x = 0$, $y = 0$, are the equations of the axis of z; that the equ. (2) may become such, we must assume $a' = b' = 0$; and these values being introduced into our formula, we shall have

$$\cos Z = \frac{1}{\sqrt{(1+a^2+b^2)}}.$$

If the straight line be made to turn about the origin, but without leaving the projecting plane, till it fall in the plane of xz, the angle of which b' is the tangent will continually decrease, and at last become nothing; thus, to obtain the angle that a straight line in space makes with another situated in the plane of xz, we must suppose $b' = 0$; which reduces the numerator to $1 + aa'$, and the second radical to $\sqrt{(1+a'^2)}$. And if this second line verge towards the axis of x, a' increases, till, on the line coinciding with the axis, it becomes infinite. In this case 1 disappears*

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* Suppose that we have the fraction $\frac{Ax^a+Bx^b+\ldots}{Mx^m+Nx^n+\ldots}$, which may be written under the form $\frac{x^a(A+Bx^{b-a}+\ldots)}{x^m(M+Nx^{n-m}+\ldots)}$, a and m being the lowest exponents of x in the two terms: three cases present themselves.

1°. If $m = a$, the factors x^a, x^m destroy each other; and the more x diminishes, the more the fraction approaches to $\frac{A}{M}$, which is the limit corresponding to $x = 0$.

2°. If $m > a$, we have $\frac{A+Bx^{b-a}+\ldots}{x^{m-a}(M+Nx^{n-m}\ldots)}$, of which infinity is evidently the limit; and the fraction therefore increases without any bounds as x is diminished.

3°. And lastly, if $m < a$, the limit is zero. This mode of taking the limit is what is termed *making x indefinitely small.*

If, on the other hand, a and m be the highest exponents in the two terms, the fraction will appear under the form $\frac{x^a(A+\frac{B}{x^{a-b}}+\ldots)}{x^m(M+\frac{N}{x^{m-n}}+\ldots)}$. But, the more x increases, the more nearly the terms $\frac{B}{x^{a-b}}$, $\frac{N}{x^{m-n}}$ approach to zero, which corresponds to x being infinite; so that if $a = m$, the limit is $\frac{A}{M}$; if $a > m$, the fraction becomes $\frac{x^{a-m}(A+\ldots)}{M+\ldots}$, which is infinite at the same time with x; and lastly, if $a < m$, the limit is zero. This mode of operation is called *making x infinitely large.*

It will easily be seen that this reasoning bears only on the 1st term of the numerator and denominator, so that we might at once have reduced the fraction to

before aa' and a'^2; so that the numerator is reduced to aa', and the second radical to $\sqrt{a'^2}$ or a'; and the quotient of these values being a, we have for the angle X, that a straight line in space makes with the axis of x,

$$\cos X = \frac{a}{\sqrt{(1+a^2+b^2)}}.$$

Cos Y is similarly found by making $a' = 0$, $b' = \infty$; and hence, the cosines of the angles that a straight line makes with the axes are

$$\cos X = \frac{a}{\sqrt{(1+a^2+b^2)}},$$
$$\cos Y = \frac{b}{\sqrt{(1+a^2+b^2)}},$$
$$\cos Z = \frac{1}{\sqrt{(1+a^2+b^2)}}.$$

2°. These values are also those of the sines of the angles that the straight line makes with the planes of yz, xz and xy; these angles evidently being the complements of X, Y, and Z.

3°. Adding the squares of these cosines, there results

$$\cos^2 X + \cos^2 Y + \cos^2 Z = 1.$$

We can therefore draw in space a straight line which makes, with the axes of x and y, any given angles X and Y; but Z is then determinate in value; as is elsewhere evident [see N°. 637].

4°. Along a straight line in space, which makes with the axes the angles X, Y, Z, take any length MN [fig. 12], and let it be projected on the axes of x and y; these projections are

$$BC = MN.\cos X, \text{ and } MN.\cos Y.$$

But mn or MP, being the projection of MN on the plane xy, we have $mn = MN.\sin Z$: and if mn also be now projected on the axes of x and y, these projections are $BC = mn.\cos\theta$ and $mn\sin\theta$, θ being the angle

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$\frac{Ax^a}{Mx^m}$; and the same will be the case for every algebraic function, as might be demonstrated by analogous reasoning.

Hence we conclude that, *to make x infinite in a function, we must retain those two terms alone in which x is affected with the highest exponents: whilst, on the other hand, to make x infinitely small, all the terms must be suppressed, except those which have the least exponents of x.*

Thus when $x = \infty$, $\frac{a+\sqrt[3]{(x^3+bx^2+c)}}{m+\sqrt{(x^2+n)}}$ reduces itself to $\frac{\sqrt[3]{x^3}}{\sqrt{x^2}} = 1$.

that mn makes with the axis of x. Our projections therefore are $BC = MN.\ \sin Z.\ \cos\theta$, and $MN.\ \sin Z.\ \sin\theta$; and equating the two values of the same projections, there results

$$\cos X = \sin Z.\cos\theta,\ \cos Y = \sin Z.\sin\theta.$$

Thus, instead of determining the direction of a line in space by means of the three angles X, Y, Z which it makes with the axes, it will be sufficient to give the angle that it makes with its projection on the plane xy (the complement of Z), and the angle θ of this projection with the axis of x; and the converse.

Adding the squares of these two equations, we have

$$\cos^2 X + \cos^2 Y = \sin^2 Z = 1 - \cos^2 Z,$$

a relation already found [3°.]

5°. Substituting the values of $\cos X, X', Y\ldots$ in $\cos A$, p. 213, we find

$$\cos A = \cos X.\cos X' + \cos Y.\cos Y' + \cos Z.\cos Z';$$

and the angle of the two straight lines is expressed in a function of the angles that each of them makes with the three axes.

6°. If the two lines are perpendicular to each other, $\cos A = 0$, and we have, for the equation which expresses this condition,

$$1 + aa' + bb' = 0,$$

or

$$\cos X.\cos X' + \cos Y.\cos Y' + \cos Z.\cos Z' = 0.$$

634. *To find the angle θ of two planes.* Their equations being

$$z = Ax + By + C,\ z = A'x + B'y + C',$$

if from the origin we let fall perpendiculars on these planes, the angle of these lines will be that of the planes. Now let $x = az$, $y = bz$ be the equations of a straight line passing through the origin; that it may be perpendicular to the 1st plane, we must have [N°. 628] $A + a = 0$, $B + b = 0$; and the equations of the perpendiculars are therefore

$$x + Az = 0,\ y + Bz = 0\ldots\ x + A'z = 0,\ y + B'z = 0.$$

Thus, the cosine of the angle of these lines, and consequently that of the plane is

$$\cos\theta = \frac{1 + AA' + BB'}{\sqrt{(1 + A^2 + B^2)}\sqrt{(1 + A'^2 + B'^2)}}$$

1°. If the 2nd plane be made to assume the position of xz, $y = 0$ is its

equation; and we must therefore make $A' = C' = 0$, and $B' = \infty$, in order to obtain the angle T, that a plane makes with that of xz. The angles U and V that it makes with yz and xy are obtained in a corresponding manner; and we have

$$\cos T = \frac{B}{\sqrt{(1 + A^2 + B^2)}},$$

$$\cos U = \frac{A}{\sqrt{(1 + A^2 + B^2)}},$$

$$\cos V = \frac{1}{\sqrt{(1 + A^2 + B^2)}},$$

whence

$$\cos^2 T + \cos^2 U + \cos^2 V = 1,$$

and

$$\cos \theta = \cos T . \cos T' + \cos U . \cos U' + \cos V . \cos V',$$

for the cosine of the angle of two planes in a function of those that they respectively form with the co-ordinate planes.

2°. If the planes are at right angles to each other, then

$$1 + AA' + BB' = 0,$$

or

$$\cos T . \cos T' + \cos U . \cos U' + \cos V . \cos V' = 0.$$

635. *To find the angle η of a straight line and a plane.* Let their equations be

$$z = Ax + By + C, \text{ and } x = az + \alpha,\ y = bz + \beta:$$

the angle required is that which the straight line makes with its projection on the plane [N°. 272]; so that if from a point of the line a perpendicular be let fall on the plane, the angle of these two lines will be the complement of η. Now, drawing from the origin any straight line $x = a'z$, $y = b'z$, that it may be perpendicular to the plane, we must have [N°. 628] $a' = -A$, $b' = -B$. And the angle at which it is inclined to the given line has for its cosine the value determined [p. 213]; consequently

$$\sin \eta = \frac{1 - Aa - Bb}{\sqrt{(1 + a^2 + b^2)}\sqrt{(1 + A^2 + B^2)}}$$

Hence we shall readily conclude that the angles which the straight line makes with the co-ordinate planes of xz, yz and xy have for their respective sines

$$\frac{b}{\sqrt{(1+a^2+b^2)}}, \quad \frac{a}{\sqrt{(1+a^2+b^2)}}, \quad \frac{1}{\sqrt{(1+a^2+b^2)}};$$

which agrees with what has been seen [N°. 633, 2°.].

TRANSFORMATION OF THE CO-ORDINATES.

636. To transfer the origin to the point $(\alpha, \beta, \gamma,)$ without changing the directions of the axes, which are supposed to be any whatever, it will be seen, from reasoning similar to that of N°. 382, that we must assume

$$x = x' + \alpha, \; y = y' + \beta, \; z = z' + \gamma.$$

The primitive axes x, y, z, are parallel to the new ones x', y', z', whatever be the angles that they make with each other. We must likewise observe to give to the co-ordinates α, β, γ, of the new origin the proper signs corresponding to its position [N°. 612]: if it be situated in the plane xy, $\gamma = 0$; if it be in the axis of z, α and β are each nothing, &c.

637. In the case for changing the direction of the axes, the origin remaining the same, we shall suppose the original axes to be rectangular; and the new ones, which we shall represent by Ax', Ay', Az', of given arbitrary direction. Taking any point, draw the co-ordinates x', y', z', of this point, and project them on the axis of x; the abscissa x will, according to N°. 383, be the sum of these three projections. Denote by (xx') the angle $x'Ax$ formed by the axes of x and x', by (yy') the angle $y'Ay$, &c.; then we shall have

$$\left.\begin{array}{l} x = x'\cos(x'x) + y'\cos(y'x) + z'\cos(z'x) \\ y = x'\cos(x'y) + y'\cos(y'y) + z'\cos(z'y) \\ z = x'\cos(x'z) + y'\cos(y'z) + z'\cos(z'z) \end{array}\right\} \dots (A),$$

the two last equations being found by projecting x', y', z', on the axis of y, and then on that of z. And such are the relations which serve to change the direction of the axes.

Since $(x'x)$, $(x'y)$, $(x'z,)$ are the angles that the straight line Ax' forms with the rectangular axes of x, y, z, we have [N°. 633, 3°.]

$$\text{and likewise} \quad \left.\begin{array}{l} \cos^2(x'x) + \cos^2(x'y) + \cos^2(x'z) = 1 \\ \cos^2(y'x) + \cos^2(y'y) + \cos^2(y'z) = 1 \\ \cos^2(z'x) + \cos^2(z'y) + \cos^2(z'z) = 1 \end{array}\right\} \dots (B).$$

The angles which the new axes form with each other give [N°. 633, 5°.]

$$\cos(x'y') = S,\ \cos(x'z') = T,\ \cos(y'z') = U\ldots\ (C),$$

making, for conciseness,

$$S = \cos(x'x).\cos(y'x) + \cos(x'y).\cos(y'y) + \cos(x'z).\cos(y'z),$$
$$T = \cos(x'x).\cos(z'x) + \cos(x'y).\cos(z'y) + \cos(x'z).\cos(z'z),$$
$$U = \cos(y'x).\cos(z'x) + \cos(y'y).\cos(z'y) + \cos(y'z).\cos(z'z),$$

If the new co-ordinates be also rectangular, we have

$$S = 0,\ T = 0,\ U = 0\ldots\ (D).$$

The equations (A), (B), (C), (D), contain the nine angles that the axes of x', y', z', make with those of x, y, z; and it appears that when a new system of co-ordinates is selected, of these nine angles but six are arbitrary, since the equations B determine three of them. Should this system also be rectangular, the equ. (D), which express this condition, leave only three of the angles arbitrary. The axis of x' makes with those of x, y, z, three angles, of which two may be any whatever, and the third follows from them: the same would be the case for y', but that it must necessarily be perpendicular to x'; and this condition really leaves but one arbitrary angle; which makes three in all, since these given angles fix the position of the axis of z', perpendicular to the plane $x'y'$.

638. Instead of determining the position of the new rectangular axes, by the angles which they form with the first, we may adopt the following method [*Mecan. Cél.* vol. i. p. 59].

Let a plane $CAy'x'$ [fig. 16] be inclined at the angle θ to the plane xAy, which it cuts in AC; let this trace AC make with Ax the angle $CAx = \psi$; and in the plane CAy', determined by θ and ψ, draw the two rectangular axes Ax', Ay'; the 1st making with the trace AC the angle $CAx' = \varphi$. Our new axes may be fixed by these angles θ, ψ, and φ; for they give the inclination of the plane $x'y'$ to xy, the direction of the trace AC and that of Ax' in the plane $x'y'$ thus determined; the axis of y' makes, in this plane, the angle $x'Ay' = 90°$; and that of z' is perpendicular to the same plane. Thus, for the transformation of the axes, it remains only to express the angles $x'x$, $y'x\ldots$, which enter into the equations A, in functions of the given angles θ, ψ, and φ.

Now, the straight lines Ax, Ax', and AC, form a trihedral of which two plane angles φ and ψ are known, as also the included dihedral angle θ. Apply therefore to this trihedral the formula (3), p. 184, of Spherical Trigonometry, making $c = (xx')$, $C = \theta$, $a = \psi$, $b = \varphi$; and we shall have

$$\cos(x'x) = \cos\psi.\cos\varphi + \sin\psi.\sin\varphi.\cos\theta.$$

It is evident that, for the angle xAy', we have only to operate in the same manner on the trihedral $x'ACy$, the plane angles of which are $(x'y)$, $CAy = 90° + \psi$, and $CAx' = \phi$; whilst, for $y'Ay$, we must take the trihedral $y'ACy$, in which $CAy' = 90° + \phi$, $CAy = 90 + \psi$; and consequently

$$\begin{aligned} \cos(y'x) &= -\cos\psi \sin\phi + \sin\psi \cos\phi \cos\theta, \\ \cos(x'y) &= -\sin\psi \cos\phi + \cos\psi \sin\phi \cos\theta, \\ \cos(y'y) &= \quad \sin\psi \sin\phi + \cos\psi \cos\phi \cos\theta. \end{aligned}$$

Taking into consideration the trihedral $z'AxC$, the axis Az' makes with AC a right angle [N°. 266], as also with the plane CAy'; the angle of the planes xy and $z'AC$ is $90° + \theta$, supposing the plane CAy' to be situated above xy; and making, in the equ. (3), p. 184.

$$c = (z'x),\ C = 90° + \theta,\ a = 90°,\ b = \psi,$$

we shall have

$$\cos(z'x) = -\sin\psi \sin\theta.$$

In like manner, the trihedral $z'ACy$ gives

$$\cos(z'y) = -\cos\psi \sin\theta,$$

ψ being augmented by 90°. And, lastly, zAC being also a right angle, and the dihedral $zACx = 90° - \theta$, the trihedral $zACx'$ gives

$$\cos(x'z) = \sin\phi \sin\theta,$$

whence $\cos(y'z) = \cos\phi \sin\theta,$

and, finally, $\cos(z'z) = \cos\theta.$

And thus we have the values of the nine coefficients of the equations (A).

The equations B and D of condition are of themselves satisfied by these values, as might easily be proved.

PLANE INTERSECTIONS.

639. When the intersection of the two surfaces is a plane curve, it is most convenient, for the ascertaining of its properties, to refer it to co-ordinates taken in this plane DOC [fig. 17], determined by the angle θ which it makes with the plane xy, and the angle ψ that Ox makes with the intersection OC of these planes: we shall take this line OC for the axis of x'; and the perpendicular OA, erected on OC in the cutting plane DOC, will be the axis of y'.

Since the question now is, to obtain the equation of the curve of intersection of the two surfaces in respect to the co-ordinates x', y', it is obvious that, one of these surfaces being referred to the axes x', y', z', by means of the transformation (A), it would be sufficient then to make $z'=0$, and we should have its intersection with the plane $x'Oy'$. In so simple a case, however, it is preferable to make $z'=0$ in the equ. (A), and then investigate in a direct manner the cosines of $(x'x)$, $(y'x)$.... Accordingly, in the trihedral $AOCB$, we know the plane angles $a=\psi$, $b=90°$, and the included dihedral angle $C=\theta$; whence the equ. (3), p. 184, becomes

$$\cos(y'x)=\sin\psi\cos\theta,\ \cos(y'y)=-\cos\psi\cos\theta.$$

Moreover,

$$x'x=\psi,\ \cos(x'y)=\sin\psi,\ (x'z)=90°;$$

and, lastly, the plane $x'Oy'$, which we suppose elevated above that of xy, makes with the axis Oz the angle $(y'z)=90-\theta$.

Thus, the equations (A) give

$$\left.\begin{array}{l} x=x'\cos\psi+y'\sin\psi\cos\theta \\ y=x'\sin\psi-y'\cos\psi\cos\theta \\ z=y'\sin\theta \end{array}\right\}\ldots (E).$$

We should equally have arrived at these results, by making use of the equations of N°. 638.

640. Let these equations be applied to the oblique cone with a circular base. We shall take the plane zAx [fig. 18], drawn through the axis SC perpendicularly to the cutting plane AB, for the plane of xz; and AB being the section of these two planes, or the *axis of the curve*, the *vertex* A, in which this axis cuts the curve, will be assumed as the origin of the co-ordinates: also, the plane xAy, parallel to the circular base of the cone, will be the plane of xy; it cuts the cone in a circle AE, of radius r, which may be considered as the *directrix itself* [N°. 621]. Thus, our cone, the vertex of which has a, o, c, for its co-ordinates, and the axis of which is in the plane xz, and the base in the plane xy, has for its equation $c^2(x^2+y^2)+2c(r-a)xz\ldots=0$, as in p. 207. On the other hand, the cutting plane AB being perpendicular to xz, it will cut the plane xy in the axis Ay, and we must therefore assume $\psi=90°$ in the equ. (E); whence

$$x=y'\cos\theta,\ y=x',\ z=y'\sin\theta\ldots(1);$$

$$y'^2[c^2\cos^2\theta+2c(r-a)\sin\theta\cos\theta+(a^2-2ar)\sin^2\theta]$$
$$+c^2x'^2+2cry'(a\sin\theta-c\cos\theta)=0\ldots(2).$$

And this is the equation of the curve, which, by varying a, c, r, and θ, may be made to represent all the sections of the oblique cone (except the parallels to the base); the values of x' being measured along Ay, those of y' along AB. It will be easy to discuss this equation [N°. 450], and to gather from it that the curves are all of the same species as for the right cone.

That the section may be a circle, the coefficients of x'^2 and y'^2 must be equal [N°. 446]; whence

$$(c^2 + 2ar - a^2)\tan^2\theta = 2c\,(r - a)\tan\theta.$$

Of the solutions of this equation, $\tan\theta = 0$ corresponds to the base AE of the cone. As to the other value of θ, for the explanation of it, we have

$$\tan SAD = \frac{SD}{AD} = \frac{c}{a},\ \tan SAB = \frac{c - a\tan\theta}{a + c\tan\theta};$$

and if, for $\tan\theta$, we substitute our second root, there results, after the different reductions,

$$\tan SAB = \frac{c^3 + a^2 c}{2a^2 r - a^3 + 2c^2 r - ac^2} = \frac{c}{2r - a} = -\frac{SD}{DE},$$

or

$$\tan SAB = -\tan SED = \tan SEA.$$

The section therefore is again a circle, when the angles SAB, SEA, formed with the opposite generatrices, are equal. The cutting plane yAB' being compared with the circle AE of the base, they form what are called *sub-contrary sections.*

To obtain the plane sections of the right cone, we have only to assume $a = r$ in (2), which gives

$$y'^2(c^2\cos^2\theta - r^2.\sin^2\theta) + c^2x'^2 + 2cry'(r\sin\theta - c\cos\theta) = 0,$$

an equation equivalent to that of N°. 386. The factors of x'^2 and y'^2 in this equation can no longer be rendered equal in two different ways; and, in fact, the two sub-contrary sections now coincide.

641. The oblique cylinder, the base of which is a circle situated as for the cone in the preceding case, with its axis lying in the plane xz, has for its equation [p. 206],

$$y^2 + (x - az)^2 = 2r\,(x - az);$$

in which introducing the values (1), the cutting plane being perpendicular to xz, we get

$$y'^2(\cos^2\theta + a^2\sin^2\theta - 2a\sin\theta\cos\theta) + x'^2 = 2ry'(\cos\theta - a\sin\theta).$$

The section generally is an ellipse; but becomes a circle when $\sin\theta = 0$, which brings us to the base of the cylinder, and also when

$$(a^2 - 1)\tan\theta = 2a, \text{ or } \tan\theta = -\tan 2\alpha,$$

[equ. L, 359], α being the angle that the axis of the cylinder makes with that of z; consequently θ is the supplement of 2α.

SURFACES OF THE SECOND ORDER.

642. The general equation of the 2nd degree is

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz = k \ldots (1).$$

To *discuss* this equation, *i. e.* to determine the nature and position of the surfaces which it represents, we shall, in the first place, simplify it by such a transformation of the co-ordinates as will make the terms in xy, xz, and yz, disappear. The axes, from being rectangular, will be rendered oblique, by the substitution of the values (A), p. 218; and the nine angles which enter into them being subjected to the conditions (B), there will remain six arbitrary ones, of which we may dispose in an infinity of ways. But if we propose that the new axes also be rectangular, since this condition is expressed by the three relations (D), the six arbitrary angles will now be reduced to three, which the three coefficients of the terms in $x'y'$, $x'z'$, and $y'z'$, equated to zero, will serve to make known; and the problem thus becomes determinate.

This calculation will be facilitated by proceeding in the manner following. Let $x = \alpha z$, $y = \beta z$ be the equations of the axis of x'; assuming, for conciseness, $l = \dfrac{1}{\sqrt{(1 + \alpha^2 + \beta^2)}}$, we find [p. 214]

$$\cos(x'x) = l\alpha,\ \cos(xy) = l\beta,\ \cos(x'z) = l;$$

and making similar assumptions for the equations $x = \alpha'z$, $y = \beta'z$, of the axis of y; as also for those of the axis of z', we have

$$\cos(y'x) = l'\alpha',\ \cos(y'y) = l'\beta',\ \cos(y'z) = l',$$
$$\cos(z'x) = l''\alpha'',\ \cos(z'y) = l''\beta'',\ \cos(z'z) = l''.$$

Thus the equations (A) of transformation become

$$x = l\alpha x' + l'\alpha'y' + l''\alpha''z',$$
$$y = l\beta x' + l'\beta'y' + l''\beta''z',$$
$$z = lx' + l'y' + l''z;$$

and the nine angles of the problems are replaced by the six unknown terms α, α', α'', β, β', β''; the equations (B) being satisfied of themselves.

These values of x, y, z, must now be substituted in the general equation of the 2nd degree, and the coefficients of $x'y'$, $x'z$ and $y'z'$, each equated to zero; whence

$$(a\alpha + d\beta + e)\alpha' + (d\alpha + b\beta + f)\beta' + e\alpha + f\beta + c = 0 \ldots x'y'$$

$$(a\alpha + d\beta + e)\alpha'' + (d\alpha + b\beta + f)\beta'' + e\alpha + f\beta + c = 0 \ldots x'z'$$

$$(a\alpha'' + d\beta'' + e)\alpha' + (d\alpha'' + b\beta'' + f)\beta' + e\alpha'' + f\beta'' + c = 0 \ldots y'z'.$$

Any one of these equations may be obtained singly, and without making the substitution throughout; also, from the symmetry of the calculation, having found one of the equations we shall be able from it to deduce the two others.

Let α' and β' be eliminated between the first of these equations and those $x = \alpha'z$, $y = \beta'z$ of the axis of y'; the result will be the following equation, which *is that of a plane*,

$$(a\alpha + d\beta + e)x + (d\alpha + b\beta + f)y + (e\alpha + f\beta + c)z = 0 \ldots (2).$$

But the first equation expresses the condition on which the term in $x'y'$ is to be got quit of; and, so long as this is the only object in view, α, β, α', β' may be assumed at pleasure, provided that this first equation be satisfied. Hence, we have but to draw the axis of y' in the plane of which we have just determined the equation, and the transformed equation will be without the term in $x'y'$.

In like manner, eliminating α'' and β'' from the second equation, by means of those, $x = \alpha''z$, $y = \beta''z$, of the axis of z', we shall obtain a plane such that, if any line drawn in it be taken for the axis of z', the transformed equation will be devoid of the term in $x'z'$. But, from the form of the two first equations, it is evident that this plane will be the same with the former; and hence, drawing in it any two lines as the axes of y' and z', *this plane will be that of* $y'z'$, and the transformed equation will have no terms either in $x'y'$ or $x'z'$. Since the directions of these axes in this plane may be any whatever, there is an infinite number of systems which will answer the end proposed. The equ. (2) will obviously be that of a plane parallel to the one which bisects all the parallels to the axis of x, and which is called the *diametral Plane.*

If, moreover, the term in $y'z'$ is to disappear, α' and β' must be determined from the third equation; and thus we see that there is an infinite number of oblique axes, that will fulfil the three conditions proposed.

643. But suppose that the axes of x', y', z', are also to be rectangular. Then the axis of x' must be perpendicular to the plane $y'z'$ of which we have just found the equation. But that $x = \alpha z$, $y = \beta z$ may be the equations of a perpendicular to this plane, it is requisite [N°. 628] that

$$a\alpha + d\beta + e = (e\alpha + f\beta + c)\,\alpha \ldots (3),$$
$$d\alpha + b\beta + f = (e\alpha + f\beta + c)\,\beta \ldots (4);$$

whence, substituting in (3) the value of α deduced from (4), we find

$$\begin{aligned}&[(a-b)fe + (f^2 - e^2)\,d]\,\beta^3 \\ &+ [(a-b)(c-b)\,e + (2d^2 - f^2 - e^2)\,e + (2c - a - b)fd]\,\beta^2 \\ &+ [(c-a)(c-b)\,d + (2c^2 - f^2 - d^2)\,d + (2b - a - c)fe]\,\beta \\ &+ (a-c)fd + (f^2 - d^2)\,e = 0.\end{aligned}$$

This equation of the 3rd degree gives for β at least one real root; and the equ. (4) then gives one for α; whence the axis of x' is so determined as to be perpendicular to the plane $y'z'$, whilst at the same time the equation is cleared of the terms in $x'z'$ and $x'y'$. And it only remains therefore to describe, in this plane $y'z'$, the two other axes at right angles to each other, and so that the term in $y'z'$ may disappear. But it is evident that we might in the same manner find a plane $x'z'$, such that the axis of y' shall be perpendicular to it, and the terms in $x'y'$, $z'y'$ be got quit of; and it turns out that the conditions which express that the axis of y' shall thus be perpendicular to the plane are again (3) and (4), so that the same equation of the 3rd degree must also give β'. And the same is the case for the axis of z'. Hence the three roots of the equation in β are real, and are the values of β, β', β''; and α, α', α'', are subsequently given by the equation (4). *There is therefore but one system of rectangular axes, by which the equation can be divested of the terms in $x'y'$, $x'z'$, $y'z'$; and there is one such in all cases; our calculation points out the mode of determining these axes.*

This system takes the denomination of the *Principal Axes of the Surface.*

644. Let us now examine the different cases that may be presented by the equation of the 3rd degree in β.

1°. If we have

$$(a-b)fe + (f^2 - e^2)\,d = 0,$$

the equation is deprived of the first term; in which case we know that one of the roots of β is infinite, as is also α, since the equ. (4) now reduces itself to $e\alpha + f\beta = 0$; the corresponding angles are right angles,

and one of the axes, that of z', for instance, lies in the plane xy; its equation is obtained by eliminating α and β by means of $x = \alpha z$, $y = bz$, and is $ex + fy = 0$. The directions of y' and z' are given by our equation in β, reduced to the 2nd degree.

2°. If, besides this first coefficient, the second be also $= 0$, deducing b from the equation above, in order to substitute it in the factor of β^2, this factor becomes reduced to the last term of the equation in β:

$$(a - c)fd + (f^2 - d^2)e = 0;$$

and these two equations express the condition proposed. But, the coefficient of β is deducible from that of β^2 by changing b into c and d into e, and the same is the case for the first and the last terms of the equation in β; so that the equation of the 3rd degree is satisfied of itself; and there is consequently an infinite number of systems of rectangular axes, which make the terms in $x'y'$, $x'z'$, and $y'z'$ vanish. Let a and b be eliminated from the equations (3) and (4) by means of the two equations of condition above; it appears, then, that they are the product of $f\alpha - d$, and $e\beta - d$ by the common factor $ed\alpha + fd\beta + fe$; the two factors therefore are each nothing; and eliminating α and β, we find

$$fx = dz,\ ey = dz,\ edx + fdy + fez = 0.$$

The two first equations are those of one of the axes; the third, that of a plane perpendicular to it, and in which the two other axes are to be drawn in arbitrary directions. This plane will cut the surface in a curve in which all the axes at right angles to each other are principal; and this curve is consequently a circle, the only one of the curves of the 2nd degree which possesses such a property. The surface is one of revolution about the axis of which we have just given the equations; as might readily be shown by transferring the origin to the centre of the circle. [See *Annales de Math.* vol. ii., for two beautiful Memoirs, by M. Bret].

645. The equation, being thus divested of the three rectangles, will be of the form

$$kz^2 + my^2 + nx^2 + qx + q'y + q''z = h \ldots (5);$$

and we shall now get quit of the terms of the 1st dimension, by a transfer of the origin [N°. 636]. It is clear that this operation will always be possible, unless one of the squares x^2, y^2, z^2 be wanting in the equation; we shall examine these cases separately, and in the first place shall discuss the equation

$$kz^2 + my^2 + nz^2 = h \ldots (6).$$

Every straight line passing through the origin cuts the surface in two points, at equal distances on each side of the origin, since the equation remains the same after changing the signs of x, y, z; thus the origin, being the middle point of all the chords drawn through it, is a *centre;* and consequently, *the surface possesses the property of having a centre, at all times when the transformed equation is not deficient in one of the squares of the variables.*

We shall assume n to be always positive; it remains then to examine the cases in which k and m are both positive, or both negative, or of different signs.

646. If, in the equation (6), k, m and n be positive, h must necessarily be so also; otherwise the problem would be *absurd, and not represent any thing;* and if h be 0, we have $x = 0, y = 0, z = 0$, simultaneously [N°. 112], and the surface is no more than *a single point.*

But, when h is positive, making x, y, or z separately $= 0$, we find equations to different ellipses, and these are the curves which result from the section of our surface by the three co-ordinate planes. Every plane parallel to these also gives ellipses, and it would be easy to show that the same is the case for all the plane sections [N°. 639]; and on this account the body has received the name of the *Ellipsoid.* The lengths A, B, C of the *three principal axes* are obtained by investigating the sections of the surfaces by the axes x, and y and z, viz.

$$kC^2 = h,\ mB^2 = h,\ nA^2 = h.$$

Eliminating k, m and n, by means of these, from equ. (6), we have

$$\frac{z^2}{C^2} + \frac{y^2}{B^2} + \frac{x^2}{A^2} = 1, \text{ or } A^2B^2z^2 + A^2C^2y^2 + B^2C^2x^2 = A^2B^2C^2,$$

the equation of the ellipsoid referred to its centre, and its three principal axes. We may conceive this surface to be generated by an ellipse, described in the plane xy, moving parallel to itself, its two axes at the same time so varying, that it shall slide along another ellipse traced in the plane xz. If two of the quantities A, B, C are equal, we have an ellipsoid of revolution; if $A = B = C$, the surface is that of a sphere.

647. Suppose now that k is negative, m and h positive, or

$$kz^2 - my^2 - nx^2 = -h.$$

Assuming x or $y = 0$, we shall perceive that the sections by the planes of yz and xz are hyperbolas, having their minor axes in the axis of z; the planes drawn through the axis of z also give this same curve; and the surface is said to be an *hyperboloid.* The sections parallel to the plane of xy are always real ellipses, in which $A, B, C\sqrt{-1}$ denoting the

lengths along the axes, measured from the origin, the equation will be the same with that above, except as to the sign of the first term, which here becomes negative.

648. Lastly, when k and h are negative,

$$-kz^2 + my^2 + nx^2 = -h,$$

all the planes drawn through the axis of z cut the surface in hyperbolas, which have their major axes in that of z; the plane xy does not meet the surface; but its parallels, beyond two opposite limits, give ellipses. We have therefore *an hyperboloid of two sheets* about the axis of z. The equation in A, B, C is still the same with the one preceding, except that the term in z^2 is the only one that is positive.

649. When $h=0$, we have, in each of these two last cases, $kz^2 = my^2 + nx^2$, *the equation of a cone*, which is to our hyperboloids what the asymptotes were to the hyperbola [see p. 206].

There still remains the case of k and m being negative; but a simple inversion of the axes serves to bring this under the two preceding cases. The hyperboloid has one or two sheets around the axis of x, accordingly as h is negative or positive.

650. When the equ. (5) is without one of the squares, x^2 for instance, it may then be divested of the constant term, and the 1st powers of x and y, viz.

$$kz^2 + my^2 = hx.$$

The sections by the planes of xz and xy are parabolas, turned one way or the other, according to the signs of k, m and h; and the planes parallel to these also give parabolas. The planes parallel to yz give ellipses, or hyperbolas, according to the sign of m. The surface is in one case an *elliptic*, in the other an *hyperbolic paraboloid*; $k=m$, when the paraboloid is one of revolution.

651. When $h=0$, the equation has the form $a^2z^2 \pm b^2y^2 = 0$, according to the signs of k and m. In the one case we have $z=0$, $y=0$, whatever x be, and the surface reduces itself to the axis of x. In the other, $(az+by).(az-by)=0$ indicates that either factor may be assumed $=0$ at pleasure; whence we have the system of two planes which cross each other in the axis of x.

652. When two squares, as x^2 and y^2, are wanting in the equation (5),

by transferring the origin in a direction parallel to z, it may be reduced to

$$kz^2 + py + qx = 0.$$

The sections by planes drawn along the axis of z are parabolas; whilst the plane xy and its parallels give straight lines that are parallel to each other. The surface therefore is *a cylinder on a parabolic base* [N°. 620].

Should the three squares be wanting in the equ. (5), it would be that of a plane.

653. We shall easily be able to distinguish the case in which the proposed equ. (1) is decomposable into two rational factors; those in which it is formed of positive squares, which resolve themselves into two equations, representing the section of two planes; and, lastly, those in which, being composed of three parts essentially positive, it is absurd. All this is analogous to what has been said in N°s. 453 and 459.

BOOK VII.

DIFFERENTIAL AND INTEGRAL CALCULUS.

I.—GENERAL RULES OF DIFFERENTIATION.

DEFINITIONS, TAYLOR'S THEOREM.

654. The more manifold the objects that a branch of science embraces, and the greater the variety of its applications, the more difficult it becomes to give a precise definition that shall convey an idea of it in its full extent, and comprehend all the subjects on which it may be brought to bear. That part of the higher analysis, denominated the *Differential Calculus*, is applicable to questions so varied, that we cannot explain its nature, without first making some preliminary observations.

An equation $y=f(x)$ between two variables x and y being given, we may consider it as represented by a plane curve BMM' [fig. 22] referred to two rectangular co-ordinate axes Ax, Ay; and it will readily be understood, that if to the abscissa x we assign a series of values, and thence deduce those of the corresponding ordinates y, we shall have a series of points M, M'... of the curve; but that these points will be separated from each other by a certain interval, however near we suppose the values of x to be to each other. Thus, in its present state, the equation $y=f(x)$ does not express that there is *continuity* between the points. And the same remark will apply to every equation between 3, 4... variables. Let us see whether analysis cannot furnish us with some artifice that will serve to evidence the fact of continuity between the points.

Let our example be the equation $y = ax^3 + bx^2 + c$. If, after considering a point M, which has the co-ordinates x and y, we wish to take another point M' and to compare it with the first, denoting its co-ordinates AP', $P'M'$ by $x + h$ and $y + k$, we shall have

$$y + k = a(x + h)^3 + b(x + h)^2 + c, \text{ and, developing,}$$
$$y + k = (ax^3 + bx^2 + c) + (3ax^2 + 2bx)h + (2ax + b)h^2 + ak^3.$$

Now, in this expression, *the coefficient of the* 1*st power of h*, viz. $3ax^2 + 2bx$, is derived from the function proposed, bears the impress of it, and corresponds to it alone; moreover, this coefficient is independent of h, which is the distance PP' of the extremities of the two abscissæ, and consequently measures the interval between the two points of the curve: this coefficient therefore is suitably composed for expressing that we consider that two points of the curve may be as close to each other as we think fit, and consequently that the function is continuous. And hence we shall conclude that whenever a question proposed, of whatever nature it may be, rests on the idea of *continuity*, it is the coefficient of the 1st power of h in the development of the function, when x is replaced in it by $x + h$, which, suitably combined and analysed, will give the solution of the problem.

Reasoning in the same manner for the general case $y = f(x)$, where the symbol f indicates any *function* of x [see note, p. 206], if x be replaced by $x + h$ and y by $y + k$, we shall have the equation $y + k = f(x+h)$; and the first point must now be to develop $f(x + h)$ so as to exhibit the terms affected with the different powers of h. This operation will depend on the nature of the function f, and we shall shortly see how it may be effected for each particular form of f: at present we shall only remark that if we take $h = 0$, which supposes $k = 0$ and makes the second point coincident with the first, all the terms of which h is a factor must disappear in the development in question; so that there will remain only the first term, which consequently must be y, or $f(x)$. It is also evident that h cannot be affected with any negative exponent; for should there exist in $f(x + h)$ a term of the form Mh^{-m}, which is equivalent to $\frac{M}{h^m}$, on making $h = 0$, this term would become infinite, and the result would no longer be $f(x)$. Thus it follows that the development of $f(x + h)$ must be of the form

$f(x + h) = f(x)$ + *a series of terms of which h, affected with different positive powers, is a factor.*

655. But it may be shown that, generally,

$$f(x + h) = f(x) + y'h + ah \dots \text{ (1)},$$

i. e. besides the term $f(x)$, of which we have just proved the existence, there will be 1°. a term $y'h$ containing the 1st power of h multiplied by a function of x alone, which we shall denote by y'; and 2°. a collection of other terms into which h enters in higher powers than the 1st, and which we denote by ah, a being a function of x and h, and also admitting the factor h in any positive power.

To prove this proposition, which serves as the basis of the whole of the Differential Calculus, let a tangent TH be drawn at the point $M(x, y)$ of the curve BMM', the equation of which is $y = f(x)$. This straight line we know is obtained by drawing through the point M any line MM', called a *secant*, and making it turn about M till the points M and M' coincide with each other. We shall represent this geometrical operation under an analytical form. Changing x into $x + h$, and y into $y + k$, in order to arrive at a second point M' of the curve, we shall have $y + k = f(x + h)$ for the ordinate $P'M'$; thus $MQ = h$, $P'M' = f(x + h)$, $M'Q = k$, or $k = P'M' - PM = f(x + h) - f(x)$; whence the rectangular triangle $MM'Q$ gives

$$\tan M'MQ = \frac{M'Q}{MQ} = \frac{k}{h} = \frac{f(x + h) - f(x)}{h};$$

and to deduce the direction of the required tangent TM, we must, in this expression, make $h = 0$, in order to express that M' verges to actual coincidence with M. The value of tan HMQ is therefore what the second side of this equation becomes, when we assume $h = 0$. And since this direction of the tangent depends on the first point M, it is clear that the result must be a function of x; this function we shall call y'.

Hence it follows that the value of this second side must consist of two parts: 1°. of the term y', which is independent of h; 2°. of other terms of which h, in different positive powers, is a factor, and which disappear when we assume $h = 0$. Denoting the aggregate of these terms by α, which will be a function of x and h, we shall have

$$\frac{f(x + h) - f(x)}{h} = y' + \alpha \dots (2),$$

an equation which reduces itself to (1), by getting quit of the denominator h, and transposing $f(x)$. This reasoning can only fail, on the supposition that the point which we have taken in the curve have no tangent, and this never happens but in certain special cases, in which, in fact, the differential calculus does present obscure results; but so long as we keep to generalities which allow of x having any value whatever, we may rest satisfied that the equ. (1) is always true.

656. It is established therefore that, whatever be the form of the function f, the expression $f(x + h)$ is susceptible of being developed, by suitable operations, in several terms, of which the 1st is the proposed function $f(x)$; the second a term $y'h$ which contains h only in the 1st power, and as a factor of a function of x; lastly, of other terms comprised

in the form αh, all of which contain the factor h in some power higher than the 1st, *i. e.* $h = 0$ gives $\alpha = 0$.

This second term $y'h$ has for its coefficient a function y' of x, which essentially results from the proposed function y or fx; and moreover, as it is independent of h, it is adapted for expressing that the function f is continuous, since it arises from our considering two points of a curve to be as near to each other as it is possible to conceive. It is to this factor y' of the 1st power of h that we give the appellation of the *derivative* or the *differential coefficient* of the function y; it is also expressed by $f'(x)$.

The function α, as we shall shortly explain, is itself susceptible of being developed in several terms, proceeding according to the powers of h, and each coefficient of which might, as well as y', be made use of to express the continuity in y: but as we shall see that these coefficients depend on y', this remark does not at all weaken our consequences; only, in treating of problems, we may be induced to prefer one or other of these coefficients, according to the object in view.

657. It appears that the more h decreases, in the equ. (2), the less α becomes, till, on h becoming zero, it is reduced to nothing; and hence we arrive at this conclusion, which frequently gives an excellent mode of calculation for deducing the function $f'(x)$ from $f(x)$, that the derivative y' of a function y is the resulting value of the 1st side of the equ. (2) when h is made nothing; *i. e. the derivative y', or the differential coefficient of a function y, is the limit of the ratio of the increment of this function to that of the variable;* for, the numerator $f(x + h) - f(x)$ is the excess of the second function above the primitive one, and the denominator is the increment h assigned to x. [See what has been said in N°. 113 on *limits*].

658. It will be well to know the origin of the word *differential.* Since, in the equ. (2), the term α decreases indefinitely along with h, whilst y', which is independent of h, remains constant, it is clear that the less h be, the more will the 2nd side verge to the value y'; so that, for very small values of h, the difference $f(x + h) - f(x)$ reduces itself to $y'h$; and since this is the difference between the second function and the primitive one, $y'h$ has received the appellation of a small difference, or a *differential.* Also, Leibnitz, the inventor of this Calculus, having denoted an infinitely small increment assigned to a variable by the symbol d, dy and dx have been fixed on as symbols for replacing the letters k and h above, and we have $y'dx$ (instead of $y'h$) for the differential of y, *viz.* $dy = y'dx$. This notation is generally received in the branch of analysis of which we are now explaining the principles. *The derivative*

or the differential coefficient of the function $y = f(x)$ *is* y', *or* $f'(x)$, *or* $\frac{dy}{dx}$; *and it is the coefficient of the 2nd term, or of the 1st power of* h, *in the development of the second function* $f(x+h)$; *or the limit of the ratio of the increment of the function* $f(x)$ *to that of the variable* x; or, lastly, it is *the coefficient of the infinitely small difference* $dy = y'dx$, *which we find when* x *is increased by* dx.

Attaching to the word *derivative* the preceding acceptation, we may now define the differential calculus to be *a branch of the higher analysis, in which are investigated the derivatives of all functions that can be proposed, their peculiar properties assigned, and application made of these derivatives to the problems in which the continuity of the functions is one of the essential conditions.*

Considering the expression y' as known, when $f(x)$ is given, since y' is a function of x, it will itself be susceptible of variation, and in its turn have a derivative, which we shall denote by y''; similarly, the derivative of y'' will be y''', that of y''' will be y^{IV}...

Thus we shall understand what is meant by the *derivatives of the first, second, third order...*

659. We are still ignorant how x and h enter into α; let us now see how this function may be developed. Representing $y' + \alpha$ by P, the equ. (2) will become

$$f(x+h) = fx + Ph = y + Ph \ldots (3);$$

and assuming $x + h = z$, whence $h = z - x$, we have

$$fz = y + P(z-x),$$

where P, which was a function of x and h, will now be so of x and z. But these variables are independent of each other, since their difference h is arbitrary; and we may therefore look upon z as a given constant number, and make x alone vary, along, however, with y and P which contain x. Thus, changing x into $x+i$ in the last equation, 1°. fz will undergo no change; 2°. y will become $y + y'i + \beta i$; 3°. $P(z-x)$ will be changed into $(P + P'i + \gamma i)(z - x - i)$. Throughout the equation, retain the coefficients of those terms alone in which i is of the 1st degree, and the result will be [see N°. 576 and 668]

$$P = y' + P'(z-x) \ldots (4).$$

Let this equation be treated in the same manner as the one preceding: without repeating the calculation, it will immediately be seen that the change of x into $x+i$ gives

$$2P' = y'' + P''(z - x) \dots (5);$$

and similarly

$$3P'' = y''' + P'''(z - x) \dots$$
$$4P''' = y^{\text{IV}} + P^{\text{IV}}(z - x) \dots$$
$$\&c. = \&c.$$

Let $P, P', P'' \dots$ be now eliminated from the equs. (3), (4), (5)...; and, h being then reinstated in place of $z - x$, we shall have

$$f(x + h) = y + y'h + \frac{y''h^2}{2} + \frac{y'''h^3}{2.3} + \frac{y^{\text{IV}}h^4}{2.3.4} + \dots (A);$$

which is the formula called *Taylor's Theorem*, from the name of the celebrated geometer by whom it was discovered.

By the theorem (1), every function fx was decomposed into $y + y'h + \alpha h$, the third part αh containing the several terms into which h enters in a higher power than the first: we are now acquainted with the formation of these terms.

It is therefore demonstrated that when, in any function fx, an increment h is given to x, the series (A), the development of $f(x + h)$, *contains only integral and positive powers of* h, so long at least as x preserves an indeterminate value [N°. 655]. This series (A) serves for finding this development, in all cases in which we can deduce from fx the successive derivatives $f'x, f''x \dots$, or, $y', y'' \dots$ For example, let $y = x^m$: we deduce from it $y' = mx^{m-1}$, such being the coefficient of h in the development of $(x + h)^m$; and similarly $y'' = m(m - 1)x^{m-2}$, $y''' = m(m - 1)(m - 2)x^{m-3}$, &c. Substituting these values in the equ. (A), $f(x + h)$ becomes $(x + h)^m$, and we again find the series of Newton. We have only to know, therefore, that the 2nd term of $(x+h)^m$ is $mx^{m-1}h$, and we shall have the complete development, whatever be the exponent m.

We have already learnt [p. 151, &c.] how to develop different functions in series; but as the investigation of these expressions is a simple application of the principles of the calculus of derivations, we shall deduce them from the formula of Taylor, and according to the rules of this calculus; and no use consequently will be made of the former series till they have been demonstrated anew.

RULES FOR THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

660. The manner in which the derivative y' is composed of x depends on the primitive function y, and bears the impress of it. We must be able, for every function proposed, to form this derivative; and there are two

modes of obtaining it. The first results from the definition itself, expressed by the equation (1):

We must change x into $x+h$ in the proposed function fx, and execute the necessary calculations for exhibiting the term affected with the 1st power of h: the coefficient of this term will be the derivative required, or $f'x$.

661. The second process is founded on the property of limits [N°. 567]. Having changed x into $x+h$, we subtract the proposed function, and divide by h, in order to form the ratio $y'+\alpha$, of the increment of the function to that of the variable; then, h being diminished indefinitely, we investigate the value to which this ratio continually tends, *i. e.* we investigate the value in the case of $h=0$; this *limit* will be y'.

The most convenient plan, however, will be to compose rules for each species of function, so that we may be able to dispense with the direct application of these processes, in the different examples that we meet with; these rules then will give the derivative for each case, without the necessity of our reasoning specially upon it, in the same manner as a multiplication, an extraction of roots, or any other algebraical operation is effected.

662. Let $y=A+Bu-Ct\ldots$; $A, B, C\ldots$ being constants, and $u,t\ldots$ functions of x. To obtain the derivative, we shall apply the first rule. Accordingly, substituting $x+h$ for x, A undergoes no change, Bu becomes $B(u+u'h+\alpha h)$, Ct is changed into $C(t+t'h+\beta h)\ldots$; and thus the second function $f(x+h)$ is in this case

$$Y=(A+Bu-Ct\ldots)+(Bu'-Ct'\ldots)h+B\alpha h-C\beta h\ldots$$

Consequently, $y'=Bu'-Ct'\ldots$; and *the derivative of a polynomial is the sum of the derivatives of the several terms, retaining the signs and the coefficients: the constant terms have zero for their derivatives.*

Thus,

$$y=a^2-x^2 \text{ gives } y'=-2x,$$

$$y=1+4x^2-5x-3x^3 \text{ gives } y'=8x-5-9x^2.$$

663. For $y=u\times t$, u and t being functions of x, substituting $x+h$ for x, there results $f(x+h)$ or

$$Y=(u+u'h+\alpha h)(t+t'h+\beta h)$$
$$y'=u't+ut',$$

Similarly, $y = u.t.v$, making $t.v = z$, becomes $y = u.z$; whence $y' = u'z + uz'$: but we also have $z' = t'v + tv'$; and therefore

$$y' = tvu' + tuv' + uvt'.$$

Our demonstration equally extends to 4, 5... factors [see N°. 680]; and thus, *the derivative of a product is the sum of the derivatives obtained by considering each factor successively as the only variable.*

$y = (a + x)\ (a - x)$ gives $y' = (a - x)\,1 - (a + x)\,1$, since $+ 1$ and $- 1$ are the derivatives of the factors; or $y' = -2x$.

$$y = (a + bx)\,x^3 \text{ gives } y' = bx^3 + 3x^2\,(a + bx).$$

664. z and u being identical functions of x, let x be changed into $x + h$ in $z = u$; we shall have then

$$z + z'h + \alpha h = u + u'h + \beta h;$$

whence [N°. 576] $z' = u'$; and we similarly have $z'' = u''$, $z''' = u'''$... Consequently, *two identical functions have their derivatives also identical for all orders.*

665. Let $y = \frac{u}{t}$; we have then $ty = u$; whence $y't + yt' = u'$, and thence

$$y' = \frac{u' - yt'}{t} \text{ or } y' = \frac{u't - ut'}{t^2}.$$

Thus, *the derivative of a fraction is equal to that of the numerator, minus the product of the proposed fraction by the derivative of its denominator, this difference being divided by the denominator.*

666. Or, otherwise, *it is equal to the denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, this difference being divided by the square of the denominator.*

These rules may also be deduced by effecting the division of

$$Y = \frac{u + u'h + \alpha h}{t + t'h + \beta h} = \frac{u}{t} + \frac{tu' - ut'}{t^2}\,h + \ldots;$$

whence $y' =$ &c.

Thus,

$$y = \frac{x}{a} - \frac{x^2}{1 - x} \text{ gives } y' = \frac{1}{a} - \frac{(2 - x)x}{(1 - x)^2}.$$

For $$y = \frac{a + \frac{1}{4}bx^2}{3 - 2x},$$

$$y' = \frac{(3-2x)\,bx + 2\,(a + \frac{1}{4}bx^2)}{(3 - 2x)^2} = \frac{(3 - x)\,bx + 2a}{(3 - 2x)^2}.$$

667. If the numerator u be constant, we must make $u' = 0$ and we shall have $y' = -\frac{ut'}{t^2} = -\frac{yt'}{t}$: *the derivative of a fraction of which the numerator is constant, is minus the product of the numerator by the derivative of the denominator, divided by the square of this denominator.*

For example,

$$y = \frac{4}{x^2} \text{ gives } y' = -\frac{4.2x}{x^4} = -\frac{8}{x^3},$$

$$y = -\frac{1}{x^3} \text{ gives } y' = \frac{3x^2}{x^6} = \frac{3}{x^4}.$$

668. Let us now investigate the derivative of powers.

1°. If m be *integral* and *positive* in $y = x^m$, since $x^m = x.x^{m-1}$, the derivative relative to the first factor is $1.x^{m-1}$; and, according to the rule for products, the same may be said for each of the m factors; whence the derivative is $y' = mx^{m-1}$.

And if we are treating of $y = z^m$, z being a function of x, since $z^m = z.z^{m-1}$, the derivative relative to the 1st factor is $z'.z^{m-1}$; and each of the m factors z gives this same quantity. Consequently, the derivative is $y' = m\,z^{m-1}.z'$.

For example, $y = (a + bx + cx^2)^m = z^m$, making

$$z = a + bx + cx^2;$$

and we have

$$z' = b + 2cx,\ y' = mz^{m-1}z' = m\,(a + bx + cx^2)^{m-1} \times (b + 2cx).$$

For $y = x^3\,(a + bx^2)$, making $a + bx^2 = z$, we have $z' = 2bx$; and by the rule of N°. 663, $y' = 3x^2z + x^3z' = x^2\,(3a + 5bx^2)$.

Lastly, for $y = (a + bx)^2$, by assuming $a + bx = z$, we have $y = z^2$; whence

$$z' = b,\ y' = 2zz' = 2b\,(a + bx).$$

2°. When m is *integral* and *negative* ($m = -n$, and $y = z^m = z^{-n}$), we have $y = \frac{1}{z^n}$; whence $y' = \frac{-nz^{n-1}.z'}{z^{2n}}$, by virtue of the rule of N°. 667, and of what has been now demonstrated, 1°. Thus

$$y' = -nz^{-n-1}.z' = mz^{m-1}.z'.$$

For $\qquad y = \frac{a}{x^p}$, we have $y' = -\frac{pa}{x^{p+1}}$.

3°. When m is *fractional* $(m = \frac{p}{q},)$ we have $y = z^{\frac{p}{q}}$, whence, raising to the power q, $y^q = z^p$: p and q are here *integers positive or negative;* so that we may take the derivatives of the two sides, which are identical functions of x [N°. 664]; it follows from the two preceding cases that $qy^{q-1}.y' = pz^{p-1}z'$; and since $p = qm$, and $y = z^m$, we have [see N°. 670],

$$qz^{mq-m}y' = qmz^{qm-1}.z'; \text{ whence } y' = mz^{m-1}.z'.$$

Whatever therefore be the exponent m, the derivative of z^m, z being a function of x, is $mz^{m-1}.z'$, z' being a derivative deduced from $z = fx$. We say nothing of the irrational or imaginary exponents, which are included among the preceding ones [see p. 14]. We shall now, in conclusion, give a demonstration which embraces all the cases.

669. Let x be changed into $x + h$ in $y = x^m$; then

$$Y = (x + h)^m = y + y'h + \&c;$$

and dividing throughout by x^m, and making $h = xz$, we have

$$\left(\frac{x+h}{x}\right)^m = \left(1 + \frac{h}{x}\right)^m = (1 + z)^m = 1 + \frac{y'z}{x^{m-1}} + \&c.$$

But, $(1 + z)^m$ is independent of x, since, h being arbitrary, z may have any value whatever, even when that of x is determined: it follows therefore that our last member must also be devoid of x, and consequently that its second term be so in particular; whence $\frac{y'}{x^{m-1}} =$ constant; *i. e.* y' must be composed of x in such a manner that, being divided by x^{m-1}, its quotient shall be a function of m, as fm, or $y' = x^{m-1}.fm$.

Let us now determine this function fm. We have

$$(x + h)^m = x^m + hx^{m-1}.fm + \&c.$$

and consequently,

$$(x + h)^n = x^n + hx^{n-1}.fn + \&c.$$

$$(x + h)^{m+n} = x^{m+n} + hx^{m+n-1}.f(m+n) + \&c.,$$

changing m into n, and then into $m + n$. But, on multiplying the two

1st equations, we find for their product the 3rd equation, except that $fm + fn$ appears, instead of $f(m+n)$; and therefore [note, p. 206].

$$f(m+n) = fm + fn.$$

But, considering n as an increment of m, we have $f(m+n) = fm + nf'm + \&c.$; consequently, $fn = nf'm + \&c.$; and since the 1st side is independent of m, the 2nd must be so also, viz. $f'm =$ an unknown number a; whence $f'' = f''' = 0$, and

$$fn = an; \text{ whence } fm = am.$$

The point now is to determine the numerical constant a. For this purpose, since

$$(x+h)^m = x^m + hx^{m-1}\,am + h^2\ldots,$$

making $m = 1$, we have

$$(x+h) = x + ha; \text{ whence } a = 1;$$

thus, $fm = m$, and the derivative is $y' = mx^{m-1}$.

And in the next place, for $y = z^m$, where z is a function of x, we have

$$f(x+h) = (z + z'h + \ldots)^m = z^m + mz^{m-1}\,z'.h + \ldots, \text{ and } y' = mz^{m-1}.z'.$$

670. For $y = \sqrt[m]{z} = z^{\frac{1}{m}}$, we have

$$y' = \tfrac{1}{m} z^{\frac{1}{m}-1}.\, z' = \frac{z'}{m\sqrt[m]{z^{m-1}}};$$

the formula for the derivative of radical functions.

For $y = \sqrt[4]{(a+bx^2)^5}$, making $z = a + bx^2$, we have

$$y = z^{\frac{5}{4}},\ y' = \tfrac{5}{4} z^{\frac{1}{4}} z' = \tfrac{5}{4}\sqrt[4]{(a+bx^2)}.\,2bx;$$

and similarly

$$y = \sqrt[5]{x^3} \text{ gives } y' = \tfrac{3}{5} x^{\frac{3}{5}-1} = \frac{3}{5\sqrt[5]{x^2}}.$$

The roots of the 2nd degree being of such constant recurrence, an express rule is formed for the case of $m = 2$: $y = \sqrt{z}$ gives $y' = \dfrac{z'}{2\sqrt{z}}$; and, *the derivative of the square root of a function is the quotient from the derivative of the function divided by the double of the radical itself.*

For example,

$$y = a + b\sqrt{x} - \frac{c}{x} \text{ gives } y' = \frac{b}{2\sqrt{x}} + \frac{c}{x^2}.$$

For

$$y = (ax^4 + b)^2 + 2\sqrt{(a^2 - x^2)}.(x - b), \text{ we have}$$

$$y' = 6ax^2(ax^3 + b) + \frac{2a^2 - 4x^2 + 2bx}{\sqrt{(a^2 - x^2)}}.$$

Lastly,

$$y = \frac{x}{-x + \sqrt{(a^2 + x^2)}} \text{ gives}$$

$$y' = \frac{a^2}{\sqrt{(a^2 + x^2)}\,(2x^2 + a^2 - 2x\sqrt{a^2 + x^2})}.$$

Had we multiplied the fraction proposed above and below by $x + \sqrt{(a^2 + x^2)}$, we should have had

$$y = \frac{x^2}{a^2} + \frac{x}{a^2}\sqrt{(a^2 + x^2)},\ y' = \frac{2x}{a^2} + \frac{a^2 + 2x^2}{a^2\sqrt{(a^2 + x^2)}}.$$

671. A complex function $y = fx$ being given, suppose that by representing one part of this function by z, or making $z = Fx$, the proposed function becomes of a simpler form and is expressed in z alone, $y = \varphi z$; we have then these three equations, of which the 1st results from the elimination of z between the two others:

$$(1)\ y = fx,\ (2)\ z = Fx,\ (3)\ y = \varphi z.$$

We shall proceed to deduce the derivative y' from the two latter, without making use of the first. As, however, there are now two variables x and z, our previous notation becomes inefficient; for y' would appear equally to denote the derivative of the 1st equation and that of the 3rd; whereas, x is the variable in the one, z in the other, and the functions f and φ are totally different. But the derivative of y is expressed as well by dy as by y' [N°. 658]; and since the derivative of x is $x' = 1$, or $dx = 1$, we shall write $\frac{dy}{dx}$, to mark that the derivative of y is taken relatively to *the principal variable* x, which receives the arbitrary increment h: *dy is called the differential of y, an expression synonymous with derivative, and differing from it only as to the notation which it implies.*

This being premised, if x be changed into $x + h$ in (2), and the corresponding increment of z be denoted by k, $Z - z = k$, we have $k = \frac{dz}{dx}h + \ldots$; and in order that y may now become, in the equation

(3), the same quantity Y, as though x had been changed into $x + h$ in (1), we must change z into $z + k$, viz.

$$Y = \phi(z+k) = y + \frac{dy}{dz}k + \ldots;$$

where the coefficient of k is the derivative of $y = \phi z$, taken as though z were the only variable, and independent of x, as our notation expresses. Substituting for k its value, we have

$$Y = y + \frac{dy}{dz}\cdot\frac{dz}{dx}h + \&c.;$$

and hence*

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}.$$

The 2nd side is the product of the derivatives of the equations (2) and (3), *i. e.* of ϕz in respect to z, and of Fx in respect to x. *The derivative of a function of z, when z is a function of x, is the product of the derivatives of these two functions.*

It is unnecessary to give examples of this theorem, which has been already applied [N°. 668], to the derivative of z^m; it will also come much into use subsequently.

672. The equation $y = fx$ may be of so complex a form, as to make it necessary to introduce two variables z and u, representing functions of x; in which case the proposed equation $y = fx$.............. (1) results from the elimination of x between the three given equations

$$(2)\ldots z = Fx,\ (3)\ldots u = \psi x,\ (4)\ldots y = \phi(z, u);$$

and from these we have to deduce the derivative of the equ. (1), as though x had been at once changed in it into $x + h$. This transformation, made in the equations (2) and (3), gives for the increments k and i received by z and u,

$$k = \frac{dz}{dx}h + \ldots,\ i = \frac{du}{dx}h + \ldots$$

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* Observe that the two dz which appear here do not destroy each other; for the dz which divides dy indicates, not only a division, but also that the derivative or differential dy is deduced from the equation $y = \phi z$, as though the increment k were assigned to z, and not to x; in which case dz is $= 1$: on the other hand, the multiplier dz indicates that the derivative of z is deduced from the equation $z = Fx$, x having received the increment h, or $dx = 1$.

Let z be now changed into $z + k$, and u into $u + i$ in equ. (4). Since this step of the operation is the same, whether k and i have a determinate value, or remain arbitrary, z and u may be treated as independent variables. We are at liberty therefore first to change z into $z + k$ without altering u; then, in the result, to substitute $u + i$ for u without making any variation in z; and this double operation will lead to the same end as though the two changes had been made simultaneously.

Thus, substituting $z + k$ for z in $y = \varphi(z, u)$, u is assimilated to the other constants of the equation, whilst y becomes $y + \frac{dy}{dz}k + \ldots$. It remains to substitute in this $u + i$ for u. The 1st term y must now be considered as containing only a single variable u, and becomes

$$y + \frac{dy}{du}i + \ldots$$

The 2nd term $\frac{dy}{du}k$ is similarly a function of the variable u; and substituting $u + i$ for u, the development will commence with this same term [N°. 654]; so that the sum is

$$Y = y + \frac{dy}{du}i + \frac{dy}{dz}k + \ldots.$$

There is no need to consider the subsequent terms, as the only object of the calculation is, to find the coefficient of h, and the terms i^2, k^2, $ik\ldots$ would lead to h^2, $h^3\ldots$ Substitute therefore the values of k and i given above; and we shall have

$$Y = y + \left(\frac{dy}{du}\cdot\frac{du}{dx} + \frac{dy}{dz}\cdot\frac{dz}{dx}\right)h + \ldots;$$

where the coefficient of h is the derivative required, and the same as though it had been directly deduced from the proposed equation $y = fx$, viz.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} + \frac{dy}{dz}\cdot\frac{dz}{dx}.$$

The remark made in the preceding note applies here.

673. Should there be three variables in the transformed function $y = \varphi(z, u, t)$, it would but be requisite to add to the 2nd side a 3rd term of the same form, $\frac{dy}{dt}\cdot\frac{dt}{dx}$; and so on.

Hence, *the derivative of a function composed of different particular functions is the sum of the derivatives relative to each, considered separately and independently one of the other, according to the rule of* N°. 671.

It is obvious that the derivations of the products and the quotients are only particular cases of this theorem [N°. 663 to 667].

Let $y = \frac{a+bx}{(1-x)^2}$: we have $y = \frac{z}{u^2}$, making

$$z = a + bx,\ u = 1 - x, \text{ whence } z' = b,\ u' = -1.$$

The derivative of y, u being considered constant, is $\frac{z'}{u^2}$; we have $\frac{-2zu'}{u^3}$ for the derivative relative to u: the sum of these is the derivative required; and consequently

$$y' = \frac{b}{u^2} + \frac{2z}{u^3} = \frac{b+bx+2a}{(1-x)^3}.$$

$$y = \frac{(1-x^2)^2-(3-2x)x}{4-5x} \text{ becomes } \frac{z^2-u}{t}, \text{ making}$$

$z = 1 - x^2$, $u = 3x - 2x^2$, $t = 4 - 5x$, whence $z' = -2x$, $u' = 3-4x$, $t' = -5$;

and taking the derivatives of y successively, in respect to but one variable z, u, or t, and adding, we have

$$y' = \frac{2zz'}{t} - \frac{u'}{t} - \frac{(z^2-u)t'}{t^2} = \frac{16x^3-15x^4-7}{(4-5x)^2}.$$

When the values that ought to be equated to the variables z, u... are not very complicated, we usually perform the operations without having recourse to this transformation, leaving it to be tacitly supposed. Thus we deduce at once from

$$y = (a-2x+x^3)^3,\ y' = 3(a-2x+x^3)^2\,(3x^2-2).$$

674. The derivative y' having been found, by treating this function of x according to the rules just explained, we shall from it deduce the derivative y'' of the 2nd order: this will in like manner give y''', then y^{IV}, &c.

For example, $y = x^{-1}$ gives $y' = -x^{-2}$, $y'' = 2x^{-3}$,

$$y''' = -2.3x^{-4} \text{ \&c., } y^{(n)} = \pm\, 2.3 \ldots nx^{-(n+1)}.$$

In like manner, $y = \sqrt{x} = x^{\frac{1}{2}}$ gives $y' = \frac{1}{2}x^{-\frac{1}{2}}$,

$$y'' = -\tfrac{1}{2}.\tfrac{1}{2}x^{-\frac{3}{2}},\ y''' = \frac{1.3}{2^3}x^{-\frac{5}{2}},\ y^{IV} = -\frac{1.3.5}{2^4}x^{-\frac{7}{2}},$$

$$y^{(n)} = \mp\frac{1.3.5\ldots(2n-3)}{2^n\sqrt{x^{2n-1}}}.$$

For $y = x^m$, we have $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$,

$$y^{(n)} = m(m-1)(m-2)\ldots(m-n+1)x^{m-n}.$$

675. It will be easy now to apply the theorem of Taylor [*A*, N°. 659] to all algebraic functions, *i. e.* to obtain their development in a series of ascending powers of h, when x is changed in them into $x+h$.

I. Let $y = x^{-1}$; we have already found $y', y''\ldots$; and hence

$$\frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} + \frac{h^2}{x^3} - \ldots \pm \frac{h^n}{x^{n+1}}.$$

This development is a progression by quotient [N°. 579].

II. $y = \dfrac{x^2-a^2}{x} = x - \dfrac{a^2}{x}$ in like manner gives

$$f(x+h) = \frac{x^2-a^2}{x} + \left(1+\frac{a^2}{x^2}\right)h - \frac{a^2h^2}{x^3} + \ldots,$$

III. For $y = \sqrt{x}$, we find $y', y''\ldots$; and substituting in the series of Taylor, we have

$$\sqrt{(x+h)} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{1.h^2}{2.4\sqrt{x^3}}\ldots \pm \frac{1.3.5\ldots(2n-3)}{2^n\sqrt{x^{2n-1}}}.\frac{h^n}{2.3\ldots n}\ldots$$

IV. Generally, $y = x^m$ gives the development of Newton's formula, whatever be the exponent m.

$$(x+h)^m = x^m + mx^{m-1}h + m.\frac{m-1}{2}x^{m-2}h^2 + \ldots$$

$$+ m.\frac{m-1}{2}.\frac{m-2}{3}\ldots\frac{m-n+1}{n}x^{m-n}h^n\ldots,$$

EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

676. To obtain the derivative of $y = fx = a^x$, following the rule of N°. 660, we find

$$f(x+h) = a^{x+h} = a^x.a^h = a^x + y'h + \&c\ldots\ [\text{N°. } 655];$$

whence, dividing by a^x, $a^h = 1 + \dfrac{y'}{a^x}h + \&c.$

But, the 1st side of this equation being independent of x, the 2nd, and the coefficient of h individually, must be so also; and consequently, y' must be so composed of x that, when divided by a^x, its quotient shall be a constant k, some function yet unknown of the base a, or $y' = ka^x$. Thus,

$$y' = ka^x,\ y'' = k^2a^x,\ y''' = k^3a^x \dots\ y^{(n)} = k^na^x;$$

and from the formula of Taylor,

$$a^x = 1 + kx + \frac{k^2x^2}{2} + \frac{k^3x^3}{2.3} + \frac{k^4x^4}{2.3.4} + \dots$$

The constant k is determined as in N°. 585: we assume $x = 1$; then, in $a = 1 + k + \frac{1}{2}k^2\dots$, we make $k = 1$, and denote the base corresponding to this by e, e being $= 2.718281828\dots$ Lastly, we assume $kx = 1$ in the first series; when the 2nd side becomes e, and we have $a^{\frac{1}{k}} = e$, $a = e^k$; and thus

$$k = \frac{\text{Log}\, a}{\text{Log}\, e} = la = \frac{1}{\log e},$$

accordingly as the system of log is any whatever, or has the base e, or, lastly, the base a. We here employ the notation agreed on p. 159; la denotes that the base of the log is e, or that the log referred to are the Napierian, &c. To conclude, the differential Calculus again affords the series demonstrated N°. 585, and of course the consequences that were drawn from them.

677. Let $y = a^z$, z being a function of x, $z = fx$; the rule of N°. 671 gives $y' = ka^z.\ z' = a^z.\ z'la$; z' being deduced from $z = fx$. Thus, *the derivative of an exponential quantity is the product of this quantity by the derivative of the exponent, and by the constant k, which is the Napierian log of the base.*

$$y = e^{mz} \text{ gives} \dots\dots\dots\dots y' = e^{mz}.\ mz'.$$
$$y = a^{3x+1} \dots\dots\dots\dots\dots\dots y' = a^{3x+1}.\ 3la$$
$$y = a^{\sqrt{(2x+1)}} \dots\dots\dots\dots\dots y' = a^{\sqrt{2x+1}}.\ \frac{la}{\sqrt{(2x+1)}}.$$

678. For $y = \text{Log}\, x$, the rule of N°. 660 leads to

$$Y = \text{Log}\,(x + h) = \text{Log}\, x + y'h + \&c.;$$

whence

$$\text{Log}\,(x + h) - \text{Log}\, x = \text{Log}\left(\frac{x + h}{x}\right) = \text{Log}\,(1 + z) = y'xz + \&c.,$$

assuming $h = xz$. But it is observable that here, as in N°. 669, z is independent of x, since by making suitable changes in the arbitrary quantity h, z may remain constant whilst x varies. The 2nd side therefore of our equation, and the term $y'xz$ individually, must not contain x; so that y' is composed of x in such a manner that the product $y'x$ is some constant M, $y'x = M$. Thus [N°. 674],

$$y' = \frac{M}{x}, y'' = -\frac{M}{x^2}, y''' = \frac{2M}{x^3}, y^{\text{IV}} = -\frac{2.3M}{x^4} \ldots,$$

$$\text{Log}\,(x+h) = \text{Log}\,x + M\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots\right),$$

$$\text{Log}\,(1+z) = \qquad M(z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 + \ldots).$$

As to the value of the unknown quantity M, it depends on the base a of the system. Let t be the log of $1 + z$; then $a^t = 1 + z = 1 + kt + \frac{1}{2}k^2t^2 \ldots$; whence $z = kt(1 + \frac{1}{2}kt + \ldots)$; and substituting this value in the series for Log $(1+z)$, we have, the common factor t being suppressed,

$$1 = Mk(1 + \tfrac{1}{2}kt \ldots) - \tfrac{1}{2}Mk^2t(1 + \tfrac{1}{2}kt \ldots)^2 \ldots$$

And this relation must subsist, whatever t be, k and M retaining their constant values. Suppose, therefore, that $z = 0$; then $t = 0$; whence $1 = Mk$, and

$$M = \frac{1}{k} = \log e = \frac{1}{la}, y' = \frac{1}{kx} = \frac{1}{xla}.$$

It will be easily seen that M is what has been denominated [p. 160] the *modulus*, the constant factor in a system of logarithms, which serves to transform them into Napierian logarithms, and the converse. We might therefore again deduce the same series and the same theory as before.

679. Let $y = \text{Log}\,z$, z being a function of x; we have [N°. 671],

$$y' = \frac{Mz'}{z} = \frac{z'}{kz} = \frac{z'}{zla}.$$

The derivative of the log of a function is the derivative of this function, multiplied by the modulus and divided by the function itself. The factor M is 1, in the case of Napierian logarithms.*

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* The derivatives of log might have been derived from those of exponentials: $y = a^z$ gives $y' = a^z.z'.la = yz'.la$; whence $z' = \frac{y'}{yla} = \frac{My'}{y}$. Conversely, from this last equation may be deduced the preceding one, *i.e.* the derivative y' of a^z.

$$y = l\left(\frac{u}{t}\right) = lu - lt \qquad \text{gives} \qquad y' = \frac{u'}{u} - \frac{t'}{t} = \frac{tu' - ut'}{ut},$$

$$y = \text{Log}\, z^n = n\,\text{Log}\, z \qquad \ldots\ldots \qquad y' = \frac{Mnz'}{z},$$

$$y = \text{Log}\,\frac{x}{\sqrt{(1+x^2)}} \qquad \ldots\ldots \qquad y' = \frac{M}{x(1+x^2)}$$

$$y = \text{Log}\,(x + \sqrt{1+x^2}) \qquad \ldots\ldots \qquad y' = \frac{M}{\sqrt{(1+x^2)}},$$

$$y = l\sqrt{\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)} \qquad \ldots\ldots \qquad y' = \frac{1}{\sqrt{(1+x^2)}},$$

$$y = l\left(\frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}\right) \qquad \ldots\ldots \qquad y' = \frac{-1}{x\sqrt{1-x^2}}$$

680. Logarithms serve frequently to facilitate the investigation of derivatives.

I. Let $y = utvz\ldots$; we deduce from it $ly = lu + lt + lv + \ldots$; and thence $\frac{y'}{y} = \frac{u'}{u} + \frac{t'}{t} + \frac{v'}{v}\ldots$ Multiplying this by the proposed equation, we have y', and the result proves that the rule of N°. 663 is true, whatever be the number of the factors.

II. $y = z^t$ gives $ly = t.lz, \frac{y'}{y} = \frac{tz'}{z} + t'lz$ and consequently,

$$y' = z^t\left(\frac{tz'}{z} + t'lz\right).$$

III. From $y = a^{b^z}$, we deduce $ly = b^z.\,la,\ y' = a^{b^z}.\,b^z.z'la\,lb.$

IV. $y = z^{t^u}$ gives $ly = t^u lz$; and consequently,

$$\frac{y'}{y} = \frac{t^u z'}{z} + lz.t^u\left(\frac{ut'}{t} + u'lt\right),\ y' = z^{t^u}.\,t^u\left(\frac{z'}{z} + u'lt.\,lz + \frac{ut'.\,lz}{t}\right).$$

CIRCULAR FUNCTIONS.

681. To determine the derivative of $y = \sin x$, the radius being 1, we have

$$\sin\,(x \pm h) = \sin x \cos h \pm \cos x \sin h = y \pm y'h + \&c.;$$

whence

$$2\cos x \sin h = 2y'h + \&c.,\ \sin h = \frac{y'}{\cos x}h + \&c.$$

The 2nd side must be devoid of x; thus, the coefficient of h must be some unknown constant A, or $y' = A\cos x$, $\sin h = Ah +$ &c.; whence we deduce $\frac{\sin h}{h} = A +$ &c.; and diminishing h, we see that A is the limit of the ratio of the sine to the arc h. But this limit [N°. 362] we know to be 1; thus, $A = 1$, $y' = \cos x$.

Similarly, for $z = \cos x$,

$$\cos(x \pm h) = \cos x \cos h \mp \sin x \sin h = z \pm z'h + \text{\&c.};$$

from this we deduce, by subtraction, $2 \sin x \sin h = -2z'h +$ &c.; and hence $\sin h = \frac{-z'}{\sin x} h +$ &c. But we have already found $\sin h = h +$ &c.; and, by the comparison of terms, we have $\frac{-z'}{\sin x} = 1$, $z' = -\sin x$.

The derivatives of $\sin x$ and $\cos x$ being thus found, we easily pass to those of the higher orders, and it appears that they recur periodically in the order

$$\sin x, \cos x, -\sin x, -\cos x.$$

The theorem of Taylor consequently gives

$$\sin(x + h) = \sin x \left(1 - \frac{h^2}{2} + \frac{h^4}{2.3.4} \cdots\right)$$
$$+ \cos x \left(h - \frac{h^3}{2.3} + \frac{h^5}{2 \ldots 5} \cdots\right).$$

Let h be now assumed $=0$ and $=90°$ successively, and we shall arrive at the same series with those of p. 165; whence result the formation of the tables, the ratio π of the diameter to the circumference, and the formulæ of N^{os}. 588 to 595.

682. For $y = \sin z$, we have $y' = z'. \cos z$.
If $y = \cos z$, we have $y' = -z'. \sin z$.

Let $y = \tan z = \frac{\sin z}{\cos z}$; then [N°. 666] $y' = \frac{z'}{\cos^2 z} = z'. \sec^2 z$.

The derivative of the tangent of an arc is the square of the secant, multiplied by the derivative of the arc, a function of x.

For $y = \cot z$, we have $y' = \frac{-z'}{\sin^2 z}$.

Since $\sqrt{\frac{1}{2}(1 + \cos x)} = \cos \frac{1}{2} x$, the derivative of this radical is $-\frac{1}{2} \sin \frac{1}{2} z$; what, in fact, the regular process gives.

Let $y = \cos mz$; we have $y' = -mz'.\sin mz$,

$$y = \sin mz \ldots\ldots\ldots y' = mz'.\cos mz.$$

From $y = \cos(lx)$, we deduce $y' = -\dfrac{\sin(lx)}{x}$.

For $y = \cos x^{\sin x}$, we have $ly = \sin x.\,l\cos x$; and hence

$$y' = \cos x^{\sin x}\left(\cos x.\,l\cos x - \frac{\sin^2 x}{\cos x}\right).$$

$y = \dfrac{1}{\cos z}$ gives $y' = \dfrac{z'.\tan z}{\cos z} = z'.\tan z.\sec z$; and this is the derivative of $y = \sec z$.

For $y = l\sin z$, we have $y' = \dfrac{(\sin z)'}{\sin z} = \dfrac{z'\cos z}{\sin z} = z'.\cot z$.

$$y = l\cos z \ldots\ldots\ldots\ldots y' = \frac{(\cos z)'}{\cos z} = -z'.\tan z,$$

$$y = l\tan z \ldots\ldots\ldots\ldots y' = \frac{z'}{\cos^2 z\,\tan z} = \frac{2z'}{\sin 2z}.$$

M. Legendre has given in the *Conn. des Temps* for 1819 series proper for the calculation of the log of sin, cos, and tan.

Let $y = \log\sin x$; denoting the modulus by M, and applying Taylor's theorem, we have

$$y' = M\cot x,\ y'' = -\frac{M}{\sin^2 x},\ y''' = \frac{2M\cos x}{\sin^3 x}\ldots$$

$$\log\sin(x+h) = \log\sin x + Mh\cot x - M\frac{h^2\cot x}{\sin 2x} + \text{\&c.};$$

and transposing log sin x, there results, for the diff. Δ between this log and that of sin $(x+h)$,

$$\Delta = Mh\cot x\left(1 - \frac{h}{\sin 2x} + \frac{h^2}{3\sin^2 x}\right) - \frac{Mh^4}{4\sin^4 x}\left(1 - \tfrac{2}{3}\sin^2 x\right).$$

We similarly find for the diff Δ_1 and Δ_2 between the log of the cosines and of the tangents of $x+h$ and x:

$$\Delta_1 = -Mh\tan x\left(1 + \frac{h}{\sin 2x} + \frac{h^2}{3\cos^2 x}\right) - \frac{Mh^4}{4\cos^4 x}\left(1 - \tfrac{2}{3}\cos^2 x\right),$$

$$\Delta^2 = \frac{2Mh}{\sin 2x}\left(1 - h\cot 2x + \tfrac{2}{3}h^2 + \tfrac{4}{3}h^2\cot^2 2x\right) + \frac{4Mh^4\cos 2x}{\sin^4 2x}\left(1 - \frac{\sin^2 2x}{6}\right);$$

h being the length of the differential arc [Vide Vol. I. p. 303].

Thus, to obtain log sin 27° 33′, knowing the log of sin 27° 30′, we make $h = 3' = 3 \sin 1'$; and the calculation is as follows:

h	$\bar{4}\cdot94085$		$\bar{4}\cdot94085-$
M........	$\bar{1}\cdot63778$		$\bar{4}\cdot86215$
cot x	$0\cdot28352$	sin $2x$... $-$	$\bar{1}\cdot91336$
1st term...	$\bar{4}\cdot86215$	2nd term...	$\bar{7}\cdot88964-$

The 3rd term does not give any thing, when we limit ourselves to seven decimals.

1st term ... =	0·00072803
2nd .... =	0·00000078
3rd........... =	0·
Δ .... =	0·0007273
log sin x......=	$\bar{1}\cdot6644056$
log sin 27° 33′=	$\bar{1}\cdot6651329$

This method is particularly useful when we wish to calculate the log to a high degree of approximation.

683. Suppose that x is the sine of an arc y; which we write thus:

$$y = \text{arc}\,(\sin = x), \text{ or } x = \sin y.$$

The variable x which receives the increment h is no longer the arc, but the sine. Now, the equation $x = \sin y$ gives

$$1 = y' \cos y,\ y = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-x^2)}},$$

Consequently,

$$\text{for } y = \text{arc}\,(\sin = z), \text{ we have } y' = \frac{z'}{\sqrt{(1-z^2)}},$$

$$y = \text{arc}\,(\cos = z)\ldots\ldots\ldots\ldots y' = \frac{-z'}{\sqrt{(1-z^2)}}.$$

Let $y = \text{arc}\,(\tan = z)$; then $z = \tan y,\ z' = \dfrac{y'}{\cos^2 y}$,

$$y = z \,.\, \cos^2 y;\ \text{and since } \cos^2 y = \frac{1}{1+z^2},\ y' = \frac{z'}{1+z^2}.$$

Thus, *the derivative of an arc, expressed by its sine, is* 1 *divided by the cosine; that of an arc, expressed by its cosine, is* -1 *divided by the sine; lastly, that of an arc expressed by its tangent is* 1 *divided by* 1 + *the square of this tangent.*

Should the radius, instead of being 1, be r, we shall have, rendering the formulæ homogeneous [N°. 347, 2°],

$$y = \text{arc}\,(\sin = z),\ y = \text{arc}\,(\cos = z),\ y = \text{arc}\,(\tan = z),$$

$$y' = \frac{rz'}{\sqrt{(r^2 - z^2)}},\ y' = \frac{-rz'}{\sqrt{(r^2 - z^2)}},\ y' = \frac{r^2 z'}{\cos^2 z}.$$

DERIVATIVES OF EQUATIONS.

684. If the equation $F(x,y) = 0$ be resolved and so brought under the form $y = fx$, it will be easy thence to deduce y', y'', y'''... This resolution, however, which is rarely possible, is by no means necessary; for suppose that, for y, its value fx be substituted in the proposed equation; there will result then a function of x identically nothing, which we shall denote by $z = F(x, fx) = 0$; and the derivatives z', z'', z'''... will also be nothing [N°. 664]. But, to obtain z', we must, according to what has been observed in N°. 672, simplify the complex expression $F(x, fx)$, by equating the group of terms fx to y, and apply the rule of that N°. to the equation $z = F(x,y) = 0$; and this equation is the very one proposed. Hence

$$z' = \frac{dz}{dx} + \frac{dz}{dy}.y' = 0,\ y' = -\frac{dz}{dx} : \frac{dz}{dy}.$$

These two terms are known functions of x and y, and are called *partial Differentials*. For example, from the equation $y^2 + x^2 - r^2 = z = 0$, we deduce

$$\frac{dz}{dx} = 2x,\ \frac{dz}{dy} = 2y,\ x + yy = 0,\ y' = -\frac{x}{y};$$

and, similarly,

$$x^2 + y^2 - 2rx = r^2 \text{ gives } yy + x - r = 0,\ y' = \frac{r-x}{y};$$

$$x^4 + 2ax^2y = ay^3 \text{ leads to } (2ax^2 - 3ay^2)\,y' + 4x^3 + 4axy = 0.$$

685. It is true that y' is here expressed in x and y, and not in x alone, as would have been the case had we resolved the equation $F(x,y) = 0$. If we wish to have y' in terms of x alone, it will remain to eliminate y between the equation $z = 0$, and its derivative $z' = 0$. Thus we see that, in the first example, $x^2 + y^2 = r^2$, we have $x + yy' = 0$; whence eliminating y, $x^2 = y'^2\,(r^2 - x^2)$, $y' = \frac{x}{\sqrt{(r^2 - x^2)}}$.

This elimination, which, however, is seldom of much service, raises y' to the same degree in which y appears in the proposed equation $z=0$; for, if $y=fx$ have n values, since the calculation for the derivative leaves in y' the same radicals that there are in fx [N°. 670], y' has also n values. If y' be only of the 1st degree in $z'=0$, the cause will be that y appears there also, and contains in itself these radicals which the subsequent elimination of y will make apparent.

686. The equation $z'=0$ contains x, y' and y, which are functions of x; and the reasoning of N°. 684 proves that we may hence deduce the equation $z''=0$, considering y and y' as containing x, and applying the rule of N°. 672. The notation that has been made use of must now be extended. For example, $\frac{d^2y}{dx\,dy}, \frac{d^3y}{dx^2\,dy}$ will signify that, in the former, the derivative has been first taken, considering x as variable, and that we have then taken the derivative of the result relatively to y; in the second, the derivatives are taken three times successively, twice in respect to x, and once in respect to y. Also, it follows from what has been said [N°. 672], that these derivatives may be taken in what order we think proper: in the 2nd case, for instance, we might take them, first in respect to y, then twice in respect to x; or, otherwise, once for x, once for y, and once again for x [vide N°. 703].

Accordingly, the equation $z'=\frac{dz}{dx}+\frac{dz}{dy}.y'=0$ gives

$$\frac{d^2z}{dx^2}+2y'.\frac{d^2z}{dx.dy}+\frac{dz}{dy}y''+\frac{d^2z}{dy^2}y'^2=0.$$

This equation of the 1st degree in y'' will give that derivative in terms of x, y, and y'; y' may be eliminated by means of the equation $z'=0$; and if y be then got quit of through the equation $z=0$, the degree of y'' will be raised.

The last example of N°. 684, $(2ax^2-3ay^2)\,y'+4x^3+4axy=0$, taking the derivatives relatively to x, y and y', as independent variables, gives

$$(2ax^2-3ay^2)\,y''+12x^2+4ay+8axy'-6ayy'^2=0.$$

687. If the proposed equation $z=0$ contain a constant term, it will disappear from the derivative $z'=0$, as we have shown in N°. 662. Thus, $x^2+y^2=r^2$ gives $x+yy'=0$, which is independent of r, and expresses a property common to all the circles which have their centre at the origin. We can also get quit of any other constant we think

proper, by eliminating it between the equations $z=0$, $z'=0$; only the constant that vanished in the first instance will make its appearance again: $y = ax + b$ gives $y' = a$, which does not contain b; and eliminating a, we have $y = y'x + b$, which is independeut of a.

The derivative of the 2nd order does away with a 2nd constant; that of the 3rd order, with a 3rd constant, &c.; and the result thus expresses a property of the proposed equation, which exists whatever these constants be: $y^2 = a - bx^2$ gives $yy' = -bx$, $yy'' + y'^2 = -b$; and eliminating b, there results this equation, cleared of a and b, $yy' = (yy'' + y'^2)x$.

We may likewise get quit of a constant c, by resolving the proposed equation, under the form $c = f(x, y)$, and differentiating. And since these two processes must lead to equivalent results, and the latter of them introduces radicals depending on the degree of c, it is evident that if we prefer eliminating c between the equations $z = 0$, $z' = 0$, the degree of y' will be raised. For example,

$$y^2 - 2cy + x^2 = c^2,\ (y - c)\,y' + x = 0$$

give

$$(x^2 - 2y^2)\,y'^2 - 4xyy' - x^2 = 0.$$

688. It appears, therefore, that any derivative of the order n, of the equation $z = F(x, y) = 0$, can contain $y^{(n)}$ only in the 1st degree; when it is otherwise, the equation does not arise from immediate differentiation; but from our having eliminated some constant, or y, or x, by means of the proposed equation.

CHANGE OF THE INDEPENDENT VARIABLE.

689. Every general question, treated on by the differential Calculus, leads to an expression in $x, y, y', y''\ldots$, such as

$$\psi\,[x, y, (y'), (y'')\ldots];$$

and if we wish then to apply it to a specific example $y = Fx$, we must deduce (y'), $(y'')\ldots$, substitute their values in ψ, and this function will be expressed solely in terms of x. The brackets [ ] are introduced to indicate that x is the *principal variable*, and receives the increment h. But it may be that, instead of $y = Fx$, there are given two equations which connect y and x with a third variable t:

$$y = \varphi t,\ x = ft\ldots\ (a).$$

The direct course would now be to eliminate t between these two equations, and having thus obtained $y = Fx$, to derive from it (y'), (y'').... and substitute in ψ. This calculation however, generally long, or even impossible to be effected, is not necessary; it will be sufficient to express the function ψ in terms of t, by means of the equations (a) and their derivatives φ', f'...; these being no longer taken in respect to x, but to t, thus become the *independent variable*. Let us see therefore how the given function ψ can be modified so as to contain t, φ', f'..., instead of $x, y, (y')$....

If h, k, i be the simultaneous increments of the variables x, y, t,

$$
\begin{aligned}
y &= Fx \text{ gives } k = (y')h + \tfrac{1}{2}(y'')h^2 + \dots \;(1),\\
y &= \varphi t \;\;\dots\;\; k = y'i \;+ \tfrac{1}{2}y''i^2 \;+ \dots \;(2),\\
y &= ft \;\;\dots\;\; h = x'i \;+ \tfrac{1}{2}x''i^2 \;+ \dots \;(3).
\end{aligned}
$$

These derivatives correspond to the respective functions F, φ, f; (y') is the derivative of Fx relatively to x; y' and x' are those of the equations (a) in respect to t, or

$$(y') = \frac{dy}{dx}, y' = \frac{dy}{dt} = \varphi' t, \; x' = \frac{dx}{dt} = f't.$$

The function ψ is given in terms of (y'), (y'')..., and our object is to express it in terms of x', y', x'', y''..., which are known functions of t.

Now, equating the values of k, and then substituting for h the series (3), confining ourselves to the two first powers of h, we have

$$(y')\,x'i + [(y')\,x'' + (y'')\,x'^2].\tfrac{1}{2}i^2 \dots = y'i + \tfrac{1}{2}y''i^2 \dots;$$

and since i is quite arbitrary, this gives [N°. 576]

$$(y')\,x' = y', \; (y')\,x'' + (y'')\,x'^2 = y'', \text{ \&c.}$$

Hence, to express ψ in terms of t alone, we must substitute for x, y, (y'), (y'')... the values $x = ft$, $y = \varphi t$,

$$(y') = \frac{y'}{x'}, \; (y'') = \frac{x'y'' - y'x''}{x'^3} \dots\dots\dots\dots \;(E).$$

(y''), (y''')... might be derived from the value of (y'), which is the quotient of the derivatives relative to t, deduced from the equations (a): $(y') = \dfrac{y'}{x'} = \dfrac{dy}{dt} : \dfrac{dx}{dt}$. For (y') represents a function of x, $(y') = F'x$; and this, since $x = ft$, may in turn be considered a function of t, such as $(y') = \varphi_{,}t$.

We may therefore apply our previous reasoning to these three last equations, and shall thence conclude that (y'') is the quotient of the

derivatives of φt and ft relatively to t. But, that of $\varphi t = \frac{y'}{x'}$ is $\frac{x'y'' - y'x''}{x'^2}$; and consequently, dividing this by x' the derivative of ft, we arrive again at the previous expression (D) for (y'').

Similarly, the derivative of this value of y'' being divided by x', we have

$$(y''') = \frac{y'''}{x'^3} - \frac{3x''y''}{x'^4} - y'\left(\frac{x'''}{x'^4} - \frac{3x''^2}{x'^5}\right) \dots (E);$$

and so on for the rest. There are three modes therefore of treating ψ. We may

1°. Eliminate t between the equations (a), from the resulting equation $y = Fx$ deduce (y'), (y'')..., and substitute these values in ψ:

2°. Substitute for (y'), (y'')... their values (D), (E)...; when ψ will be expressed in terms of x, y, y', x'..., and subsequently of t, by means of the equations (a) and their derivatives:

3°. And lastly, transform the fraction $(y') = \frac{y'}{x'}$ into a function of t, then take the derivatives relatively to t, dividing each time by x' or $f't$, and substitute the values thus obtained for (y'), (y'')... in ψ.

690. Let r be a given function of t, $r = ft$; and suppose that the equations (a) are $x = r \cos t$, $y = r \sin t$;* then

$$x' = r' \cos t - r \sin t \qquad y' = r' \sin t + r \cos t,$$
$$x'' = r'' \cos t - 2r' \sin t - r \cos t \qquad y'' = r'' \sin t + 2r' \cos t - r \sin t,$$
&c....

and if, in ψ, we substitute, first the expressions (D), which will introduce y', x', y''... instead of (y'), (y'')...; then the values that we have just obtained, there will appear in the result only t and r, r', r''..., which are known in terms of t, from the equation $r = ft$. Thus, if

$$\psi = \frac{x(y') - y}{y(y') + x}, \text{ we have } \psi = \frac{y'x - x'y}{yy' + xx'},$$

* These are the equations by means of which rectangular co-ordinates are transformed into polar; and when a differential formula ψ has been found for the 1st system, this calculation will reduce it to one suitable for the 2nd system. Of the following values of ψ, the first expresses the tangent of the angle β, that a radius vector makes with the tangent to any curve; the other is its radius of curvature [N°. 724, 733]. These expressions therefore are transformed, by our process, into others corresponding to polar co-ordinates.

from the value (D) of (y'); and since those of x' and y' give $y'x - x'y = r^2$, $yy' + xx' = rr'$ (this equation is the derivative of $y^2 + x^2 = r^2$), we finally find $\psi = \frac{r}{r'}$.

Similarly, let $\psi = \frac{[1 + (y')^2]^{\frac{3}{2}}}{(y'')} = \frac{(x'^2 + y'^2)^{\frac{3}{2}}}{x'y'' - y'x''}$: we have

$$x'^2 + y'^2 = r^2 + r'^2, \; x'y'' - y'x'' = r^2 + 2r'^2 - rr'';$$

and therefore

$$\psi = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{r^2 + 2r'^2 - rr''}.$$

Thus ψ is known for every value of t, since the formulæ will be expressed in t alone, when we have $r = ft$.

691. When ψ has been thus transformed, t is the independent variable. If we wish to restore x to its previous state, we have only to assume $x' = 1$, whence... $x'' = x''' = ... = 0$; for y' in this case again becomes (y'), and y'' is consequently changed into (y''), &c. We may see this verified in our examples.

ψ having been once generalized and adapted to the principal variable t, it is indifferent whether or not x were that variable originally, and we may suppose that it was some other variable u that was independent. But, when we make $x' = 1$, we at once imply that the principal variable is x; $t' = 1$ establishes the same thing therefore for t; or *the condition which expresses that t is the principal variable is* $t' = 1$; whence $0 = t'' = t'''...$; that is to say, *the differential of t is constant.* When ψ has been generalized so as to suit any principal variable, *no differential is constant.*

Since the series (3), p. 255, is derived from the equation $x = ft$, x' denotes $\frac{dx}{dt}$, and $x' = 1$ shows that the differential of x relative to any third variable t is constant.

Similarly, if we assume $t' = 1$, in order that t may become the principal variable, it must be understood that *the derivative of t, relative to any other variable u is constant.* The use of this proposition will be seen from what follows.

692. In case ψ do not contain x, or $\psi = [y, (y'), (y'')...]$, the equation $x = ft$ is no longer necessary; it will be sufficient if we have its derivative $x' = f't$; for the relations (D) do not introduce x into ψ, but only x', y'..., and the preceding calculations are easy. But, if

this given derivative equation should contain t, instead of x, as the dependent variable, should we, for instance, have $F(t, t', x) = 0$, it will be necessary first of all to generalize this equation, so that no differential shall be constant in it, by substituting $\frac{t'}{x'}$ for t'; and we must then, in order that t may be the principal variable, make $t' = 1$; which is tantamount to at once replacing t' by $\frac{1}{x'}$.

Suppose, for example, that the equations (a) are $y = \varphi t$, $y = (t')$, the derivative being here relative to x; that it may become so to t, we must assume

$$y = \frac{1}{x'}; \text{ whence } x' = \frac{1}{y}, x'' = -\frac{y'}{y^2}, \text{ \&c.};$$

and ψ having been generalized by means of the relations (D), we must introduce these values, and ψ will then be expressed in terms of t and of derivatives relative to t, if x do not enter into it. Thus

$$\psi = \frac{[1+(y')^2]^{\frac{3}{2}}}{(y'')} \text{ becomes } \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''}, \text{ and then } \frac{(1+y^2y'^2)^{\frac{3}{2}}}{y(yy''+y'^2)}.$$

It is obvious that ψ will be expressed in a function of t, since y', y'' have derivatives relative to t, deduced from $y = \varphi t$; and ψ therefore will be known for any value of t.

Similarly, if the equations (a) be $y = \varphi t$, $t'^2 = 1 + (y')^2$, the derivatives being relative to x, we shall change the latter of these expressions into $t'^2 = x'^2 + y'^2$; and t' being then assumed $= 1$, so that t may be the independent variable, we have $x'^2 + y'^2 = 1$; whence $x'x'' + y'y'' = 0$. Our generalized value of ψ becomes therefore, eliminating x'' or y'', $\psi = \frac{x'}{y''}, = -\frac{y'}{x''}$. And that ψ may now be expressed in a function of t, it remains only to deduce, from $y = \varphi t$, the derivatives y', y'', relative to t; then $x' = \sqrt{(1-y'^2)}$, and to substitute in $\psi = x' : y''$. If, instead of $y = \varphi t$, $x = ft$ had been given, we should have operated in the same manner on the second value of ψ.

Again, let $\psi = \frac{(y''')}{x(y'')}$, x being the principal variable originally, and it being required that t should become so, and also that $t'^2 = 1 + (y')^2$. The formulæ D, E give, after having multiplied above and below by x'^5,

$$\psi = \frac{y'''x'^2 - 3x'x''y'' - y'x'x''' + 3y'x''^2}{xx'^2(x'y''-y'x'')};$$

whilst from $x'^2 + y'^2 = 1$, we derive

$$x'x'' + y'y'' = 0,\ x'x''' + x''^2 + y'y''' + y''^2 = 0.$$

From the last of these equations eliminating x'', then y'^2, by means of the two preceding ones we find $x''' = -\frac{y'y'''}{x'} - \frac{y''^2}{x'^3}$; and thus the expression ψ finally becomes

$$\psi = \frac{y'''}{xx'y''} + \frac{4y'y''}{xx'^3}.$$

693. These principles will serve to simplify some demonstrations. If we have the equation $y = fx$*, and its derivatives (y'), (y'')... relatively to x, and we wish to gain the derivatives of $x = \phi y$ relatively to y, without solving the first equation, we must make $y' = 1$, $0 = y'' = y'''$... in the equations D, *i. e.* it will be enough if we assume $(y') = \frac{1}{x'}$, $(y'') = -\frac{x''}{x'^3}$....

For example, $y = a^x$ gives $(y') = ka^x$; and hence we derive $\frac{1}{x'} = ka^x = ky$; whence $x' = \frac{1}{ky}$, when y is the independent variable. It is evident that we thus have the derivative of $x = \text{Log}\, y$.

For $y = \sin x$, we have $(y') = \cos x$; and we consequently find $\frac{1}{x'} = \cos x$, $x' = \frac{1}{\cos x} = \frac{1}{\sqrt{(1 - y^2)}}$, the derivative of the equation $x = \text{arc}\,(\sin = y)$ p. 251.

Lastly, from $y = \tan x$, we deduce the derivative of $x = \text{arc}$ $(\tan = y)$:

$$(y') = \frac{1}{\cos^2 x} = \frac{1}{x'},\ x' = \cos^2 x = \frac{1}{1 + y^2}.$$

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* This admits of a direct demonstration; for let k and h be the simultaneous increments of y and x; then

$$y = fx \text{ gives } k = y'h + \tfrac{1}{2} y''h^2 + \ldots,$$
$$x = \phi y \ \ldots\ h = x'k + \tfrac{1}{2} x''k^2 + \ldots;$$

whence

$$h = x'y'\,h + (x'y'' + x''y'^2) . \tfrac{1}{2} h^2 + \ldots;$$

consequently

$$1 = x'y',\ x'y'' + x''y'^2 = 0,\ \&c. \ldots;$$

and hence we have the values of y', y''...

694. To generalize a function ψ of the 1st order, we change

$$(y') \text{ into } \frac{y'}{x'}, \text{ or } \frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt}; \text{ or } \frac{dy}{dx} = \frac{dy}{dx},$$

suppressing the common divisor dt; only it must be borne in mind that, on the 2nd side, dy and dx denote differentials taken relatively to t. Hence, *when a derivative function ψ is of the 1st order, and is expressed by the differential notation d, it will not require to undergo any alteration, when we wish to change the independent variable; only dy, dx... will denote differentials taken relatively to this new variable.* It is this which renders the differential notation so highly convenient in the integral Calculus, and in every operation in which we are called on to change the principal variable, provided that the derivatives are but of the 1st order.

Let $y = \sin z$; then $y' = \cos z.z'$ reduces itself to $dy = \cos z.dz$; whence $dz = \frac{dy}{\cos z} = \frac{dy}{\sqrt{(1-y^2)}}$; and this mode is preferable to that of N°. 693, for obtaining the derivative of the equation $z = \text{arc}(\sin = y)$.

To conclude, the advantage of which we speak does not extend to the 2nd order; for the 2nd formula (D) becomes

$$\frac{d^2y}{dx^2} = \frac{dx.d^2y - dy.d^2x}{dx^3}.$$

The derivatives are here relative to a 3rd variable t, of which x and y are supposed to be given functions. But it follows from the foregoing principles, that the 1st side is no other than the derivative of $\frac{dy}{dx}$, afterwards divided by dx; and considering dy and dx as functions of t, we may assume [N°. 689, 3°].

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}, \quad \frac{d^3y}{dx^3} = \frac{d\left(\frac{d^2y}{dx^2}\right)}{dx} \ldots;$$

so that the equations (D), (E)... are easily retrieved, and may indeed be borne in memory.

CASES OF FAILURE IN TAYLOR'S THEOREM.

695. The formula [A, N°. 659] may not always be true, when for x we substitute a number a; for $y = fx$ becoming $f(a + h)$ when x is changed into $a + h$, it is possible that the constants of the function

f may destroy a; in which case, should x be found involved under any radical signs, the value $a + h$, substituted for x, would leave h under these signs; and thus h would have fractional powers. It will be seen likewise that, $f(a+h)$ containing no other variable than h, it is not always developable according to integral and positive powers of h. Thus $\cot h$, $\log h$... must have negative exponents for h, since $h = 0$ renders them infinite.

Let $y = \sqrt{x} + \sqrt[3]{(x-a)^4}$; for $x = a + h$, we have

$$Y = \sqrt{(a+h)} + \sqrt[3]{h^4} = \sqrt{a} + \frac{h}{2\sqrt{a}} + h^{\frac{4}{3}} - \frac{h^2}{8\sqrt{a^3}} \ldots$$

Similarly, $\dfrac{1}{(x-a)^2} + \sqrt{x}$ gives $\dfrac{1}{h^2} + \sqrt{(a+h)}$, or $h^{-2} + \sqrt{a} \ldots$;

and, lastly, $\dfrac{1}{\sqrt{(x-a)}} + \sqrt{x}$ becomes $h^{-\frac{1}{2}} + \sqrt{a} + \dfrac{h}{2\sqrt{a}} , \ldots$

Thus our rules have hitherto been free from exceptions, because x has preserved its general value; but when we come to apply these rules to particular cases in which x shall be a given number, we may accidentally meet with an exception to the theorem of Taylor. It will be well to ascertain the characteristics which betoken this circumstance, and to learn what must then be done, in order to arrive at the true series for $f(a+h)$.

696. The development of $f(a+h)$ being arranged according to h, let the least fractional power of h be m, comprised between the integers l and $l + 1$; we may assume then

$$f(a+h) = A + Bh + Ch^2 + Dh^3 \ldots + Lh^l + Mh^m \ldots.$$

If m be negative, Mh^m will be the 1st term of the series. $A, B, C\ldots$ are, in general, finite and unknown constants.

Now, since this equation exists, whatever h be, let the derivatives be taken relatively to this variable:

$$f'(a+h) = B + 2Ch + 3Dh^2 \ldots + lLh^{l-1} + mMh^{m-1} \ldots,$$
$$f''(a+h) = 2C + 2.3\,Dh \ldots + l(l-1)\,Lh^{l-2} + m(m-1)\,Mh^{m-2} \ldots$$
$$f'''(a+h) = 2.3D \ldots + l(l-1)(l-2)Lh^{l-3} + m(m-1)(m-2)Mh^{m-3} \ldots$$
$$\&. = \&c.$$

Making $h = 0$, we find

$$A = fa,\ B = f'a,\ C = \tfrac{1}{2}f''a,\ D = \tfrac{1}{6}f'''a \ldots;$$

and the coefficients $A, B \ldots L$ are therefore the values respectively assumed by fx and its derivatives, when we make $x = a$, precisely as

in Taylor's series. But, at each derivation, the 1st term disappears, as being constant; thus, at the lth derivation, we obtain L: at the $(l+1)$th we have

$$f^{(l+1)}(a+h) = m(m-1)\ldots Mh^{m-l-1} + \ldots;$$

and since m is a fraction $< l+1$, this 1st term has a negative exponent, whence $h=0$ gives $f^{(l+1)}a = \infty$. And all the derivatives, onward from $y^{(l+1)}$, will be infinite in like manner, since this exponent always continues negative in them [N°. 668, 2°.]. Hence,

1°. If the value $x = a$ do not render any of the functions $y, y', y''\ldots$ infinite, there is no failure in the development of Taylor's theorem. [N°. 659].

2°. *If any one of the functions $y, y', y''\ldots$ become infinite for $x = a$, all the following ones are so also; and in this case the theorem fails, but only from the term which contains the first infinite derivative; h at this point acquires a fractional exponent.*

3°. If y is infinite, $y', y''\ldots$ are so also, and h has negative powers.

4°. Since, for $y = x^m$, the derivative of the nth order is of the form Ax^{m-n}, which no value of x renders infinite, unless it be $x = 0$, when m is not integral and positive, we have it proved that the formula for the binomial $(x+h)^m$ is never in fault (this case excepted). The same may be said of the series for a^x, Log $(1+x)$, sin x and cos x.

697. It remains to show how we are to find the development which must replace the faulty part, when this case exists. For this purpose, x must be changed into $a+h$ in fx, and the development of $f(a+h)$ effected by means of the series already known. For example,

$$y = c + (x-b)\sqrt{(x-a)} \text{ gives } y' = \frac{3x-2a-b}{2\sqrt{(x-a)}}:$$

$x = a$ renders this value of y' infinite; those therefore of $y'', y'''\ldots$ are so also, and h must have an exponent between 0 and 1, in the development of $Y = f(a+h)$; the 1st term is $y = c$. Let x in fact be changed into $a+h$ in the expression proposed, and we have $Y = c + (a-b)h^{\frac{1}{2}} + h^{\frac{3}{2}}$.

Again, let $y = c + x + (x-b)(x-a)^{\frac{3}{2}}$;

then

$$y' = 1 + (x-a)^{\frac{3}{2}} + \tfrac{3}{2}(x-b)\sqrt{(x-a)},$$

$$y'' = 3\sqrt{(x-a)} + \frac{3(x-b)}{4\sqrt{(x-a)}};$$

$x = a$ gives $y = c + a$, $y' = 1$; the other derivatives are infinite. The development therefore of $f(a+h)$ commences with $c + a + h$, but the other terms no longer proceed according to $h^2, h^3 \ldots$ In effect, substituting $a + h$ for x, y becomes

$$Y = (c + a) + h + (a - b)h^{\frac{3}{2}} + h^{\frac{5}{2}}.$$

698. Having found the different terms that are not faulty of the series Y, to obtain the succeeding ones, subtract the known part from $f(a+h)$; the remainder, being reduced, will be a function S of h, which we shall have to develop in a series that no longer proceeds according to integral powers of h.

Let A be the value of S for $h = 0$; we have then $S = A + Mh^m$, m being the highest power of h, which divides $S - A$, so that the quotient $M = \frac{S-A}{h^m}$ be not 0, or infinity, for $h = 0$. This condition will make known the number m, and the function M of h. We must now, in like manner, make $h = 0$ in M; and B being the value that then results for M, assume $M = B + Nh^n$, and determine N and n; and so on. Thus, the development required will be

$$S = A + Bh^m + Ch^{m+n} + Dh^{m+n+p} \ldots$$

If S ought to have negative powers of h, we shall assume $h = h'^{-1}$, develop according to h'; and then change the signs of the exponents of h' [See *les Fonct. analyt.* Nos. 11 and 120].

699. Let us inquire now into what takes place, when $x = a$ causes a term P of the function fx to disappear. P must in this case have for a factor some power m of $x - a$ [No. 500], or $P = Q(x-a)^m$.

1°. If m be integral and positive, the mth derivative will contain a term disengaged of the factor $x - a$, since the exponent is successively reduced down to ...2, 1, 0; and thus, the factor Q, which has vanished from all the preceding derivatives, will re-appear in the mth and the following ones: the theorem of Taylor, therefore, will hold good, and nothing particular presents itself in this instance.

Let $y = (x-a)^2.(x-b) - ax^2$; we have

$$Y = -a^3 - 2a^2h - bh^2 + h^3.$$

2°. When m is a fraction comprised between l and and $l+1$, $x = a$ causes Q to disappear from all the derivatives; also, that of the order $l+1$

having the factor $(x-a)^{m-i+1}$, the exponent of which is negative, the derivative becomes infinite, and the series of Taylor is in fault from this term. And, in fact, since the radical indicated by $(x-a)^m$ disappears from the whole series, but still continues in $f(a+h)$, the two sides could not one of them have as many values as the other, if h did not become affected with this same root.

Thus, $$y = x^3 + (x-b)(x-a)^{\frac{5}{3}} \text{ gives}$$

$$Y = a^3 + 3a^2h + 3ah^2 + (a-b)h^{\frac{5}{3}} + h^3 + h^{\frac{8}{3}}.$$

See also the examples, N^{os}. 695 and 697.

3°. If m be negative, P and all its derivatives, having $x-a$ in the denominator, are infinite for $x=a$, and the development of Taylor being in fault from the very first, h has negative powers. This is the case for

$$y = \frac{x^2}{x-a}, \text{ whence } Y = a^2h^{-1} + 2a + h;$$

$$y = \frac{1}{\sqrt{(x^2-ax)}}, \quad Y = \frac{1}{a}\left(\frac{h}{a}\right)^{-\frac{1}{2}} - \frac{1}{2a}\left(\frac{h}{a}\right)^{\frac{1}{2}} + \frac{3}{8a}\left(\frac{h}{a}\right)^{\frac{3}{2}} \ldots$$

700. Suppose that $x=a$ causes a radical to disappear from y, whilst it still remains in y', *i. e.* that this radical has the 1st power of $x-a$ for a factor: it follows that, for $x=a$, y' will have more values than y, on account of the radical, which exists only in y'. Now, by elevating the equation $y=fx$ to a suitable power, we can destroy this radical, which will no longer enter into the equation $z = F(x,y) = 0$. Take the derivative of this [N°. 684]

$$\frac{dz}{dx} + \frac{dz}{dy}.y' = 0,$$

and in it substitute a for x, and for y the unique value referred to; then the coefficients will become certain numbers A and B, viz, $A + By' = 0$. But, by supposition, y' has at least two corresponding values α and β, viz. $A + B\alpha = 0$, $A + B\beta$; whence $B(\alpha-\beta) = 0$, or $B=0$ and $A=0$, since α is different from β. Thus our derivative equation from $z=0$ is satisfied of itself, and is independent of any value of y':

$$\frac{dz}{dx} = 0, \frac{dz}{dy} = 0, y' = \frac{0}{0}.$$

Passing now to the derivative equation of the 2nd order, which has the form [N°. 686]

$$\frac{dz}{dy}y'' + My'^2 + 2Ny' + L = 0;$$

its 1st term disappears; and since M, N, L are functions of x and y free from radicals, and which subsequently become known constants, the equation $My'^2 + 2Ny' + L = 0$ will determine the two values of y': unless for $x = a$, there ought to be more than two values of y', corresponding to one of y; for, in that case, M, N, L will each be found to be nothing simultaneously, and recourse must be had to the equation of the 3rd order: y'' and y''' will disappear from this, their coefficients being $3(My' + N)$ and $\frac{dz}{dy}$, which are nothing; y' will enter into it in the cube.

Generally, we must proceed to a derivative of the same order as the radical that $x = a$ removes from y.

For example, let $y = x + (x - a)\sqrt{(x - b)}$;

then $$y' = 1 + \sqrt{(x - b)} + \frac{x - a}{2\sqrt{(x - b)}};$$

and $x = a$ gives $y = a$, $y' = 1 \pm \sqrt{(a - b)}$. But the proposed equation is also equivalent to

$$(y - x)^2 = (x - a)^2.(x - b);$$

whence $$2(y - x)y' = 2(y - x) + (x - a)(3x - 2b - a);$$

each side of which becomes 0, when $x = y = a$. The derivative of the 2nd order is

$$(y - x)y'' + (y' - 1)^2 = 3x - 2a - b;$$

and this gives $(y' - 1)^2 = a - b$, and the same value of y' as above.

Similarly, $y = (x - a).(x - b)^{\frac{1}{3}}$ gives $y = 0$,

$y' = \sqrt[3]{(a - b)}$, when $x = a$. But if we get quit of the radical, and take the derivatives of the three first orders:

$$\begin{aligned} y^3 &= (x-a)^3(x-b), \\ 3y^2y' &= (x-a)^2(4x-3b-a), \\ y^2y'' + 2yy'^2 &= 2(x-a)(2x-a-b), \\ y^2y''' + 6yy'y'' + 2y'^3 &= 8x-6a-2b: \end{aligned}$$

$x = a$ and $y = 0$ satisfy the three first equations, and the 4th gives $y' = \sqrt[3]{a - b}$, as before.

Should the radical disappear from y and y', but remain in y'', $(x-a)^2$ is a factor of y and y', which have each the same number of values, whilst y'' has more than either, for $x=a$. If the radical therefore be made to vanish from the proposed equation $y=fx$, and we proceed to investigate y'' by means of the derivative of the 2nd order of the implicit equation $z=0$, it must give $y''=\frac{0}{0}$, as being satisfied independently. We must pass on to the 3rd, 4th... derivatives, whence alone y'' can be determined.

The same reasoning will apply when $(x-a)^3$ is a factor of a radical in $y=fx$, &c.

For example, $y=x+(x-a)^2\sqrt{x}$ gives, when $x=a$,

$$y=a,\ y'=1,\ y''=\pm 2\sqrt{a}:$$

But the proposed equation is also equivalent to

$$(y-x)^2=x(x-a)^4;$$

whence

$$2(y'-1)(y-x)=(x-a)^3(5x-a),$$
$$(y'-1)^2+y''(y-x)=2(x-a)^2(5x-2a),$$
$$3y''(y'-1)+y'''(y-x)=6(x-a)(5x-3a),$$
$$3y''^2+4y'''(y'-1)+y^{\mathrm{IV}}(y-x)=12(5x-4a).$$

When $x=a$, we find $y=a$; the equation of the 1st order wholly destroys itself; that of the 2nd gives $y'=1$; the following one is $0=0$, and, finally, the last gives $y''=\pm 2\sqrt{a}$.

LIMITS OF TAYLOR'S SERIES.

701. Let Ah^{α} be a term of the series $f(a+h)$, α being positive; this term and all the succeeding ones give a sum of the form $h^{\alpha}(A+Bh^{\beta})$ [N°. 698]. But, $A+Bh^{\beta}$ reduces itself to A when h is nothing, and increases by insensible degrees along with the factor h: if therefore h be very small, A will exceed Bh^{β}. Thus, *h may be taken so small, that any term of the series $f(a+h)$ shall be greater than the sum of all those that follow.*

702. When a increases and becomes $a+h$, fa may be of such a nature as to increase or to decrease according to circumstances, h continuing positive. For, in the series $f(a+h)=fa+hf'a...$, since h may be taken exceedingly small, the sign of $f'a$ will determine that of the development of $f(a+h)-fa$; if therefore $f'a$ be positive, fa is

increasing; and the contrary when $f'a$ has the sign $-$. Thus it is that $\sin a$ increases up to 90°, and then begins to decrease, because $\cos a$, the derivative, is positive in the 1st quadrant, negative in the 2nd. Hence, *if $f'x$ remain positive from $x = a$ onward to $x = a + b$, without in the mean time becoming infinite, fx goes on increasing through the whole of this extent.*

Suppose that, in $f'(a + h)$, h is made to increase from zero up to b, and let $a + h = p$, $a + h = q$ be the values which give the least and the greatest result; then

$$f'(a + h) - f'p, f'q - f'(a + h)$$

will be positive. But, these are the derivatives, relative to h, of*

$$f(a + h) - fa - hf'p, fa + hf'q - f(a + h).$$

These functions therefore must go on increasing in the interval from $h = 0$ to $h = b$; and since $h = 0$ reduces them to nothing, they are positive for this interval, or

$$f(a + h) > fa + hf'p \text{ and } < fa + hf'q.$$

The contrary would be the case if h were negative. Hence, $f(a + h) = fa$ + some number comprised between $hf'p$ and $hf'q$, *i. e. if we limit the series for $f(a + h)$ to the 1st term fa, the error is $> hf'p$ and $< hf'q$.*

Suppose now that Taylor's series holds good for its three first terms, $f(a + h) = fa + hf'a + \frac{1}{2}h^2f''a\ldots$; and let p and q be the values of $a + h$ for which $f''(a + h)$ is respectively the least and the greatest, from $h = 0$ to $h = b$; for this extent, then, the quantities

$$f''(a + h) - f''p, f''q - f''(a + h)$$

are positive, as also their primitives

$$f'(a + h) - f'a - hf''p, f'a + hf''q - f'(a + h),$$

since $h = 0$ reduces each of them to nothing. And the same may be said for the primitives of these last functions, which are

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* Assuming $x + h = z$, $F(x + h)$ becomes Fz; and if the derivative be taken relatively either to x, or to h, since $z' = 1$, it will equally be $F'z$ [N°. 672]. $F'(x+h)$ therefore may be indifferently supposed to have arisen from the variation of x, or of h in $F(x + h)$. Thus, though we have here considered the derivatives as relative to h, they will result the same as if we had taken them for x, and had then made $x = a$.

$f(a+h) - fa - hf'a - \frac{1}{2}h^2f''p,\ fa + hf'a + \frac{1}{2}h^2f''q - f(a+h)$;

whence, consequently,

$$f(a+h) = fa + hf'a + \tfrac{1}{2}h^2A,$$

A being some number comprised between $f''p$ and $f''q$.

Taylor's Series therefore being confined to its two first terms, the error is comprised between the limits $\frac{1}{2}h^2f''p$ and $\frac{1}{2}h^2f''q$.

And generally, if the series for $f(a+h)$ be stopped at the term which precedes h^n, the error will lie between the products of $\frac{h^n}{1.2.3\ldots n}$ by $f^{(n)}p$ and $f^{(n)}q$, or by numbers, the one of which is less than the 1st, the other greater than the 2nd of these quantities; p and q being the values of $x+h$ which render $f^n(x+h)$ the least and the greatest in the interval from $h=0$ to h any whatever. But it is incumbent that no one of the functions $fx, fx' \ldots f^{(n)}x$ become infinite, from $x=a$ to $x=a+h$.

And since p and q are values intermediate to a and $a+h$, the error is $\frac{h^n.f^{(n)}(a+j)}{1.2.3\ldots n}$, j being some suitable and unknown number. Provided therefore that no one of the derivatives be infinite, we may assume exactly

$$f(x+h) = fx + hf'x + \frac{h^2}{2}f''x \ldots + \frac{h^{n-1}.f^{(n-1)}x}{1.2.3\ldots n-1} + \frac{h^n f^{(n)}(x+j)}{1.2.3\ldots n}.$$

Thus we have a new demonstration of Taylor's series, and can measure the error committed in stopping it at a specified term, or obtain a finite expression which shall be its value.

For example, $y = a^x$ gives $y^{(n)} = k^n.a^x$; $f^{(n)}(x+h) = k^n.a^{x+h}$; the least and the greatest values of which correspond to $h=0$ and h any whatever. The limits of the error therefore are the products of $\frac{k^nh^n}{2.3\ldots n}$ by a^x and a^{x+h}. For a^h, these last factors are 1 and a^h.

For $\log(x+h)$, the limits are $\pm \frac{h^n}{n} \times \left(\frac{1}{x^n} \text{ and } \frac{1}{(x+h)^n}\right)$.

Lastly, $y = x^m$ gives $y^{(n)} = [mPn]x^{m-n}$ [N°. 475]; the error therefore lies between these limits

$$\frac{h^n[mPn]}{1.2.3\ldots n} \times [x^{m-n} \text{ and } (x+h)^{m-n}], \text{ or } [mCn]h^n(x+h)^{m-n}.$$

DEVELOPMENT OF FUNCTIONS OF SEVERAL VARIABLES.

703. Let z be a function of two independent variables x and y, $z=f(x,y)$; and, x being changed into $x+h$, y into $y+k$, let it be proposed to develop according to the powers of these arbitrary increments h and k. Proceeding in the same manner as in N°. 672, instead of carrying these two changes into effect at once, we shall first substitute $x+h$ for x, without supposing y to vary; when z, considered as a fuction of a single variable x, will become

$$f(x+h,y)=z+\frac{dz}{dx}h+\frac{d^2z}{dx^2}\frac{h^2}{2}+\frac{d^3z}{dx^3}\frac{h^3}{2.3}+\&c.$$

and in this result, $y+k$ must then be substituted throughout for y, x being in turn left unchanged. The 1st term z will thus become

$$f(x,y+k)=z+\frac{dz}{dy}k+\frac{d^2z}{dy^2}\frac{k^2}{2}+\frac{d^3z}{dy^3}\frac{k^3}{2.3}+\&c.$$

Similarly, if u be taken to represent the function of x and y denoted by $\frac{dz}{dx}$, on substituting $y+k$ for y, u will be changed into $u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{2}+\&c$; and thus, replacing the value of u,

$$\frac{dz}{dx}h \text{ will become } \frac{dz}{dx}h+\frac{d^2z}{dx\,dy}hk+\frac{d^3z}{dx\,dy^2}\frac{k^2h}{2}+\ldots$$

In like manner,

$$\frac{d^2z}{dx^2}\frac{h^2}{2} \text{ will become } \frac{d^2z}{dx^2}\frac{h^2}{2}+\frac{d^3z}{dx^2\,dy}\frac{h^2k}{2}+\&c.$$

&c....................&c.

And hence, combining these several parts, we have

$$\begin{aligned}f(x+h,y+k)=z&+\frac{dz}{dy}k+\frac{d^2z}{dy^2}\frac{k^2}{2}+\frac{d^3z}{dy^3}\frac{k^3}{2.3}+\ldots\\&+\frac{dz}{dx}h+\frac{d^2z}{dx\,dy}kh+\frac{d^3z}{dx\,dy^2}\frac{k^2h}{2}+\ldots\\&+\frac{d^2z}{dx^2}\frac{h^2}{2}+\frac{d^3z}{dx^2dy}\frac{h^2k}{2}+\ldots\\&+\frac{d^3z}{dx^3}\frac{h^3}{2.3}+\ldots\\&+\&c.\end{aligned}$$

The general term is $\frac{d^{m+n}z}{dy^m dx^n}\times\frac{k^m.h^n}{(2.3\ldots m)(2.3\ldots n)}$.

It is obvious that we might have first changed y into $y + k$, and then in the result x into $x + h$. But in this way we should have obtained a series, which, though necessarily identical with the one just found, would have been different in form: the several derivatives relative to x would have preceded those of y. To arrive at this series, nothing more is requisite than, in the one above, to change y into x, k into h, and the converse. The identity of this last result with the one preceding gives, by a comparison of the corresponding terms,

$$\frac{d^2z}{dy\,dx} = \frac{d^2z}{dx\,dy}, \quad \frac{d^3z}{dy^2\,dx} = \frac{d^3z}{dx\,dy^2}, \quad \frac{d^3z}{dy\,dx^2} = \frac{d^3z}{dx^2\,dy},$$

and generally, $$\frac{d^{m+n}z}{dy^m\,dx^n} = \frac{d^{m+n}z}{dx^n dy^m}.$$

And hence we conclude that *when we have to take the successive derivatives of a function z relatively to two variables, it is indifferent in what order we perform this double operation.*

For instance, $z = \frac{x^3}{y^2}$ gives $\frac{dz}{dx} = \frac{3x^2}{y^2}$, $\frac{dz}{dy} = -\frac{2x^3}{y^3}$; and the derivative of the 1st, in respect to y, that of the 2nd, relatively to x, are equally $-\frac{6x^2}{y^3}$.

The derivatives of the 2nd order are

$$\frac{d^2z}{dx^2} = \frac{6x}{y^2}, \quad \frac{d^2z}{dy^2} = \frac{6x^3}{y^4};$$

and hence $-\frac{12x}{y^3}$ is the derivative of the first of these, relatively to y, and at the same time the derivative of the 2nd order of $\frac{dz}{dy}$ relatively to x; $\frac{18x^2}{y^4}$ is the derivative of $\frac{d^2z}{dy^2}$ in respect to x, and also that of the 2nd order of $\frac{dz}{dx}$ in respect to y; and similarly for the other derivatives.

704. Since x and y are supposed to be independent of each other in the equation $z = f(x, y)$, the derivative may be taken in respect to x alone or y alone; let the functions of x and y, that are found for these respective derivatives, be denoted by p and q, $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$.

But if there should be a dependence established between x and y, such as $y = \varphi x$, these partial differences could no longer be taken separately,

since the variation of x would imply that of y. In order to include both these cases in a single one, it is usually supposed that this relation $y=\varphi x$ does exist, and the derivate then appears under the form $dz=pdx+qdy$ [N°. 684]; but since this function φ is left arbitrary, it will be necessary to take it into consideration, whenever this equation comes to be applied. If the question require the dependence to be established, we must, from $y=\varphi x$, deduce $dy=y'dx$, and substituting we shall have $dz=(p+qy')dx$. If the dependence do not exist, the differential equation will spontaneously separate itself into two others: for dz represents the differential of z taken relatively to both x and y at the same time, or $\frac{dz}{dx}dx+\frac{dz}{dy}dy$; and, since the equation subsists whatever φ, or its derivative y', be, we shall have

$$\frac{dz}{dx}+\frac{dz}{dy}y'=p+qy',\text{ whence }\frac{dx}{dx}=p,\ \frac{dz}{dy}=q.$$

$$z=\frac{ay}{\sqrt{(x^2+y^2)}}\text{ gives }dz=\frac{-axydx+ax^2dy}{(x^2+y^2)^{\frac{3}{2}}},$$

an equation which we subdivide into two others

$$\frac{dz}{dx}=-\frac{axy}{(x^2+y^2)^{\frac{3}{2}}},\ \frac{dz}{dy}=\frac{ax^2}{(x^2+y^2)^{\frac{3}{2}}}.$$

$$z=\text{arc}\left(\tan=\frac{x}{y}\right)\text{ gives }dz=\frac{ydx-xdy}{y^2+x^2},$$

whence we deduce

$$\frac{dz}{dx}=\frac{y}{y^2+x^2},\quad\frac{dz}{dy}=\frac{-x}{y^2+x^2}.$$

Let $u=0$ be, in general, an equation between the three variables x, y and z; if moreover we have a second relation $z=F(x,y)$, there must no longer be considered to be more than a single independent variable in the equation proposed: thus we have in the first place [N°. 673]

$$\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz=0\ldots(1),$$

and hence, $z=F(x,y)$ giving $dz=pdx+qdy$,

$$\left(\frac{du}{dx}+p\frac{du}{dz}\right)dx+\left(\frac{du}{dy}+q\frac{du}{dz}\right)dy=0.$$

We shall easily therefore deduce the value of $\frac{dy}{dx}$, which is the deriva-

tive that would have been obtained by eliminating z from the equation $u = 0$.

But should there be no other relation than $u = 0$, we shall be at liberty to suppose one, provided that it remain arbitrary; so that, since y' may now be of any value whatever, our last equation will separate itself into two others

$$\frac{du}{dx} + p\frac{du}{dz} = 0, \frac{du}{dy} + q\frac{du}{dz} = 0,$$

where p and q are the partial derivatives or differentials of z relative to x and y. Now this is, in fact, what the equation $u = 0$ would have given, had we successively considered y and x as constant in it, as in N°. 684; and the equation (1) therefore is the derivative of $u = 0$, whether there be or be not any other dependence between the variables x, y and z.

There is no necessity to say any thing expressly on the derivatives of the higher order; it is evident that each equation of the 1st order may be differentiated relatively either to x, or to y, which will give three equations of the 2nd order; and similarly for the other orders.

We shall easily be able to find the development of functions of 3, 4... variables according to the powers of their increments, since nothing more will be requisite than to repeat the same operations separately for each variable.

705. It has been mentioned that the derivative of an equation between two variables may serve for the elimination of a constant. Something of a more extensive nature presents itself in the case of three variables; and it is here we have the germ of the calculus of partial differences, become so celebrated for its applications to Mechanics, Astronomy, &c.

Let $z = ft$, t denoting a known function of two variables $t = F(x, y)$. The derivatives relative to x and y, separately, are [N°. 671]

$$\frac{dz}{dx} \text{ or } p = f't \times \frac{dt}{dx}, \quad \frac{dz}{dy} \text{ or } q = f't \times \frac{dt}{dy};$$

where $f't$ is the same in each equation, and the derivatives $\frac{dt}{dx}$, $\frac{dt}{dy}$, are presumed to be known in terms of x and y. Dividing the equations, $f't$ disappears, and we find $p\frac{dt}{dy} = q\frac{dt}{dx}$, a relation which expresses that z is a function of t, $z = ft$, whatever may otherwise be the form of this function f.

For example, $\qquad z = f(x^2 + y^2)$ gives

$$p = f'(x^2 + y^2) \times 2x,\ q = f'(x^2 + y^2) \times 2y;$$

whence $$py - qx = 0.$$

And in whatever manner $x^2 + y^2$ may enter into the value of z, this last equation will continue the same; it will accord with

$$z = \log(x^2 + y^2),\ z = \sqrt{(x^2 + y^2)},\ z = \frac{x^2 + y^2}{\sin(x^2 + y^2)}, \text{\&c....}$$

It follows, therefore, that every function of $x^2 + y^2$ must be a particular case of *the equation of partial differences* $py - qx = 0$.

Similarly, $y - bz = f(x - az$, when we differentiate separately, first in regard to z and x, then z and y, gives

$$-bp = (1 - ap) \times f',\ (1 - bq) = -aq \times f';$$

and eliminating f', we have $ap + bq = 1$ for the equation of partial differences corresponding to the proposed equation, whatever may be the form of the function f.

Treating $\dfrac{y - b}{z - c} = f\left(\dfrac{x - a}{z - c}\right)$ in the same manner, we find

$$z - c = p(x - a) + q(y - b).$$

We shall have an opportunity subsequently of showing the importance of this theory; for the present we shall confine ourselves to observing that the three equations of the 2nd order will serve to eliminate two arbitrary functions, &c.

II. APPLICATION OF THE DIFFERENTIAL CALCULUS.

DEVELOPMENT OF FUNCTIONS OF A SINGLE VARIABLE IN SERIES.

706. Making $x = 0$ in the series of Taylor [p. 235], and denoting by $f, f', f''...$ the constant values then assumed by $fx, f'x, f''x...$, we have

$$fh = f + hf' + \tfrac{1}{2}h^2 f'' + \tfrac{1}{6}h^3 f''' + ...;$$

a formula, however, which only holds good so long as $x = 0$ do not render any one of the quantities $fx, f'x...$ infinite. Let h be now

changed into x; then, f, f', f''... being independent of h, there results

$$y = fx = f + xf' + \frac{x^2}{2}f'' + \frac{x^3}{2.3}f''' + \frac{x^4}{2.3.4}f^{IV} + \dots;$$

and this is the formula, due to Maclaurin, which serves to develop a function of x in a series of integral and positive powers of x, whenever it is susceptible of such a development.

For example, $y = (a + x)^m$ gives

$$y' = m(a + x)^{m-1}, y'' = m(m - 1)(a + x)^{m-2} \dots;$$

whence

$$f = a^m, f' = ma^{m-1}, f'' = m(m - 1)a^{m-2}, \dots;$$

which affords another proof of Newton's series [p. 245].

From $y = \sin x$, we derive $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x \dots$; whence we have, 0, 1, 0 and -1 for the alternate values of f, f', f''... on to infinity; and substituting we find the series for $\sin x$ given in p. 165.

The same calculations can readily be made for $\cos x$, a^x, $\log(1 + x)$...; and, generally, for every function of x. If we assume $y = \text{arc}$ $(\tan = x)$, we shall arrive again at the series N [p. 170].

707. If one of the functions f, f', f''... be infinite, the formula of Maclaurin can no longer be employed, the proposed function not proceeding according to the integral and positive powers of the variable. We must then either submit it to the processes of N°. 698, or, which is the preferable mode, transform it into a shape adapted to our calculation; the supposition of $y = x^k z$ will frequently answer this purpose, the constant k being so determined, that $x = 0$ shall not render any one of the functions z, z', z''... infinite.

For example, the series for $\cot x$ cannot proceed according to the positive powers of x, since $\cot 0 = \infty$. But, making $y = \frac{z}{x} = \cot x$, we have $z = \frac{x \cos x}{\sin x}$, or, by reason of the formulæ G and H, p. 165.

$$z = \frac{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots}{1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \dots},$$

a function of which we shall easily obtain the successive derivatives, which are no longer infinite when x is nothing. We now find $f = 1$, $f' = 0$, $f'' = -\frac{2}{3}$, $f''' = 0 \dots$; whence

$$z = 1 - \frac{x^2}{3} - \frac{x^4}{3^2.5} - \dots,$$

and $\frac{z}{x}$ or $\cot x = x^{-1} - \frac{x}{3} - \frac{x^3}{3^2.5} - \frac{2x^5}{3^3.5.7} - \frac{x^7}{3^3.5^2 7}\dots$

This method however labours under the disadvantage of not making known the law of the series, though it is here exhibited.

We shall hereafter show how the differential Calculus may be employed for the development of y in a continued fraction, its terms being functions of x [See N°. 835]. We may likewise find y under the form of a series, according to the method of the note p. 116.

708. The theorem of Maclaurin may also be applied to equations of two variables. Thus, for $mz^3 - xz = m$, taking z', z''... [N°. 684], and making $x = 0$, we shall have

$$f = 1, f' = \frac{1}{3m}, f'' = 0, f''' = \frac{-2}{27m^3}\dots;$$

whence

$$z = 1 + \frac{x}{3m} - \frac{x^3}{81m^3} + \frac{x^4}{243m^4} - \dots$$

We may also develop according to the descending powers of x. We must substitute t^{-1} for x, and deduce the series according to the increasing powers of t; then replace x^{-1} for t, and we shall have the series required. Thus, for $my^3 - x^3y - mx^3 = 0$, making $x^3 = t^{-1}$, we shall have $my^3t - y = m$; of this we must take the derivatives y', y''... relative to t, then make $t = 0$ throughout; and, finally, substitute the results for f, f', f''... in the series of Maclaurin, in which t will take the place of x. This calculation, x^3 being replaced for t, will give

$$y = -m - m^4x^{-3} - 3m^7x^{-6} - 12m^{10}x^{-9} + 55m^{13}x^{-12}\dots$$

709. Let it be proposed to develop $u = fy$ according to the powers of x, y being connected with x by the equation

$$y = a + x.\,\varphi y\dots \quad (1),$$

the functions fy and φy being also given. We shall first observe that if, by means of the equation (1), we eliminated y, u would then contain only x, and the theorem of Maclaurin would become applicable. We should consequently investigate u, u', u''...; and then f, f', f''... by making $x = 0$. But the differential calculus furnishes us with the means of determining the derivatives u', u''..., without having recourse to elimination. In effect, the derivatives [N°. 671] relative to x are, for equation (1),

$$y' = \varphi y + xy'\varphi' y,\ y'' = 2y'\varphi' y + xy''\varphi y + xy'^2\varphi'' y, \text{ \&c.};$$

those for $u = fy$ are

$$u' = y'f'y,\ u'' = y''f'y + y'^2f''y,\ \&c.;$$

making $x = 0$, $y, y', y''...$ become

$$a,\ \varphi a,\ 2\varphi a\ \varphi' a = (\varphi^2 a)',\ (\varphi^3 a)'',\ \&c.;$$

so that, substituting these values, $u, u', u''...$ become

$$fa,\ \varphi a f' a,\ (\varphi^2 a)'.f'a + \varphi^2 a.f''a = (\varphi^2 a.f'a)',\ \&c.;$$

and these are the values of $f, f', f''...$, and we find*

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* Though by this method, we may find as many terms of the series as we wish, yet the law is not evident. We shall here give the demonstration of M. Laplace. [*Méc. Cél.* Vol. i. p. 172].

Let x and a be considered as variables in equation (1), and the derivatives taken relatively to each [N°. 704]; those in respect to x being still represented by $y', u', u''...$ There will result

$$\frac{dy}{da} = 1 + x\varphi' y.\frac{dy}{da};\ y' = \varphi y + xy'\varphi' y = \varphi y.\frac{dy}{da},$$

eliminating $\varphi' y$. Let $u = fy$ be treated in the same manner, and we shall have

$$\frac{du}{da} = f'y.\frac{dy}{da},\ u' = y'f'y;\ \text{whence } u'\frac{dy}{da} = y'\frac{du}{da},$$

and, substituting for y' its value above,

$$(2)...\ u' = \varphi y.\frac{du}{da} = \varphi y.f'y.\frac{dy}{da},$$

the derivatives $f'y$, y', u' being relative to x. Now, since u' is the product of $\frac{dy}{da}$ by a function of y, we may suppose that the value (2) of u' is also the derivative relative to a of some function of y, such as $z = Fy$, so that

$$u' = \frac{dz}{da};\ \text{and consequently, } u'' = \frac{d^2z}{dadx} = \frac{dz'}{da},$$

z being here the derivative of Fy, relative to x. But, in the same manner that

$$u = fy,\ y = a + x\varphi y \text{ give the equation (2),}$$

taking $z = Fy$ instead of the first of these, we see that (2) will become

$$(3)...\ z' = \varphi y.\frac{dz}{da} = \varphi y.u' = \varphi^2 y.\frac{du}{da};$$

and therefore

$$u'' = \frac{dz'}{da} = \left(\varphi^2 y.\frac{du}{da}\right)'.$$

the $'$ indicating a derivative relative to a.

$$fy = fa + x\phi af'a + \frac{x^2}{2}(\phi^2 af'a)' + \frac{x^3}{2.3}(\phi^3 a.f'a)'' + \ldots$$

By fa, ϕa are meant the resulting values of the given function fy, ϕy, when we make $y = a$; by $\phi^2 a$ the square of ϕa, by $f'a$ the derivative of fa relative to a, by $(\phi^2 af'a)'$ that of the function $\phi^2 a.f'a$... [See an application of this equation, *Méc. Cél.* v. i. p. 177.]

Let the developed value of $u = y^m$ be required, supposing that $y = a + xy^n$. Comparing with the equation (1), we have

$$fa = a^m, f'a = ma^{m-1}, \phi a = a^n, \phi af'a = ma^{m+n-1},$$
$$\phi^2 af'a = ma^{m+2n-1}, \phi^3 af'a = ma^{m+3n-1}, \ldots;$$

whence

$$y^m = a^m + mxa^{m+n-1} + m.\frac{m+2n-1}{2}x^2a^{m+2n-2} + \ldots.$$

We might also obtain the value of y^n, supposing the equation (1) to be replaced by $\alpha + \beta y + \gamma y^m = 0$; it would be sufficient to make

$$a = -\frac{\alpha}{\beta}, x = \frac{\gamma}{\beta}.$$

710. Making $x = 1$ in the preceding series, we arrive at the development of fy, when $y = a + \phi y$,

$$fy = fa + \phi af'a + \tfrac{1}{2}(\phi^2 a.f'a)' + \tfrac{1}{6}(\phi^3 a.f'a)'' + \ldots$$

Similarly, considering $\phi^2 y.\frac{du}{da}$ as being the derivative relative to a of a function $t = \psi y$, viz.

$$\phi^2 y.\frac{du}{da} = \frac{dt}{da} = z', u'' = \frac{d^2t}{da^2}, u''' = \frac{d^3t}{da^2dx} = \frac{d^2t'}{da^2};$$

if $u = fy$ be now changed above into $t = \psi y$, we shall see that the equation (2) will become

$$t' = \phi y.\frac{dt}{da} = \phi y.z' = \phi^3 y.\frac{du}{da}; \text{ and therefore } u''' = \left(\phi^3 y.\frac{du}{da}\right)''.$$

And in like manner we shall have... $u^{IV} = \left(\phi^4 y.\frac{du}{da}\right)'''$, &c.... the derivatives being throughout relative to a. Assume then $x = 0$, whence

$$y = a, u = fa, \frac{du}{da} = f'a,$$

we shall thus obtain for u', u''... values of which it is easy to recognise the law, and hence we shall finally deduce the series above.

Hence may be obtained the power n of the least root y of the equation $y = a + \phi y$, by making $fy = y^n$, and therefore $fa = a^n, f'a = na^{n-1}$; whence

$$y^n = a^n + n\,[\phi a.a^{n-1} + \tfrac{1}{2}\,(\phi^2 a.a^{n-1})' + \tfrac{1}{6}\,(\phi^3 a.a^{n-1})''\ldots].$$

The accents indicate derivatives relative to a; the numerical value of a is not inserted, till after the operations [See *Résol. numér.* note XI].

For example, the equation $\gamma y^2 - \beta y + \alpha = 0$ is reduced to the form $y = a + \phi y$, by assuming

$$a = \frac{\alpha}{\beta},\ \phi a = \frac{\gamma}{\beta} a^2, \text{ whence}$$

$$\phi a.a^{n-1} = \frac{\gamma}{\beta} a^{n+1},\ \phi^2 a.a^{n-1} = \frac{\gamma^2}{\beta^2} a^{n+3}\ldots;$$

and taking the requisite derivatives, we finally find

$$y^n = \left(\frac{\alpha}{\beta}\right)^n \left[1 + n\left(\frac{\alpha\gamma}{\beta^2}\right) + n\,\frac{n+3}{2}\left(\frac{\alpha\gamma}{\beta^2}\right)^2 + n\,\frac{n+4}{2}.\frac{n+5}{3}\left(\frac{\alpha\gamma}{\beta^2}\right)^3\ldots\right],$$

the general term being

$$\left(\frac{\alpha}{\beta}\right)^n \frac{n}{i} \times [(2i + n - 1)\ C\,(i-1)] \times \left(\frac{\alpha\gamma}{\beta^2}\right)^i.$$

To obtain the power n of the greatest root y, we must change y into y^{-1}, *i. e.* replace, in our result, α by γ, γ by α, and y^n by y^{-n}.

711. When we wish for the 1st power of y, the equation being $y = a + \phi y$, we must make $n = 1$ above; whence

$$y = a + \phi y = a + \phi a + \tfrac{1}{2}\,(\phi^2 a)' + \tfrac{1}{6}\,(\phi^3 a)'' + \ldots.$$

This series is especially applicable to the *inverse method of series*, which consists in deriving the value of y from the equation

$$\alpha + \beta y + \gamma y^2 + \ldots = 0;$$

this we reduce to the form $y = a + \phi y$, by assuming

$$a = -\frac{\alpha}{\beta},\ \phi a = -\frac{\gamma a^2 + \delta a^3\ldots}{\beta},\ \phi^2 a = \frac{\gamma^2 a^4 + 2\gamma\delta a^5\ldots}{\beta^2}\ldots,$$

and there finally results

$$y = -\frac{\alpha}{\beta} - \frac{\alpha^2\gamma}{\beta^3} + \frac{\alpha^3\delta}{\beta^4} - \frac{\alpha^4\epsilon}{\beta^5}\ldots - \frac{2\alpha^3\gamma^2}{\beta^5} + \frac{5\alpha^4\gamma\delta}{\beta^6}\ldots - \frac{5\alpha^4\gamma^3}{\beta^7}\ldots$$

ON THE SOLUTION OF EQUATIONS.

712. We shall demonstrate anew several theorems respecting equations.

I. Let y be a function of x, admitting the factors $(x-a)^m$, $(x-b)^n$..., so that we have

$$y = (x-a)^m . (x-b)^n ... \times P,$$

P containing only unequal factors of the 1st degree: taking the log of the two sides and the derivatives of these logs, we find

$$y' = (x-a)^{m-1} (x-b)^{n-1} ... [mP(x-b)... + nP(x-a)... \&c.].$$

Thus the proposed function of x has $(x-a)^{m-1} (x-b)^{n-1}$... for the greatest common divisor between itself and its derivative, which leads again to the theorem on equal roots [p. 61].

II. The derivative of $l(\cos x \pm \sin x \sqrt{-1})$ is [N°. 679]

$$\frac{-\sin x \pm \cos x \sqrt{-1}}{\cos x \pm \sin x \sqrt{-1}}, \text{ which reduces itself to } \pm\sqrt{-1}.$$

But $\sqrt{-1}$ is also the derivative of $x\sqrt{-1} + A$, A being an arbitrary constant [N°. 678]; so that

$$l(\cos x \pm \sin x \sqrt{-1}) = \pm x \sqrt{-1} + A;$$

and since this equation must subsist whatever x be, making $x = 0$, we find $A = 0$. Hence results the theorem [1, p. 168], whence it will be easy to deduce the formulæ K, L, M, and subsequently, the factors of $x^m \pm a^m$ [p. 86].

III. The equation $x^m + px^{m-1} + ... + u = 0$ being decomposed into its simple factors, $(x-a)(x-b)(x-c)$..., the log of these functions will be identical; whence

$$l(x^m + px^{m-1} + ...) = l(x-a) + l(x-b) + ...;$$

and taking the derivatives on each side, we arrive again at the equation of p. 101, and consequently at Newton's theorem respecting the sums of the powers of the roots, which form a recurring series, the scale of relation of which is $-p, -q..., -u$.

IV. Fx denoting a rational and integral function of x, let k be the approximate part of one of the roots of the equation $Fx = 0$, and y the correction due to it; whence $x = k + y$, and

$$F(k+y) = Fk + yF'k + \tfrac{1}{2}y^2F''k + ... = 0.$$

When y^2, y^3... are neglected, in consideration of y being a very small quantity, we find $y = -\frac{Fk}{F'k}$, which agrees with the method of Newton [p. 64].

But, without neglecting any term, we may deduce the value of y from this equation, by means of the series of N°. 711. We shall make in it $\alpha = Fk$, $\beta = F'k$, and assuming, for conciseness, $z = \frac{Fk}{F'k}$, which is the first correction only with a contrary sign, there results

$$y = -z - \frac{z^2}{z} \cdot \frac{F''k}{F'k} + \frac{z^3}{2.3} \cdots;$$

consequently the root required, or $k + y$, is

$$x = k - z - \frac{z^2}{z} \cdot \frac{F''k}{F'k} + \frac{z^3}{2.3}\left[\frac{F'''k}{F'k} - 3\left(\frac{F''k}{F'k}\right)^2\right] + \ldots$$

Thus, from the equation $x^3 - 2x = 5$, we deduce $k = 2\cdot1$ for the approximate value of one of the roots [p. 64]; so that

$Fk = k^3 - 2k - 5 = 0.061$, $F'k = 3k^2 - 2 = 11.23$, $F''k = 6k = 12.6$;

$$\text{whence } z = \frac{Fk}{F'k} = \frac{61}{11230}, \frac{F''k}{F'k} = \frac{1260}{1123},$$

and

$$x = 2\cdot1 - 0\cdot00543188 - 0\cdot00001655 = 2\cdot09455157.$$

ON THE VALUES $\frac{0}{0}$, $0 \times \infty$, &c.

713. It has already been observed [p. 36, 2°.] that, when $x = a$ changes a proposed fraction into $\frac{0}{0}$, $x - a$ is a common factor of the two terms; and that the fraction must be divested of this factor, which may enter into it in different powers. The differential calculus gives an easy mode of accomplishing this, and of obtaining the value of the fraction, in the case of $x = a$, a value which may be *nothing*, or *finite*, or *infinite*. Let x be changed into $x + h$, and the proposed fraction $\frac{P}{Q}$ will become

$$\frac{P + hP' + \frac{1}{2}h^2P'' + \ldots}{Q + hQ' + \frac{1}{2}h^2Q'' + \ldots} \ldots (A).$$

In this make $x = a$; then P and Q vanish, and dividing above and below by h, we have

$$\frac{P' + \frac{1}{2}hP'' + \ldots}{Q' + \frac{1}{2}hQ'' + \ldots} = \frac{P'}{Q'} \text{ when } h = 0.$$

But these suppositions of $x = a$ and $h = 0$ are tantamount to having in the first instance changed x into a. Thus, when $x=a$, $\frac{P}{Q}=\frac{P'}{Q'}$. In case therefore that P' or Q' be still $= 0$, the fraction is nothing or infinite respectively; whilst, if P' and Q' both vanish in the developments (A), we must divide them by $\frac{1}{2}h$ and again make $h = 0$; when we shall have, for $x = a$, $\frac{P}{Q} = \frac{P''}{Q''}$; and so on.

Hence, *to obtain the value of a fraction which becomes $\frac{0}{0}$ when $x = a$, we must differentiate the numerator and the denominator the same number of times, until one or the other no longer become zero when a is substituted for x.* We need be under no apprehensions lest all the derivatives P', Q', P'', Q''... should turn out nothing; for in that case, whatever h be, we must have $f(a + h) = 0$, which is impossible.

714. The following are some examples of this theory.

I. The sum of the n first terms of the progression $\div 1 : x : x^2 : x^3 \ldots$ is $\frac{x^n - 1}{x - 1}$ [N°. 144], a fraction which, if $x = 1$, becomes $\frac{0}{0}$: take the derivatives of the two terms, which are nx^{n-1} and 1, then make $x = 1$, and there results n for the sum required, as is evidently the case.

II. Let the fraction be $\frac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}$, which becomes $\frac{0}{0}$ for $x=c$: the derivatives of the 1st order still give $\frac{ax-ac}{bx-bc} = \frac{0}{0}$; and we must therefore proceed to a second derivation, whence we have $\frac{a}{b}$. Two successive operations have been necessary, in consequence of $(x-c)^2$ being a common factor.

III. Similarly, $\frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}$ gives $\frac{0}{0}$ for $x = a$: the derivatives of the two terms are $3x^2 - 2ax - a^2$ and $2x$; the first of which is 0 for $x=a$; and zero therefore is the value required, which arises from the factor of the numerator being $(x - a)^2$, whilst that of the denominator is $x - a$. For a similar reason, the same fraction, reversed, would have had infinity for its value. This is the case, when $x = a$ in

$$\frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4}.$$

IV. $x = 0$ renders $\dfrac{a^x - b^x}{x} = \frac{0}{0}$; and the derivatives give

$$\frac{a^x la - b^x lb}{1} = la - lb = l\left(\frac{a}{b}\right).$$

V. For $y = \dfrac{1 - \sin x + \cos x}{\sin x + \cos x - 1}$, in the case of the arc x being the quadrant, we have

$$y = \frac{-\cos x - \sin x}{\cos x - \sin x} = 1.$$

VI. When $x = a$, $\dfrac{\sqrt{(2a^3x - x^4)} - a\sqrt[3]{(a^2x)}}{a - \sqrt[4]{(ax^3)}}$ becomes $\dfrac{0}{0}$: the derivatives of the two terms give

$$\frac{a^3 - 2x^3}{\sqrt{(2a^3x - x^4)}} - \frac{a^2}{3\sqrt[3]{(ax^2)}} : - \frac{3a}{4\sqrt[4]{(a^3x)}} = \frac{16a}{9}.$$

VII. We shall similarly see that $x = 1$ gives $\dfrac{0}{0}$ for

$$\frac{1 - x + lx}{1 - \sqrt{(2x - x^2)}} = -1, \text{ and } \frac{x^2 - x}{1 - x + lx} = -2.$$

715. The method now explained will cease to be applicable if Taylor's theorem be *faulty within the order of the terms that we are obliged to retain;* as may easily be conceived, since one of the derivatives to which we are led will result infinite. In this case we must change x into $a+h$ in P and Q, and effect the developments [N°. 698], confining ourselves to the 1st terms of each; whence we shall have $\dfrac{P}{Q} = \dfrac{Ah^m + \ldots}{Bh^n + \ldots}$, where m and n may be fractional or negative. We must now divide the two terms by the lowest power of h, and in the result make $h = 0$. If $m = n$, we have the finite value $\dfrac{A}{B}$; otherwise, the fraction is nothing or infinite, accordingly as m is $>$ or $<$ n.

I. Let the fraction be $\dfrac{(x^2 - a^2)^{\frac{1}{2}}}{(x - a)^{\frac{1}{2}}}$: $x = a$ gives $\dfrac{0}{0}$; and it is useless

to have recourse to the derivatives of the two terms, since they become infinite [N°. 699, 2°]. But, making $x = a + h$, we find, for $h = 0$,

$$\frac{(2ah + h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}} = \frac{(2a + h)^{\frac{3}{2}}}{1} = 2a^{\frac{3}{2}}.$$

II. $\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{(x^2 - a^2)}}$ becomes $\dfrac{0}{0}$ for $x = a$: making $x = a + h$, developing by the binomial theorem, dividing above and below by $h^{\frac{1}{2}}$, and then making $h = 0$, we have

$$\frac{(a + h)^{\frac{1}{2}} - a^{\frac{1}{2}} + h^{\frac{1}{2}}}{(2ah + h^2)^{\frac{1}{2}}} = \frac{h^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}h + \ldots}{h^{\frac{1}{2}}(2a + h)^{\frac{1}{2}}} = \frac{1}{\sqrt{(2a)}}.$$

III. For $x = c$ in $\dfrac{(x - c)\sqrt{(x - b)} + \sqrt{(x - c)}}{\sqrt{2c} - \sqrt{(x + c)} + \sqrt{(x - c)}}$, we shall substitute $c + h$ for x; we may indeed make use of Taylor's theorem in the investigation of the terms arising from $(x - c)\sqrt{(x - b)}$ and $\sqrt{(x + c)}$, for which it is not faulty [N°. 699]; when we shall have

$$\frac{\sqrt{h} + h\sqrt{(c - b)} + \ldots}{\sqrt{h} - \frac{1}{2}h(2c)^{-\frac{1}{2}}\ldots}$$

and dividing by $\sqrt{h}$ and then making $h = 0$, we find 1 for the value required.

IV. Changing x into $a + h$ and developing,

$$\frac{(x^2 - a^2)^{\frac{3}{2}} + x - a}{(1 + x - a)^3 - 1} \text{ becomes } \frac{h + (2ah)^{\frac{3}{2}}\ldots}{3h + 3h^2\ldots};$$

and dividing above and below by h, and making $h = 0$, we have $\frac{1}{3}$ for the value of the proposed fraction, when $x = a$.

716. When $x = a$ gives to a product $P \times Q$ the form $0 \times \infty$, we must substitute for Q a value $\dfrac{1}{R}$, such that R may be nothing for $x = a$; when we shall have a fraction $\dfrac{P}{R}$ which becomes $\dfrac{0}{0}$.

For example, $y = (1 - x)\tan(\frac{1}{2}\pi x)$ comes under this case, when $x = 1$: since $\tan = \dfrac{1}{\cot}$, we have $y = \dfrac{1 - x}{\cot(\frac{1}{2}\pi x)} = \dfrac{2}{\pi}$, this fraction being treated according to the prescribed rules.

When $\frac{P}{Q}$ becomes $\frac{\infty}{\infty}$, P and Q have each the form $\frac{1}{R}$, R becoming 0 for $x = a$, and thus this fraction is included in the case of $\frac{0}{0}$.

For example, let $P = \tan\left(\frac{\pi.x}{2.a}\right)$ and $Q = \frac{x^2}{a(x^2 - a^2)}$: the fraction $\frac{P}{Q}$ then becomes $\frac{\infty}{\infty}$ when $x = a$; but it may be changed into

$$\frac{P}{Q} = \frac{a(x^2 - a^2)}{x^2 \cot\left(\frac{\pi}{2}.\frac{x}{a}\right)};\ \text{whence}\ \frac{2a^2}{-\frac{1}{2}\pi a} = -\frac{4a}{\pi}.$$

Lastly, if we have $\infty - \infty$ for $x = a$, we must transform the expression into $\frac{1}{P} - \frac{1}{Q}$, P and Q being each nothing, or $\frac{P-Q}{PQ}$, which comes under the case just discussed. Thus $x \tan x - \frac{1}{2}\pi \sec x$, in the case of $x = 90^\circ$, becomes

$$\frac{x \sin x - \frac{1}{2}\pi}{\cos x} = \frac{0}{0},\ \text{whence}\ \frac{x\cos x + \sin x}{-\sin x} = -1.$$

MAXIMA AND MINIMA.

717. When, on assigning to x different successive values in the function $y = fx$, it first goes on increasing and then begins to decrease, we give the name of *maximum* to that value of the function which separates the incremental state from the decremental; and if fx first decrease and then increase, the *minimum* is the value that separates these two states. Consequently, *a function fx is rendered a maximum or a minimum by the supposition of $x = a$, when it is greater in the 1st case, and less in the 2nd, than the values that we should have by assuming for x two numbers, the one immediately $>$, the other immediately $< a$.*

If, for example, the equation $y = fx$ be represented by a curve *CENM...* [fig. 2], the ordinates *immediately* adjoining the *maxima CB, GF* are less than these latter; and the contrary is the case for the *minimum IR*. It will be seen also that a function fx may have several *maxima* and *minima* that are unequal to each other.

Thus, the condition which determines fa to be a *maximum* or a *minimum*, is that $f(a + h)$ and $f(a - h)$ be both $> fa$, or both $< fa$, however small h be. But

$$f(a \pm h) = fa \pm hf'a + \frac{h^2}{2}f''a \pm \&c.;$$

in which developments h may always be taken so small that the term $hf'a$ shall exceed the sum of all those that follow [N°. 701], so that the sign of $hf'a$ will be that of the whole series subsequent to this term; and we shall therefore have $f(a \pm h) = fa \pm \alpha h$. But fa is comprised between these values, and cannot therefore under these circumstances be either a *maximum* or a *minimum*. Thus, it is incumbent that $f'a = 0$; and in order therefore to find the values of x, which alone are capable of rendering fx a *maximum* or a *minimum*, we must solve the equation $y' = f'x = 0$.

Our developments will now become

$$f(a \pm h) = fa + \tfrac{1}{2}h^2 f''a \pm \tfrac{1}{6}h^3 f'''a + \ldots;$$

and if $f''a$ be positive, we see that $f(a \pm h) = fa + \alpha h^2$; whence it follows that there is a *minimum*: we have a *maximum* when $f''a$ is negative.

But if $f''a$ be also $= 0$,

$$f(a \pm h) = fa \pm \tfrac{1}{6}h^3 f'''a + \tfrac{1}{24}h^4 f^{IV}a + \ldots,$$

and the development again becomes similar to that of the 1st case; whence it follows that there is neither a *maximum*, nor *minimum*, unless $f'''a$ also vanish; if this be the case, $f^{IV}a$ is negative for the 1st of these states, and positive for the 2nd, and so on.

Having found the roots of the equation $f'x = 0$, we must substitute each in $f''x, f'''x\ldots$, till we arrive at a derivative that does not vanish: the root will correspond to a maximum or a minimum, accordingly as this derivative shall be negative or positive, provided only that it be of an even order; for otherwise it will not lead to either one or the other.

718. We shall now give some examples.

I. For $y = \sqrt{(2px)}$, we have $y' = \dfrac{p}{\sqrt{(2px)}}$; and since this quantity cannot be rendered $= 0$, the function $\sqrt{(2px)}$ is not susceptible of either a *maximum* or a *minimum*.

II. $y = b - (x - a)^2$ gives $y' = -2(x - a) = 0$, whence $x = a$, $y'' = -2$; thus, y'' being negative, $x = a$ gives the *maximum* $y = b$; as is in fact evident. $y = b + (x - a)^2$ has on the contrary a *minimum*.

Generally, $y = X(x - a)^n = 0$ gives $x = a$,

$$y' = [X'(x - a) + nX](x - a)^{n-1},\ y'' = \&c.;$$

and it will be readily seen that $x = a$ gives a *maximum* or a *minimum*,

accordingly as X becomes on this supposition negative or positive, provided that n be odd.

III. Let $y = \frac{x}{1+x^2}$; we derive [Nos. 665 and 666],

$$y' = \frac{1-x^2}{(1+x^2)^2},\ y'' = -2x\ \frac{1+2y'(1+x^2)}{(1+x^2)^2}:$$

$y' = 0$ gives $x = \pm 1$; in which case $y = \pm \frac{1}{2}$ and $y'' = \mp \frac{1}{2}$; consequently $x = 1$ corresponds to the *maximum* $\frac{1}{2}$, and $x = -1$ to the *minimum* $-\frac{1}{2}$; or rather to the *negative maximum*, since we have agreed to consider quantities as being the less the farther they are advanced towards the negative infinity.

IV. For $y^2 - 2mxy + x^2 = a^2$, we find [Nos. 684 and 665]

$$y' = \frac{my - x}{y - mx},\ y'' = \frac{2my' - y'^2 - 1}{y - mx} \dots;$$

$y' = 0$ gives $my = x$; eliminating x and y by means of the proposed equation, we find

$$x = \frac{\pm ma}{\sqrt{(1-m^2)}},\ y = \frac{\pm a}{\sqrt{(1-m^2)}},\ y'' = \frac{\mp 1}{a\sqrt{(1-m^2)}};$$

and we therefore have both a *maximum* and a *minimum*.

V. Similarly, $x^3 - 3axy + y^3 = 0$ gives

$$y' = \frac{ay - x^2}{y^2 - ax},\ y'' = \frac{2(ay' - x - yy'^2)}{y^2 - ax} \dots;$$

and it appears that $x = 0$ corresponds to the *minimum* $y = 0$, and $x = a\sqrt[3]{2}$ to the *maximum* $y = a\sqrt[3]{4}$ [See fig. 27].

VI. To divide a number a into two parts, so that the product of the power m of the one, by the power n of the other, may be the greatest possible. Assuming x for one of the parts, we must find a maximum value for the quantity

$$y = x^m(a-x)^n;$$

whence

$$y' = x^{m-1}(a-x)^{n-1}[ma - x(m+n)],$$
$$y'' = x^{m-2}(a-x)^{n-2}[(m+n-1)(m+n)x^2 - \&c.].$$

$y' = 0$ gives $x = 0$, $x = a$ and $x = \frac{ma}{m+n}$; the last of these roots corresponds to the *maximum* which is $m^m n^n \left(\frac{a}{m+n}\right)^{m+n}$; the two others correspond to *minima* when m and n are even.

To divide a number a into two parts the product of which shall be the greatest possible, the half must be taken [N°. 97, 3°].

VII. What is the number x the x^{th} root of which is a *maximum*? We have [N°. 680]

$$y = \sqrt[x]{x},\ y' = y.\frac{1 - lx}{x^2} = 0 \text{ and } lx = 1\,;$$

the number required therefore is the base of the Napierian logarithms, or $x = e = 2{\cdot}71828...$

VIII. Of all fractions which is that which exceeds its m^{th} power by the greatest number possible? Let x be this fraction; we have $y = x - x^m$,

$$y' = 1 - mx^{m-1} = 0, \text{ whence } x = \sqrt[m-1]{\frac{1}{m}}.$$

IX. Of all the supplemental chords of an ellipse, which are those which form the greatest angle? Denoting the semi-axes by a and b, and the tangent of the angle that one of these chords makes with the axis of x by α, the angle of the chords [N°. 409] has for its tangent $\frac{a^2\alpha^2 + b^2}{\alpha(a^2 - b^2)}$; and this quantity we must render a maximum by a suitable value of α, or rather (neglecting the constant divisor $a^2 - b^2$)

$$y = a^2\alpha + \frac{b^2}{\alpha}, \text{ whence } y' = a^2 - \frac{b^2}{\alpha^2} = 0,\ \alpha = \pm\frac{b}{a};$$

the chords in question therefore are directed to one of the extremities of the minor axis: their parallels, drawn through the centre, are the conjugate diameters which form the greatest possible angle: these diameters are equal [See p. 366, Vol. i].

X. Of all the triangles constructed on a given base a, and *Isoperimetrical, i. e.* of the same perimeter $2p$, required that the area of which is the greatest. Denoting the area by y, and one of the unknown sides by x, the third side is $2p - a - x$; and we have [N°. 318, III]

$$y^2 = p(p - a)(p - x)(a + x - p).$$

To render y^2 a *maximum*, take the log of this and the derivative, and we shall have

$$\frac{-1}{p - x} + \frac{1}{a + x + p} = 0, \text{ whence } 2x = 2p - a\,;$$

thus the triangle required is isosceles.

Generally, of all the isoperimetrical polygons, that of the greatest area is equilateral; for let $ABCDE$ [fig. 19] be the *maximum* polygon: if AB be not $= BC$, form the isosceles triangle AIC, such that $AI + IC = AB + AC$; we shall have the triangle $AIC > ABC$, whence $AICDE > ABCDE$, which is contrary to the hypothesis.

XI. The base $AC = a$ [fig. 20] being given, which is the least of all the triangles that can be described about the circle OF? Let the radius $OF = r$, $AF = AD = x$, the perimeter $= 2p$; $CF = CE$ will be $= a - x$, $BE = BD$ will be $= p - a$; and the three sides being $a, p - x$ and $p - a + x$, we have for the area y of the triangle [N°. 318, III]

$$y^2 = px(p - a)(a - x),$$

or, y being $= pr$ [N°. 318, IV],

$$yr^2 = x(y - ar)(a - x).$$

Taking the derivative, and making $y' = 0$, we shall find $(y - ar)(a - 2x) = 0$; whence $x = \frac{1}{2}a$; thus, F is the middle point of AC; the other two sides are equal, and the triangle is isosceles.

XII. On the sides of a square $ABCD$ [fig. 21], let any equal parts Aa, Bb, Cc, Dd be taken; the figure $abcd$ will be a square: for 1°. $aB = bC\ldots$, and the triangle $dAa = aBb = \ldots$; whence $ab = bc = cd = ad$; 2°. a is the vertex of two complemental angles, and of the angle dab; this last therefore is a right angle; and similarly for the angle abc, &c....

This being premised, of all the squares inscribed in a given square, required that which is the least.

Let $AB = a$, $Aa = x$, and consequently $aB = a - x$; then the triangle Aad gives

$$(ad)^2 = 2x^2 - 2ax + a^2,\ 4x - 2a = 0;$$

whence $x = \frac{1}{2}a$: and thus the point a must be taken in the middle of AB.

XIII. Of all the rectangular parallelipipeds that are equal to a given cube a^3, and have the line b for an edge, which is that of which the surface is the least? Let x and z be the other edges; then bxz will be the volume $= a^3$; and the dimensions therefore of the parallelipiped are b, x and $\frac{a^3}{bx}$; consequently $\frac{a^3}{b}$, bx and $\frac{a^3}{x}$ are the areas of the faces, and the double of their sum is the total surface:

$$y = \frac{2a^3}{b} + 2bx + \frac{2a^3}{x},\ y' = 2b - \frac{2a^3}{x^2} = 0,\ x = \sqrt{\frac{a^3}{b}} = z;$$

thus the two other dimensions x and z must be equal.

If the side b be not given, x being always one of them, each of the others must be $\sqrt{\dfrac{a^3}{x}}$; $\dfrac{2a^3}{x} + 4\sqrt{a^3x}$ is therefore the total area, whence

$$\frac{a^3}{x^2} = \sqrt{\frac{a^3}{x}}, \text{ and } x = a;$$

the cube proposed is therefore the rectangular parallelipiped of the least surface.

719. To apply this theory to curves, we form [N°. 684] the derivative of their equation: the real roots of x and y, which satisfy the proposed equation and its derivative, are obtained by elimination; and they alone can correspond to *maxima* or *minima* of the ordinates. We next take the derivative of the 2nd order, and in it substitute for x and y one of the pairs of values deduced from $y' = 0$; if then $x = AF$, $y = FG$, [fig. 2] render y'' negative, the point G will be a *maximum;* if, on the other hand, the co-ordinates AR, RI, render y'' positive, I will be a *minimum* [See examples IV and V].

When the developments of $f(a \pm h)$ prove faulty within the range of the terms to which we are obliged to have recourse for the purpose of ascertaining the *maxima* or *minima*, these developments must be found under their proper form [N°. 698], and we must then examine whether in effect they are both $>$ or both $< fa$. Thus, $y = b + (x - a)^{\frac{5}{3}}$ gives

$$y' = \tfrac{5}{3}(x - a)^{\frac{2}{3}},\ y'' = \tfrac{10}{9}(x - a)^{-\frac{1}{3}};$$

$y' = 0$ gives $x = a$, which renders $y'' = \infty$; and thus Taylor's formula fails. But $f(a \pm h) = b \pm h^{\frac{5}{3}}$; and there consequently is neither a *maximum* nor a *minimum*. On the contrary, from $y = b + (x - a)^{\frac{4}{3}}$, we deduce

$$f(a + h) = b + h^{\frac{4}{3}} = f(a - h);$$

and therefore $x = a$, $y = b$ correspond to a minimum. We should have a maximum for $y = b - (x - a)^{\frac{4}{3}}$.

720. As to the functions of two variables, $z = f(x, y)$, we must adopt the reasonings of N°. 717. Changing x into $x + h$, y into $y + k$, and developing as in N°. 703, we shall have, making $k = \alpha h$,

$$Z = z + h\left(\frac{dz}{dx} + \alpha\frac{dz}{dy}\right) + \frac{h^2}{2}\left(\frac{d^2z}{dx^2} + 2\alpha\,\frac{d^2z}{dy\,dx} + \alpha^2\,\frac{d^2z}{dy^2}\right)\ldots$$

But, that we may always have $Z < z$, or $Z > z$, whatever h and k be, the second term must be nothing independently of α, whence

$$\frac{dz}{dy} = 0,\ \frac{dz}{dx} = 0 \ldots (1);$$

and, moreover, the following term must be positive in the case of the *minimum*, negative for the *maximum*. We must therefore eliminate x and y between the equations (1), and their roots alone will answer the end proposed: these roots being then substituted in the following term $\frac{h^2}{2}\left(\frac{d^2z}{dx^2}\ldots\right)$, it must constantly have the same sign, whatever value we give to α, and whatever be its sign. But, a quantity $A + 2\alpha B + C\alpha^2$ cannot retain the same sign whatever α be, unless its factors be imaginary [N°. 139, 9°], which requires that $AC - B^2$ be > 0. We must therefore have

$$\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2} - \left(\frac{d^2z}{dx\,dy}\right)^2 > 0 \ldots (2);$$

and $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{dy^2}$ must consequently be of the same sign: if this sign be $-$ for $k = 0$, or $\alpha = 0$, our trinomial then becoming $\frac{d^2z}{dx^2}$, *i.e.* negative, the trinomial always retains this sign; and there is therefore a *maximum*: there is a *minimum*, when $\frac{d^2z}{dx^2}$ and $\frac{d^2y}{dy^2}$ are positive. And if the condition (2) be not fulfilled, there is neither a *maximum* nor a *minimum*.

When the roots of the equations (1) reduce the terms of our trinomial to nothing, recourse must be had to the 4th term of the development, which must also be nothing, and so on.

721. Required, for example, the shortest distance between two given straight lines. We shall take one of these lines for the axis of x, and the other will have for its equations

$$z = ax + \alpha,\ y = bx + \beta.$$

Take in the 1st line a point, the abscissa of which is x': the distance from it to any point of the second line will be R, viz. [N°. 614]

$$R^2 = (x - x')^2 + y^2 + z^2,$$

or

$$R^2 = (x - x')^2 + (bx + \beta)^2 + (ax + \alpha)^2;$$

and denoting this second side by t, we shall have

$$\frac{dt}{dx} = 2(x - x') + 2b(bx + \beta) + 2a(ax + \alpha) = 0,$$

$$\frac{dt}{dx'} = -2(x - x') = 0; \text{ whence } x = x' = -\frac{a\alpha + b\beta}{a^2 + b^2}.$$

Since $x = x'$, the line in question is perpendicular to the axis of x, and consequently is so also to the 2nd line which might have been taken for this axis: this we have found to be the case already [N°. 274]. In the last place

$$\frac{d^2t}{dx^2} = 2(1 + a^2 + b^2), \frac{d^2t}{dx'^2} = 2, \frac{d^2t}{dx\,dx'} = -2;$$

so that the condition (2) is satisfied, $4(a^2 + b^2)$ being > 0; and there is a *minimum*. The length of the line is $R = \dfrac{a\beta - b\alpha}{\sqrt{(a^2 + b^2)}}$. The equation of its projection on the plane yz being $y = Az$, since it passes through a point (x, y, z) of the 2nd given straight line, we have

$$A = \frac{y}{z} = \frac{bx + \beta}{ax + \alpha} = -\frac{a}{b};$$

and these lines therefore satisfy the condition [N°. 633, 6°], and are perpendicular to each other, as has been already proved.

METHOD OF TANGENTS.

722. Let it be proposed to draw a tangent TM [fig. 22] at the point $M(x, y)$ of the curve BMM', having given its equation $y = fx$. The equation of the straight line TM is

$$Y - y = \tan\alpha(X - x),$$

X and Y being the variable co-ordinates of the straight line, x and y those of the point of contact M, and α the angle T. Now it has been proved, N°. 655, that the tangent of the angle T, is the derivative $y' = f'x$ the limit of the ratio of the increments MQ and $M'Q$ of the co-ordinates x and y; and it is indeed on this principle that we have established the existence of the derivatives for all the functions of x, and consequently the differential Calculus itself. Hence [N°. 346]

$$\tan\alpha = y', \cos\alpha = \frac{1}{\sqrt{(1 + y'^2)}}, \sin\alpha = \frac{y'}{\sqrt{(1 + y'^2)}};$$

$$Y - y = y'(X - x).$$

1°. The normal MN makes with the axis of x an angle [N°. 370] the tangent of which is $-\frac{1}{y'}$; and its equation therefore is

$$y'(Y-y)+X-x=0.$$

2°. Making $Y=0$, we have the abscissæ AT, AN, of the feet of the tangent and normal; whence we deduce $x-X$, or

$$\text{sub-tangent } TP=\frac{y}{y'},\ \text{sub-normal } PN=yy'.$$

When these values have a negative sign, it indicates that these lines fall in directions opposite to those of our figure; we must examine then whether it is y or y' that is negative, and we shall easily ascertain the situation of the lines [see N°. 339].

3°. The hypotenuses TM and MN give

$$\text{tangent } TM=\frac{y}{y'}\sqrt{(1+y'^2)},$$

$$\text{normal } MN=y\sqrt{(1+y'^2)}.$$

4°. Applying our previous reasoning to the case in which the angle of the co-ordinates is any whatever [see N°. 420], we shall find that the equation of the tangent and the value of the sub-tangent continue the same.

723. The following are some examples of these formulæ:

I. In the parabola, $y^2=2px$; whence $yy'=p$, $\frac{y}{y'}=2x$; the normal $MN=\sqrt{(2px+p^2)}$ [N°. 404].

II. For the ellipse and hyperbola $a^2y^2\pm b^2x^2=\pm a^2b^2$; whence $y'=\mp\frac{b^2x}{a^2y}$; and hence we have the sub-tangents, &c. [see N°s. 408 and 414]. For example, making $c^2=a^2\mp b^2$, we find for the length of the normal,

$$N=\frac{b\sqrt{\pm(a^4-c^2x^2)}}{a^2}.$$

III. For the equation $y^m=x^na^{m-n}$, we find $\frac{y}{y'}=\frac{mx}{n}$. The parabola is a particular case of this; on which account it is that the name of *parabolas* has been given to all the curves comprised under this equation, m and n being positive: $y^3=a^2x$ is called the *first cubical parabola*; $y^3=ax^2$ is the *second*.

And, similarly, the name of *hyperbola* is given to the curves the equation of which is $x^n y^m = a^{m+n}$; their sub-tangent is $\frac{y}{y'} = -\frac{mx}{n}$; the same as in the preceding case, only with a contrary sign.

IV. For the curve the equation of which is $x^3 - 3axy + y^3 = 0$, we have

$$y' = \frac{ay - x^2}{y^2 - ax}, \text{ sub-tangent} = \frac{y^3 - axy}{ay - x^2}, \text{ \&c.}$$

V. In the logarithmic curve [N°. 468], $y = a^x$ gives $\frac{y}{y'} = \frac{1}{la}$; so that the sub-tangent is equal to the modulus [N°. 585].

VI. Let $AP = x$, $PM = y$, $MQ = z = \sqrt{(2ry - y^2)}$ [fig. 23], the equation of the cycloid AMF is $x = \text{arc}\,(\sin = z) - z$ [N°. 471]; the arc being here taken in the generating circle MGD, the radius of which is r. The derivative therefore is [N°. 683]

$$1 = \frac{rz'}{\sqrt{(r^2 - z^2)}} - z', \text{ where } z' = \frac{(r-y)\,y'}{\sqrt{(2ry - y^2)}};$$

and consequently, eliminating z and z', the cycloid has for its derivative equation

$$yy' = \sqrt{(2ry - y^2)}, \text{ or } y' = \sqrt{\left(\frac{2r - y}{y}\right)},$$

the origin being at the point of reflexion A.

To draw a tangent TM, we shall observe that the sub-normal $= yy' = \sqrt{(2ry - y^2)} = z = MQ$.

Thus, the line MD drawn to the point of contact D of the generating circle with the axis AE is the normal; and the chord MD is its length; we find, in fact, $y\sqrt{(1 + y'^2)} = \sqrt{(2ry)}$. The supplemental chord MG is consequently the tangent; and it appears therefore that to draw a tangent at M, we must draw MN parallel to the axis AE, then the chord KF, and lastly MG parallel to KF.

If the origin be situated at the highest point F, so that we have to take $FS = x$, $SM = y$, the equation of the cycloid is

$x = \text{arc}\,(\sin = z) + z$ [N°. 471], and the derivative

$$y' = \sqrt{\left(\frac{y}{2r - y}\right)}.$$

This equation might also have been found by transferring the origin to F, x being changed for this purpose into $\pi r - x$ and y into $2r - y$.

724. We may on these principles solve a great number of problems relative to tangents, such as drawing them from an exterior point, or in a direction parallel to a given straight line, or &c. [See N^{os}. 407 and 413].

Let us, for instance, investigate the angle β formed by the tangent TM [fig. 24] and the *radius vector* AM drawn from the origin to the point of contact M (x, y). The angle θ which this radius vector makes with the axis is given by $\tan\theta = \frac{y}{x}$; also $\tan\alpha = y'$; and consequently

$$\tan(\alpha - \theta) \text{ or } \tan\beta = \frac{y'x - y}{x + yy'}.$$

In applying this, attention must be paid to the sign which this fraction takes.

For the equation $y^2 + x^2 = r^2$, which belongs to the circle, we find $\tan\beta = \infty$, as is otherwise evident.

725. When a curve BM [fig. 24] is referred to polar co-ordinates $AM = r$, $MAP = \theta$, the preceding formulæ cannot be made use of until the equation $r = f\theta$ of the curve has been suitably transformed into one between x and y, by means of the relations [N°. 385].

$$x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2.$$

We may, contrarywise, transform the formulæ for the tangent, &c., into others for r and θ. Thus, take θ for the independent variable instead of x; and the calculation that has been already made [p. 256] gives

$$\tan\beta = \frac{r}{r'}.$$

726. We might in like manner convert the values yy', $\frac{y}{y'}$, &c. into r, r' and θ; but, on account of their complexity, we prefer the following method.

The name of *sub-tangent* is given to the length of the part AT, taken along the perpendicular to AM; and the point T being thus determined, the tangent TM follows of course. But, the triangle ATM gives $AT = AM\tan\beta$, or

$$\text{sub-tangent} = AT = \frac{r^2}{r'}.$$

For the spiral of Archimedes [N°. 472, fig. 25], we have

$$r = \frac{a\theta}{2\pi}, \frac{r^2}{r'} = \theta r, \frac{r}{r'} = \theta.$$

Thus, the sub-tangent AT is equal in length to the circular arc described with the radius $AM = r$, and which measures the angle $MAx = \theta$. As to the angle β, it increases continually with θ; and since it is not till after an infinite number of revolutions of the radius vector that θ becomes infinite, the right angle is the limit of β.

In the hyperbolic spiral [N°. 473]

$$r = \frac{a}{\theta}, \text{ sub-tan } = -a, \tan\beta = -\theta:$$

thus the sub-tangent is constant; the asymptote is the limit of all the tangents; and lastly, the angle of the radius vector with the tangent is obtuse, and decreases as θ increases [See fig. 160, vol. i.].

For the logarithmic spiral [N°. 474]

$$r = a^\theta, \tan\beta = \frac{1}{la}, \text{ sub-tan} = \frac{r}{la}.$$

the curve cuts all its radii vectores at the same angle, which is that of 45°, when a is the base of the Napierian logarithms; the sub-tangent increases proportionally to the radius vector.

RECTIFICATIONS AND QUADRATURES.

727. When the equation $y = fx$ of a curve BMM' [fig. 22] is given, the length $BM = s$ of a developed arc may be determined when its extremities B and M are known: let us investigate this length. For this purpose, we shall observe that, B remaining fixed, s varies with the point M; thus s is a function of $x = AP$, $s = Fx$, and this function we are now to find. If x be increased by $h = PP'$, y will increase by $M'Q = k$, s by $MM' = l$; and

$$y = fx \text{ gives } f(x+h) = y + y'h + \tfrac{1}{2}y''h^2 + \ldots,$$
$$s = Fx \ \ldots \ F(x+h) = s + s'h + \tfrac{1}{2}s''h^2 + \ldots;$$

whence

$$k = y'h + \tfrac{1}{2}y''h^2 + \ldots, \quad l = s'h + \tfrac{1}{2}s''h^2 + \ldots;$$
$$\text{chord } MM' = \sqrt{(h^2 + k^2)} = h\sqrt{(1 + y'^2 + y'y''h + \ldots)}.$$

On the other hand, the tangent MH gives [N°. 722]

$$QH = y'h, \ MH = h\sqrt{(1 + y'^2)}, \ M'H = -\tfrac{1}{2}y''h^2 \ldots;$$

whence

$$\frac{\text{chord } MM'}{MH + M'H} = \frac{\sqrt{(1 + y'^2 + y'y''h + ...)}}{\sqrt{(1 + y'^2)} - \frac{1}{2}y''h...}$$

But the more h decreases, the more this last ratio verges towards unity; 1 therefore is also the limit of the 1st side; and since the arc MM' is comprised between its chord and the broken line $MH + M'H$, 1 is the limit too of the chord to the arc, or of

$$\frac{\text{chord}}{\text{arc}} = \frac{\sqrt{(1 + y'^2 + y'y''h...)}}{s' + \frac{1}{2}s''h...}; \text{ whence } 1 = \frac{\sqrt{(1 + y'^2)}}{s'},$$

$$s' = \sqrt{(1 + y'^2)}, \text{ or } ds = \sqrt{(dx^2 + dy^2)}.$$

This formula serves for the rectification of all curvilinear arcs. We substitute for y' its value $f'x$, deduced from the given equation $y = fx$ of the curve, and we thus obtain the derivative s' of the equation $s = Fx$; it will then remain to *integrate* $F'x$, *i. e.* to trace back this derivative to its primitive function Fx. The mode of effecting this will be explained subsequently [N°. 809].

The equation of the circle, its centre being at the origin, is

$$y^2 + x^2 = r^2, \text{ whence } yy' + x = 0;$$

$$s' = \sqrt{\left(1 + \frac{x^2}{y^2}\right)} = \pm \frac{r}{y} = \frac{\pm r}{\sqrt{(r^2 - x^2)}};$$

and this is the derivative of the circular arcs, expressed in a function of its sine or cosine [which is x, N°. 683]. In order therefore to rectify the circular arc, we must integrate this function [N°. 809, III].

By means of our value of s', we may simplify the formulæ of p. 291, which now become

$$\tan \alpha = y' = \frac{dy}{dx}, \cos \alpha = \frac{1}{s'} = \frac{dx}{ds}, \sin \alpha = \frac{y'}{s'} = \frac{dy}{ds},$$

$$\text{tangent} = \frac{ys'}{y'} = \frac{yds}{dy}, \text{ normal} = ys' = \frac{yds}{dx}.$$

728. To obtain the area $BCPM = t$ [fig. 22], reasoning as before, we shall see that t is a function of x, or $t = \varphi x$; whilst the increments k and i of the ordinate and the area for the abscissa $x + h$ are

$$k = M'Q = y'k + ..., i = MPP'M' = t'h + ...$$

We also have rectangle $MPP'Q = yh$, $LP' = (y + k)h$; and the limit of their ratio $\frac{y}{y + k}$ is unity; unity therefore is also the limit of

the ratio between the rectangle $MPP'Q = yh$ and the increment $MPP'M' = i$ of the area t. This ratio is

$$\frac{yh}{i} = \frac{y}{t' + \frac{1}{2}t''h + ...};\ \text{whence}\ \frac{y}{t'} = 1,\ \text{or}\ t' = y.$$

In this expression we must substitute fx for y, and integrate the equation $t' = fx$ [See N°. 805].

Had the co-ordinates been supposed to make an angle α, we should have found

$$t' = y \sin \alpha.$$

729. Let us now investigate the area $AKM = \tau$ [fig. 24], comprised between two radii vectores AM, AK, the latter of which remains fixed, whilst the other varies with M. We have the area AKM or

$$\tau = ABMK - ABM;$$

of which

$$ABM = ABCD + DCMP - AMP = ABCD + t - \tfrac{1}{2}xy;$$

and consequently

$$\tau = ABMK - ABCD - t + \tfrac{1}{2}xy.$$

But, the variation of the point M produces no change in the points B, C and K; so that the derivative may be taken, considering $ABMK$ and $ABCD$ as constant, and we have

$$\tau' = -t' + \tfrac{1}{2}(xy' + y) = \tfrac{1}{2}(xy' - y).$$

To transform the values of s' and τ' into those for polar co-ordinates r and θ, substituting $\frac{s'}{x'}, \frac{y'}{x'}, \frac{\tau'}{x'}$ for s', y' and τ' [N°. 689] we have

$$s'^2 = x'^2 + y'^2,\ \tau' = \tfrac{1}{2}(xy' - yx'),$$

where the independent variable may be any whatever. That it may be θ, we have only, in these expressions, to substitute for x, y, x', y' the values of N°. 690, when there will result

$$s' = \sqrt{(r^2 + r'^2)},\ \tau' = \tfrac{1}{2}r^2;$$

and these are the formulæ of rectification and quadrature for curves referred to polar co-ordinates, the equation being $r = f\theta$. They might also have been obtained directly by the method of limits.

OSCULATIONS.

730. Suppose that, at a point M [fig. 26] of a curve BMZ, a tangent TM and a normal MN be drawn; and that, from different points a, b... of the normal, circles be described passing through the point M, and consequently having TM for their common tangent. It is obvious then, from the disposition of these circles, that some of them will lie within, and others without the curve, and that there will be one which approaches more nearly than any other to the curve BMZ, on each side of the point M. To this we give the name of the *osculating Circle;* its centre D and radius DM are called the *Centre* and *Radius of Curvature;* and as, on changing the point M, the circle also changes its centre and radius, we give the name of *Evolute* to the curve IOD, which passes through all the centres of curvature: the given line BMZ is the *Involute* of IOD.

To find the osculating circle of a curve, at any given point M, the conditions which determine it must be expressed analytically; but we shall first generalize these considerations. Suppose that there are any two curves which cut each other; their equations $y = fx$, $Y = FX$ give $y = Y$ for the abscissa $x = X$ of the common point. As yet there is no more than a simple intersection; to compare the courses of the two lines beyond this point, substitute $x + h$ for x and X, in y and Y; the corresponding ordinates will be

$$y + y'h + \tfrac{1}{2}y''h^2 \ldots,\ Y + Y'h + \tfrac{1}{2}Y''h^2 + \ldots;$$

whence

$$\delta = h(y' - Y') + \tfrac{1}{2}h^2(y'' - Y'') + \ldots,$$

for the distance between the two points of our curves the abscissa of which is $x+h$: X must be replaced by x in Y', Y''... And the less δ be for a given value of h, the nearer will the corresponding points be to each other; so that the degree of approach of our curves depends on the minuteness of δ, for a fixed length of h. In case now that the value of x, for which $y = Y$, also render $y' = Y'$, we have

$$\delta = \tfrac{1}{2}h^2(y'' - Y'') + \tfrac{1}{6}h^3(y''' - Y''') + \ldots;$$

and our two curves approach more closely to each other than a third could be made to do which, passing through the same point (x, y), did not fulfil this same condition. For, let $\gamma = \phi\xi$ be the equation of this last curve; the distance Δ, between the points of this curve and the first, which have $x + h$ for their abscissa, is

$$\Delta = h(\gamma' - y') + \tfrac{1}{2}h^2(\gamma'' \quad y'') + \ldots$$

supposing $\varphi x = fx$, in order that they may have the common point (x, y). But these values of δ and Δ have the forms

$$\delta = bh^2 + ch^3 + \ldots, \Delta = Ah + Bh^2 + Ch^3 \ldots;$$

whence $\quad \Delta - \delta = Ah + (B - b)h^2 + (C - c)h^3 + \ldots;$

and if h therefore be taken so small [N°. 701] that the term Ah give its sign to these latter series, $\Delta - \delta$ having the sign of Δ, we shall have $\Delta > \delta$ for this value of h, and for all those which are less, whatever be the sign of h. Thus the curve $y = Fx$ approaches to the one $y = fx$, for the whole of this extent h, and on each side of the common point, more closely than the third curve $\gamma = \varphi\xi$ can be made to do, whatever be its nature.

If, besides $y' = Y'$, we likewise have $y'' = Y''$, it will in the same manner be seen that our two curves approach to each other, in the points immediately contiguous to the one that is common, more nearly than a third for which these two conditions are not fulfilled; and so on. We shall say of two lines that they have a *Contact* or an *Osculation of the 1st order*, when they satisfy the conditions $y = Y$, $y' = Y'$, for the same abscissa x. Similarly $y = Y$, $y' = Y'$, $y'' = Y''$ will be the conditions of the *contact of the 2nd order*, &c.; and it is demonstrated that these two curves are nearer to each other in the neighbourhood of the common point, than any 3rd curve, unless it form a similar osculation.

731. These principles being premised, if certain of the constants $a, b, c \ldots$ contained in the equations $y = fx$, $Y = FX$ of the two curves, be arbitrary, the nature of these lines is fixed, whilst their position and some of their dimensions are not so. We may therefore determine these $(n + 1)$ constants by an equal number of conditions $y = Y$, $y' = Y'$, $y'' = Y'' \ldots$, and the curves will thus have a contact of the nth order; and will approach to each other more closely than any other curve which does not form an osculation of the same order.

732. To apply this to the straight line, let $y = fx$ be the given equation of a curve, and take a straight line, the position of which is undetermined; our equations then are

$$y = fx, \; Y = aX + b,$$

a and b being any whatever. If we assume $y = Y$, $y' = Y'$, or

$$y = ax + b, \; y' = a,$$

there will be an osculation of the 1st order. The straight line will be a tangent; in fact, that any other straight line may approach more nearly to the curve, on each side of the common point, it must fulfil the same conditions, *i. e.* have the same values for its constants. Thus y' is the tangent of the angle that our straight line makes with the axes; and eliminating a and b, the equation of the tangent is

$$Y - y = y'(X - x),$$

as in N°. 722. We easily deduce from this the equation of the normal, the value of the sub-tangent, &c.

733. Reasoning in the same manner for the circle, the equations of the given curve, and of a circle taken in any situation whatever, are

$$y = fx, \quad (y - b)^2 + (X - a)^2 = R^2;$$

a and b being the co-ordinates of the centre, and R the radius. To determine these constants, we shall establish a contact of the 2nd order. The derivatives of the latter equation are

$$(Y - b)\,Y' + X - a = 0, \quad (Y - b)\,Y'' + Y'^2 + 1 = 0;$$

and therefore

$$\begin{aligned} (y - b)^2 + (x - a)^2 &= R^2 \ldots (1), \\ (y - b)y' + x - a &= 0 \ldots (2), \\ (y - b)y'' + y'^2 + 1 &= 0 \ldots (3), \end{aligned}$$

Deducing $y - b$ and $x - a$ from the two last of these,

$$y - b = -\frac{1 + y'^2}{y''}, \quad x - a = \frac{y'(1 + y'^2)}{y''};$$

whence the 1st gives

$$R = \pm \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}\,*:$$

also

$$a = x - \frac{y'}{y''}(1 + y'^2), \quad b = y + \frac{1 + y'^2}{y''};$$

* The value of R must take the sign $\pm$; but since this expression has no meaning except when it is positive [N°. 336], that of the two signs must be preferred which will give to the value of R the sign $+$. If y'' be positive, which is the case when the curve turns its convex side towards the axis of x, we shall take the sign $+$; whilst the sign $-$ must be preferred in the contrary case [See N°. 743].

and thus therefore we have the radius and the centre of curvature. Every other circle will approach to our curve in a less degree than this; as otherwise it must fulfil the same conditions, *i. e.* be coincident with it.

734. Hence we see that

1°. The tangent to the curve is so also to the osculating circle, since y' has the same value for both.

2°. The equation of the normal being $y'(Y-y)+X-x=0$, if in it we substitute a and b for X and Y, it is satisfied, since we thus arrive at the relation (2), which supposes only a contact of the 1st order between the curve and the circle: hence *the centre of curvature is on the normal*, as is also the centre of every circle which has the same tangent TM [fig. 26].

3°. If we eliminate x and y between the equation $y=fx$ of the curve, and these (2), (3) which determine a and b, we shall have a relation between the co-ordinates of the centre of the curvature which exists, wherever the point M be, and this therefore will be the *equation of the evolute*.

4°. Since R, a and b are functions of x, easily determined by differentiation, if we were to substitute them in (1) and (2), the results would be respectively identical with those equations; and we may consequently differentiate (1) and (2), considering R, a and b as variable. Operating first on the equation (2), there results

$$(y-b)y''+y'^2-b'y'-a'+1=0;$$

whence, subtracting from (3),

$$b'y'+a'=0;$$

which, as was to be expected, is the derivative of the equation (2), relatively to a and b alone. We consequently have $-\frac{1}{y'}=\frac{b'}{a'}$ for the tangent of the angle which the normal makes with the axis of x. Now let $b=\varphi a$ be the equation of the evolute; its tangent at the point (a,b) makes with the axis of x an angle the trigonometrical tangent of which is $\frac{db}{da}=\frac{b'}{a'}=-\frac{1}{y'}$ [N°. 689], since, in our calculation, we have considered b and a as functions in which x is the principal variable. Hence, *the normal to the involute is a tangent to the evolute*.

5°. Let the same thing be done for the equation (1), *i. e.* let the deri-

tive be taken supposing all the letters to vary, and the result be subtracted from the equation (2); or, otherwise, let the derivative of (1) be taken relatively to a, b and R alone: there results

$$-(y-b)b' - (x-a)a' = RR'.$$

To deduce from this a relation which belongs generally to all the points of the evolute, x and y must be eliminated. Now the values of $x-a$ and $y-b$ being derived from (1) and (2), and $-\frac{a'}{b'}$ substituted in them for y', we find

$$x-a = -\frac{y'R}{\sqrt{(1+y'^2)}} = \frac{a'R}{\sqrt{(a'^2+b'^2)}},$$

$$y-b = \frac{R}{\sqrt{(1+y'^2)}} = \frac{b'R}{\sqrt{(a'^2+b'^2)}};$$

and hence

$$\frac{a'^2R+b'^2R}{\sqrt{(a'^2+b'^2)}} = -RR', \text{ or } R' = \sqrt{(a'^2+b'^2)};$$

so that if a be taken as the principal variable, $R' = \sqrt{(1+b'^2)}$ is the derivative of the radius of curvature relatively to a. But that of the arc s of the evolute is also $s' = \sqrt{(1+b'^2)}$ [N°. 727]; and consequently $R' = s'$, an equation which is the derivative of $R = s + A$, A being an arbitrary constant [N°. 768].

For any other arc S of the evolute, the radius of curvature is in like manner $S + A$, the fixed origin of this arc being the same; thus $s - S$ is the difference of the two radii; and it follows therefore that if O and D [fig. 26] be the centres of curvature of the points B and M, the arc OD of the evolute is the difference of the radii of curvature BO, MD. Hence, if a thread be wrapped round the evolute OD, one end being stretched in the direction BO, and it be then unwound from OD, the extremity B will describe the involute BM; it is to this property that these curves owe their denominations.

6°. The expressions for the radius of curvature and the co-ordinates of the centre present themselves under different forms, accordingly as one or other of the variables is taken for the independent one. Thus it has been seen [N°. 692] that

$$R = \frac{(x'^2+y'^2)^{\frac{3}{2}}}{x'y''-y'x''},\ R = \frac{x'}{y''} = -\frac{y'}{x''},$$

accordingly as the principal variable is arbitrary, or is the arc s; if this

variable be the abscissa x, the values of R, a and b may be written thus:

$$R=\frac{s'^3}{y''},\ a=x-\frac{y's'^2}{y''},\ b=y+\frac{s'^2}{y''}.$$

7°. If the co-ordinates are polar, we must express x and y in functions of these new co-ordinates $AM=r$, $MAP=\theta$ [fig. 24]; and substitute for the resulting values of $x, x'\ldots$ in that of R in which there is no principal variable [See the formulæ, N°. 690]. All the reductions being gone through, we have

$$R=\frac{(r'^2+r^2)^{\frac{3}{2}}}{2r'^2-rr''+r^2}=\frac{s'^3}{2r'^2-rr''+r^2}.$$

735. We shall now apply this theory to some examples.

I. For the parabola, $y^2=2px$, $y'=\frac{p}{y}$, $y''=-\frac{p^2}{y^3}$; and substituting in our formulæ, we find

$$s'=\sqrt{\left(\frac{2x+p}{2x}\right)},\ R=\frac{(2px+p^2)^{\frac{3}{2}}}{p^2}=\frac{N^3}{p^2},$$

N being the length of the normal [N°. 723, I.] Hence, *the radius of curvature of the parabola is equal to the cube of the normal, divided by the square of the semi-parameter.* At the vertex A [fig. 26], where $x=0$, we have $R=p$; and thus the distance AI from the vertex to its centre of curvature is the double of that from the focus. The more x increases, the more the curvature decreases, and that indefinitely. The co-ordinates of the centre of curvature are

$$a=3x+p,\ b=-\frac{2xy}{p}.$$

Eliminating x and y from $y^2=2px$, we have, for the equation of the evolute, $b^2=\frac{8}{27p}(a-p)^3$, whence $b^2=\frac{8a^3}{27p}$, the origin being transferred to I: this is the second cubical parabola; we shall proceed shortly to discuss it.

II. For the ellipse we have $m^2y^2+n^2x^2=m^2n^2$,

$$m^2yy'+n^2x=0,\ m^2yy''+m^2y'^2+n^2=0,$$

$$y'=-\frac{n^2x}{m^2y},\ y''=-\frac{n^4}{m^2y^3},\ 1+y'^2=\frac{n^2}{m^4}\cdot\frac{m^4-c^2x^2}{y^2},$$

c being the distance from the focus to the centre, $c^2=m^2-n^2$;

$$R = -\frac{(m^4 - c^2x^2)^{\frac{3}{2}}}{m^4 n}, \; a = \frac{c^2x^3}{m^4}, \; b = -\frac{c^2y^3}{n^4};$$

and these are the values of the radius and of the co-ordinates of the centre of curvature for the ellipse. The values of R, of the normal [p. 292] and the parameter p being compared, it will be seen that $R = \frac{m^2N^3}{n^4} = \frac{N^3}{(\frac{1}{2}p)^2}$: the same theorem as for the parabola.

Since an arc of the evolute is the difference between the radii of curvature which start from its extreme points [p. 302], and these radii are certain finite quantities, the arc is rectifiable. The same is the case for all the algebraic curves, and we can always find a straight line of the same length as a given arc of the evolute.

Since R decreases when x increases, it is at the four extremities of the axes that R is a *maximum* or *minimum*: at the vertices O, O' of the ellipse [fig. 53] the curvature is the greatest, $R = \frac{n^2}{m}$, $a = \pm\frac{c^2}{m}$, $b=0$; at D and D', the curvature is the least, $R = \frac{m^2}{n}$, $b = \pm\frac{c^2}{n}$, $a = 0$; the points h, h', i, i', thus determined, are the centres of curvature at the extremities of the axes. To obtain the equation of the evolute, the values of x and y must be deduced from those of a and b, and substituted in the equation of the ellipse; when we have

$$\sqrt[3]{\left(\frac{b^2n^2}{c^4}\right)} + \sqrt[3]{\left(\frac{a^2m^2}{c^4}\right)} = 1, \text{ or } \sqrt[3]{\left(\frac{b}{p}\right)^2} + \sqrt[3]{\left(\frac{a}{q}\right)^2} = 1,$$

making $Ch = q$, $Ci = p$. It will subsequently be found that the curve undergoes reflexion at the four points h, h', i, i', and that it is formed of four arcs convex towards the two arcs, in respect to which it is symmetrical: the evolute is represented by the dotted line in fig. 53.

For the hyperbola [N°. 397], n must be changed into $n\sqrt{-1}$.

III. The cycloid [fig. 23] gives [p. 293]

$$y' = \sqrt{\left(\frac{2r-y}{y}\right)} = \sqrt{\left(\frac{2r}{y} - 1\right)}, y'' = -\frac{r}{y^2},$$

whence $$s'^2 = \frac{2r}{y} \text{ and } R = 2\sqrt{(2ry)} = 2N.$$

Thus, the radius of curvature being double of the normal, if we produce MD and take $M'D = MD$, M' will be the centre of curvature. It would be easy from this to deduce the figure of the evolute; but we shall prefer following the general method, which gives

$$a = x + 2\sqrt{(2ry - y^2)}, \; b = -y,$$

for the elimination of x and y. Since the equation of the cycloid is a derivative, we must take those of a and b,

$$a' = \frac{2r-y}{y},\ b' = -y';$$

and dividing these values, and substituting $-b$ for y, we have

$$\frac{b'}{a'} = \frac{-yy'}{2r-y} = -\sqrt{\frac{y}{2r-y}} = -\sqrt{\frac{-b}{2r+b}}.$$

If now the positive ordinates b be taken in the contrary direction, there results $\frac{b'}{a'} = \sqrt{\frac{b}{2r-b}}$; which is precisely the equation of the cycloid itself, when the origin is in F. Thus the evolute LA of the cycloid is an equal cycloid; the arc AL is identical with FA'; and the vertex F is transferred to A.

IV. In the logarithmic spiral [fig.25], $r = a^\theta$; whence

$$R = r\sqrt{(1+l^2a)} = r \sec \eta = \frac{r}{\cos \eta},$$

the tangent of the angle $AMN = \eta$ made by the radius vector with the normal being $= la$ [N°. 726]. The projection of the radius of curvature MN on the radius vector is $= r$; so that the perpendicular AN, erected on this radius at the pole, meets the normal in the centre N of curvature. AM therefore is the sub-tangent of the evolute, AN its radius vector, and AM forms with the curve MI, at each point, the same angle β that AN makes with the evolute. Hence, the evolute is the same curve differently situated.

The theory of osculations might be similarly applied to curves of a higher order [See *Fonct. anal.*, N°. 117]; and it is evident that two curves which have a contact of the 2nd, 3rd, 4th... order, have the same tangent and the same osculating circle at this point.

736. The difference between the ordinates of the two curves being $\delta = Mh^m + Nh^{m-1} + \ldots$, and the sign of δ, when h is very small, being that of the first term Mh^m, it follows that, as Mh^m is positive or negative, the ordinate of the curve is greater or less than that of its osculate; and hence we shall be able to decide whether the first is above or below the other. Putting $-h$ for h, Mh^m will change its sign when m is odd, and the curve will be cut by its osculate at the common point. It appears therefore that *a curve is always cut by its osculating circle.*

ASYMPTOTES.

737. If the development of $f(x+h)$ be faulty, no osculation can then be established, unless the series for $F(x+h)$ proceed according to the same law, within the range at least of the initial terms which it is requisite to compare. This condition depends on the nature of the functions fx and Fx, and can only subsist accidentally, *i. e.* for some particular values of x; when it does subsist, the same reasoning that has been already employed will prove that the first terms must be equated in order that osculation may take place [See *Fonct. analyt.*, N°. 120].

Let $y=fx$, $y=Fx$ be the equations of two curves; and suppose that fx and Fx have been developed in series of descending powers of x [see p. 275], so that each of these functions may appear under the form

$$Ax^a + Bx^{a-b} + \ldots + Mx^{-m} + Nx^{-m-n} + \ldots$$

If then the exponents of these two developments be the same as far as a certain term Mx^{-m}, and some constants can be so disposed of that the first coefficients may also be rendered equal without introducing imaginary quantities, the difference between any two ordinates will be $M'x^{-m}+\ldots$ Hence it follows that, as x increases, one of our curves will go on continually approaching the other, though without ever reaching it; and there will be a term, beyond which no other curve, that does not also fulfil these conditions, will be able to make a nearer approach. Our curves will therefore be *Asymptotes* one of the other.

Thus, *when a curve is indefinite in extent, it has an infinite number of asymptotes*, which are found by developing $y=fx$ in a descending series, and taking for the ordinate of the line required the sum of the first terms, as far as some rank the exponent of which is negative; or, rather, composing a function Fx, the development of which commences with these same first terms.

I. For example, for the hyperbola [N°. 416]

$$y = \pm\frac{b}{a}\sqrt{(x^2-a^2)} = \pm\frac{bx}{a} \mp \tfrac{1}{2}bax^{-1} + \ldots$$

Consequently the straight lines which have for their equations $y=\pm\dfrac{bx}{a}$, are the rectilinear asymptotes, and they alone possess this property.

It is the same with $x=0$ and $y=0$, for $xy=m^2$.

II. The curve, the equation of which is $y = \frac{k}{\sqrt{(x^2 - a^2)}}$, is composed of four branches symmetrical in respect to the axes, and we shall shortly be able to determine their figure. We have [N°. 135]

$$y = kx^{-1} + \&c., \text{ or } x = a + \tfrac{1}{2} \cdot \frac{k^2}{a} y^{-2} + \ldots,$$

as we form the development according to the powers of x or y. And hence, the straight lines, which have for their equations $y = 0$ and $x = a$, are asymptotes. The hyperbola which has the axes of x and y for its asymptotes, and k for its power, is also an asymptote to our curve; the degree of approach is in this case much greater.

III. Let $y^3 - 3axy + x^3 = 0$, fig. 27 [N°. 708]: we have

$$y = -x - a + \tfrac{1}{3}a^3x^{-2} - \tfrac{1}{3}a^4x^{-3} \ldots;$$

and the straight line $y = -x - a$ is therefore an asymptote; it is constructed by taking $AB = AC = a$, and drawing BC.

IV. Lastly, let $y^4 - 2x^2y^2 - x^4 + 2axy^2 - 5ax^3 = 0$; then

$$y = \pm px \pm \frac{a(3\sqrt{2} - 4)}{8p} + Ax^{-1} + \ldots,$$

p denoting $\sqrt{(1 \pm \sqrt{2})}$. And hence, constructing the straight lines GF, GH [fig. 28], which have these two first terms for ordinates, we shall have the rectilinear asymptotes of the curve proposed.

MULTIPLE AND CONJUGATE POINTS.

738. When different branches of a curve pass through any the same point, either as cutting, or as touching each other, this point is called *double, triple,... multiple,* accordingly as it is common to two, three... or several branches. The equation of a curve being given, let it be proposed to determine these points, if there be any, and their nature.

Let $$V = 0, \; My' + N = 0,$$

be the equation in x and y of the curve, and its derivative: V being supposed to be clear of radicals.

1st Case. If the branches of the curve cut each in the point in question, there will be several tangents at this point: thus, for one particular value of x, and for that of y which corresponds to it, y' must have as

many values as there are branches. But, it has been seen [N°. 700] that this condition requires M and N to be each nothing.

2nd Case. If the branches of the curve touch each other, there is but one value of y'; and when the contact is of the $(n-1)$th order, there is likewise only one value [N°. 731] of $y', y'' \ldots y^{(n-1)}$; but there must be several for $y^{(n)}$. Now, the derivative equation of the order n has the form $My^{(n)} + \ldots = 0$, M being here the same coefficient [N°. 686] as for $y', y'' \ldots$, in the successive derivatives; and since this equation is of the 1st degree, and clear of radicals, it cannot give several values of $y^{(n)}$ for a single one of x and of y: we therefore must again have $M = 0$, and consequently $N = 0$, by the same reasoning as that of N°. 700.

Hence we shall conclude that, *to find the multiple points of a curve, we must equate to zero the derivatives M and N of its equation $V = 0$, taken successively in respect to x and to y. Then, eliminating x and y between two of the equations*

$$M = 0, N = 0, V = 0 \ldots (1);$$

the real values that satisfy the 3rd, and they alone, may belong to multiple points.

We say *may belong;* for, as we shall see, it is possible that these points may not exist, although the equations be established.

Passing on to the derivative of the 2nd order [N°. 686], $My'' + Py'^2 + \&c. = 0$; if we take one of the pairs of values of x and y that we suppose to be just found, and substitute them here, y'' will disappear, and y' will be given by an equation of the 2nd degree. Should the roots of this equation be real, there will be a *double point;* the two tangents to the branches will be determined by these values of y', and they will give the directions of the curves at this point.

739. But if the roots be imaginary, there will be a point without a tangent, and which is consequently altogether isolated from the branches of the curve; and this we denominate a *conjugate point.* If, in fact, there be such a point for the abscissa a, the ordinates immediately adjoining must be imaginary; and supposing the equation $V = 0$ to be put under the form $y = fx$, if we substitute $a \pm h$ for x, the corresponding value of y, or $f(a \pm h)$, must be imaginary, taking h very small. Let $y^{(n)}$ be the 1st coefficient that is imaginary in this series; since then the equation $My^{(n)} + \&c. = 0$ cannot present $y^{(n)}$ under this form, seeing that it does not contain any radicals, either in its original form or after the elimination of $y', y'' \ldots y^{(n-1)}$, it follows that we must have $M = 0$, and consequently $N = 0$.

Thus, the conjugate points are comprised among those given by the equations (1); but they have this distinction that the curve cannot

have a tangent at such points: y' must be imaginary, x and y being real.

740. It may happen that the terms of the derivative of the 2nd order all disappear; in which case recourse must be had to the derivative of the 3rd order, whence y''' and y'' will vanish, and it will contain y' in the 3rd degree. There will be a *triple point* if the three roots be real, and no multiple point will exist in the contrary case.

When we are obliged to have recourse to the equation of the 4th order, in which y' is of the 4th degree, the curve has a *quadruple, double or conjugate point*, accordingly as the four roots are real, or two imaginary, or, lastly, not one be real; and so on.

741. We annex some examples.

I. Let $ay^3 - x^3y - bx^3 = 0$; then

$1^\circ \ldots\ (3ay^2 - x^3)\,y' - 3x^2(y + b) = 0,$

$2^\circ \ldots\ 6ayy'^2 - 6x^2y' - 6x(y + b) = 0,$

$3^\circ \ldots\ 6ay'^3 - 18xy' - 6y - 6b = 0;$

omitting the terms in y'', $y'''\ldots$, which would disappear in the course of the calculation. From

$$3ay^2 - x^3 = 0,\ x(y + b) = 0,$$

we derive $y = -b$, $x = \sqrt[3]{(3ab^2)}$, which do not satisfy the proposed equation; and $x = 0$, $y = 0$, which do: the origin therefore may be a multiple point. But the terms of the derivative of the 2nd order all disappear on this supposition; whilst that of the 3rd becomes $ay'^3 = b$, which gives only a single real root for y'; and our curve therefore has no multiple point.

II. Take $y^4 - x^5 + x^4 + 3x^2y^2 = 0$, whence

$$2yy'(2y^2 + 3x^2) + 4x^3 - 5x^4 + 6y^2x = 0.$$

Assuming

$$y(2y^2 + 3x^2) = 0,\ x(4x^2 - 5x^3 + 6y^2) = 0,$$

we find that $x = 0$ and $y = 0$ are the only values that can fulfil these conditions and also satisfy the equation proposed. On this supposition the derivatives of the 2nd and 3rd orders vanish of themselves; that of the 4th order becomes $y'^4 + 3y'^2 + 1 = 0$, the roots of which are imaginary; and thus the origin is a conjugate point.

III. For $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$ [fig. 29], we have

$$-6a(y+a)yy' + 4x(x^2 - a^2) = 0,$$

$$-6a(2y+a)y'^2 + 12x^2 - 4a^2 = 0.$$

We shall now give a mode of determining the figure of the curve, which, it may be observed, is symmetrical in respect to the axis of y, since x enters into the proposed equation only in even powers. Making

$$(y+a)y = 0,\ x(x^2 - a^2) = 0;$$

and combining these with the equation proposed, it will be found that there are three multiple points, viz.

at D and D', where $y = 0$ and $x = \pm a$,

and at E, where $x = 0$ and $y = -a$.

These points are double; and the tangents Ec, Ef, Da, Db make, with Ax, angles the tangents of which are $y' = \sqrt{\tfrac{2}{3}}$ for the point E, and $y' = \sqrt{\tfrac{4}{3}}$ for D and D'.

For the points at which the tangent is parallel to x, we must make y' nothing, or $x(x^2 - a^2) = 0$: in the first place, $x = 0$ corresponds to $y = -a$, whence we again have the point E, for which y is $\tfrac{0}{0}$, a value which in the present case is not $= 0$; we also find the *maximum* at F, $y = \tfrac{1}{3}a$: secondly, $x = \pm a$ gives, besides the points D and D', the *minima* O and H for which $y = -\tfrac{2}{3}a$. Lastly, $y' = \infty$, or $y(y+a) = 0$ makes known the points I and G, at which the curve has its tangent parallel to y; we find $AB = AC = DE$.

IV. Let $x^4 + 2ax^2y - ay^3 = 0$ [fig. 30]; whence

$$ay'(2x^2 - 3y^2) + 4x(x^2 + ay) = 0.$$

Having found that the origin alone can be a multiple point, we are led to the derivative of the 3rd order, which gives $y' = 0$ and $y' = \pm\sqrt{2}$. Thus, there is a triple point at A: the curve has for tangents the axis of x and the lines Ab, Ac inclined at 45°.

We have the *minima* H and O by making $y' = 0$, or $x(x^2 + ay) = 0$, whence $y = -a$ and $x = \pm a$.

Lastly, the limits G and F are found by assuming $y' = \infty$, or $2x^2 = 3y^2$; whence

$$x = \pm \tfrac{4}{9}a\sqrt{6} \text{ and } y = -\tfrac{8}{9}a.$$

V. The equation $y^4 - axy^2 + x^4 = 0$ [fig. 31] gives

1°... $2yy'(2y^2 - ax) + 4x^3 - ay^2 = 0$,

2°... $2(6y^2 - ax)y'^2 - 4ayy' + 12x^2 = 0$,

3°... $24yy'^3 - 6ay'^2 + 24x = 0$.

It appears that the origin is a triple point; and since we have $y' = 0$ and $y' = \infty$, the axes are tangents to the curve.

VI. We may take for additional practice the equation

$$y^4 + x^4 - 3ay^3 + 2bx^2y = 0;$$

the curve [fig. 32] corresponding to which has also a triple point at the origin [See too the ex. IV, p. 307, fig. 28].

742. When the equation is explicit, the investigation of the multiple points is much easier. It has been seen [p. 263] that the corresponding abscissa must remove a radical from the value of y, by rendering its coefficient $= 0$. The degree of this radical depends on the number of the branches, and the exponent of the coefficient determines whether there is a simple intersection or an osculation.

The equation $y = (1 - x)\sqrt{(2 - x)}$ gives $y' = \dfrac{3x-5}{2\sqrt{(2-x)}}$.

For $x = 1$, y loses a radical, which does not disappear from y'. Thus, the origin being at I [fig. 33], $IC = 1$ gives a double point at C, in which the branches cut each other at right angles, since $y' = \pm 1$. Likewise, $x = \frac{5}{3}$ gives the *maxima* towards D and D'; and $IA = 2$ fixes the limit A of the curve.

For the equation $y = (2 - x)\sqrt{(1 - x)}$, the curve has a conjugate point the abscissa of which is $x = 2$, since y is imaginary for the adjoining points. The origin is in like manner a conjugate point for the curve, the equation of which is $y = x\sqrt{(x - b)}$.

Lastly, $y = (x - a)^2\sqrt{(x - b)} + c$, where $a > b$, is the equation of the curve $EDFG$[fig. 34] composed of two branches which have at D the same tangent ED. Had $x - a$ appeared in the cube, the two branches would also have had the same osculating circle, &c.... To conclude, a triple, quadruple... point is announced by a radical of the 3rd, 4th... degree.

A circle being described on the diameter $AI = 2r$ [fig. 33], a straight line AF revolves about A, whilst PN, perpendicular to AI, moves parallel to itself: required the curve AMC of the points M of section of these two straight lines, supposing the point N to be always in the middle of the arc ANF subtended by AF. The origin being in C, the equations of the moveable straight lines PN, AF are $x = \alpha$, $y = \beta(x - r)$; the co-ordinates of the point M are $CP = \alpha$, $PM = \beta(\alpha - r)$; and since PN is an ordinate to the circle, we have $PN^2 = r^2 - \alpha^2$. But, N being the middle point of the arc ANF, the radius CN is perpendicular to AF, and the triangles APM, CPN are similar; whence

$$\frac{AP}{PM} = \frac{PN}{PC}, \frac{r-\alpha}{\beta(\alpha-r)} = \frac{\sqrt{(r^2-\alpha^2)}}{\alpha} = -\frac{1}{\beta};$$

and this is the equation of condition between the constants α and β [N°. 462]: eliminating them by means of the equations $x = \alpha$ and $y = \beta(x - r)$, there results, for the equation of the curve proposed

$$y = \pm x\sqrt{\left(\frac{r-x}{r+x}\right)}; \text{ whence } y' = \frac{r^2 - x^2 - rx}{(r+x)\sqrt{(r^2-x^2)}}.$$

We shall easily recognise the fig. 33. The origin C has a double point, for which $y' = \pm 1$; and the tangents are there inclined at 45° to AI. The *leaf* AC has a *maximum* towards D, and does not extend beyond the vertex A, which is a limit. In the same manner that the point M is given by the middle point N of the arc ANF, the middle N' of the arc $AN'F$ gives M'; and we thus have two infinite branches CO, CO'; the points O, O' of section with the circle having for their abscissæ $x = -\frac{1}{2}r$. These branches have, for asymptotes, the tangent of the circle at the point I.

CONCAVITY, CONVEXITY AND SINGULAR POINTS OF CURVES.

743. The different positions of the tangent may be made use of for investigating the figure of the curve [406, 411]. The equation $y = fx$, and the tangent at the point (x, y) being given, let us compare the ordinates for the same abscissa $x + h$ [N°. 722] fig. 22:

$$P'H = y + y'h,\ f(x+h) = P'M' = y + y'h + \tfrac{1}{2}y''h^2 + \ldots$$

Since h may be taken so small that the sign of $\frac{1}{2}y''h^2$ shall be that of the rest of the series, the ordinate of the curve will be greater or less than that of the tangent, accordingly as y'' is positive or negative; so that the curve is convex to the axis of x in the 1st case, and concave in the 2nd. If the ordinates were negative, the contrary would obviously be the case; and hence *a curve is convex or concave to the axis of x, accordingly as y and y'' have the same or different signs.*

It is easy to see that at the point of *inflexion* M [fig. 39 and 40], where the curve from being concave becomes convex, y'' must also change its sign, which requires that at this point y'' be nothing or infinite; unless however y change its sign at the same time with y'', in which case the point under consideration will be situated in the axis of x.

We shall proceed to the exposition of this fact.

744. Having taken a point (α, β) in our curve, to determine whether it presents any peculiarity, *i. e.* whether it is *Singular*, we must compare the parts of the curve on each side of this point, or the ordinates $f(\alpha \pm h)$. We shall distinguish two cases.

1st Case. *The development of $f(\alpha + h)$ not containing for h any fractional exponent the denominator of which is even:* we have

$$f(\alpha + h) = \beta + Ah^a + Bh^b + \ldots;$$

in which the coefficients are real, since, were they imaginary, the point (α, β) would be conjugate [N°.739]. Moreover (whatever be the sign of h) h^a, h^b... are real, so that the curve extends on each side of the point (α, β).

1°. If the development of $f(\alpha + h)$ be faulty in the second term Ah^a, or if a be a fraction > 0 and < 1, y' is infinite [N°. 696], and at the point (α, β) the tangent is perpendicular to the axis of x. Taking the derivatives relative to h, we have

$$f'(\alpha + h) = aAh^{a-1} + \ldots, f''(\alpha + h) = a(a-1)Ah^{a-2}\ldots;$$

and we shall fix on this value of $f'(\alpha + h)$ to give us the direction of the tangent at the point of the curve the abscissa of which is $\alpha + h$, since it is indifferent whether x or h be supposed to vary in $f(x + h)$. [See the note, p. 267].

This being premised, the sign of Ah^a and its derivatives decides that of the whole series, when h is very small. Let a be a fraction $\frac{m}{n}$, in which n is odd: if m be so also, the ordinate $f(\alpha + h)$ increases on one side and decreases on the other of the ordinate of the tangent, since $A\sqrt[n]{h^m}$ changes its sign along with h. There is consequently an inflexion, disposed after the manner of fig. 35 or 36, accordingly as A is positive or negative.

And, in fact, $f''(\alpha + h)$ also changes its sign at the same time with h, because $a - 2$ gives to h, in the 1st term of the series, an odd exponent $m - 2n$: thus, the curve is on one side concave, and on the other convex to the axis of x [N°. 743]. We have constructed the equations:

$$y = \beta + (x - \alpha)^{\frac{3}{5}}\ldots \text{[fig. 35]},$$

$$y = \beta - (x - \alpha)^{\frac{3}{5}}\ldots \text{[fig. 36]}.$$

The same conclusions may be drawn for $y^3 = a^2x$ and $(y-1)^3 = 1 - x$.

But if m be even, $A \sqrt[n]{h^m}$ has always the same sign as A, whatever be that of h, so that the ordinates, contiguous to our tangent on each side, increase when A is positive, and decrease in the contrary case, in almost the same manner as for the *maxima*: the curve takes the form indicated by fig. 37 and 38, to which we shall give the name of *Ceratoid.** The sign of $f''(\alpha + h)$ is obviously negative for the one, and positive for the other, so that the curve must present to the axis of x, on each side of the ordinate of the tangent, its concavity or its convexity, accordingly as A has the sign $+$ or the sign $-$.

The equations $y = \beta + (x - \alpha)^{\frac{2}{3}}$ and $y = \beta - (x - \alpha)^{\frac{2}{3}}$ give the fig. 37 and 38; and we have another example also in the Cycloid.

2°. But if the development be not faulty in the two first terms, then $a = 1$, $b > 1$, y' is no longer infinite, and we have A for the tangent of the angle made with the axis of x by the straight line which touches the curve at the point (α, β): this line is parallel to x, if $A = 0$; inclined at 45°, if $A = 1$, &c.

$$f(\alpha + h) = \beta + Ah + Bh^b + \ldots$$
$$f'(\alpha + h) = A + bBh^{b-1} + \ldots$$
$$f''(\alpha + h) = b(b-1)Bh^{b-2} + \ldots$$

If now the exponent b be an even number, or a fraction with an even numerator, the curve does not present any thing particular at the point b, (α, β), since it extends, on each side, above the tangent if B is positive, and below if B be negative; the difference between the ordinates of these two lines being $Bh^b +$ &c. It appears likewise that the sign of $f''(\alpha + h)$ is then the same as that of B.

This is the case for the equation $y = \beta + x^2 + (x - \alpha)^{\frac{4}{3}}$.

If, however, $A = 0$, there is a *maximum* or a *minimum* [See p. 289].

This occurs for $y = \beta + k(x - \alpha)^{\frac{4}{3}}$.

When b is an odd number, or a fraction of which the numerator m is odd, $b = \frac{m}{n}$; Bh^b, or $B \sqrt[n]{h^m}$, changes its sign at the same time with h, and the ordinates increase on one side, decrease on the other; $f''(\alpha + h)$ is likewise similarly circumstanced, since the exponent of its 1st term is also an odd number $b - 2$, or a fraction of which the

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* We have preferred the denominations of *Ceratoid* and *Ramphoid* to those of *rebroussement* (or *Cusps*) of the 1st and 2nd species under which these points are known. The terms are derived from Κέρας, *a horn*, Ῥάμφος, the *beak of a bird*, and Εἶδος, *form*.

numerator $m - 2n$ is odd: there consequently is an *inflexion* at the point (α, β), the disposition of which depends on the direction of the tangent, and the sign of B.

The following are some examples:

$$1^\circ.\ y = x + (x-\alpha)^3; \qquad 2^\circ.\ y = \tfrac{1}{2}x + (x-\alpha)^{\frac{1}{3}}\ \text{[fig. 39]};$$

$$3^\circ.\ y = x - (x-\alpha)^3; \qquad 4^\circ.\ y = -(x-\alpha)^{\frac{5}{3}}\ \text{[fig. 40]};$$

$$5^\circ.\ y = -x + (x-\alpha)^{\frac{7}{3}}\ \text{[fig. 43]}:$$

the tangent is inclined at 45° in the examples 1° and 3°; at 135° in the 5th; and is parallel to x in the 4th.

If b be integral (*i. e.* 3, 5, 7...), y'' is nothing. We may compare our theorem with that of the *maxima* [N°. 717]; and thence infer that no one of the roots of $y'' = 0$ can correspond to an inflexion, unless the 1st of the derivatives y''', y^{IV}..., which it does not render nothing, be of an odd order. If b be not integral, since it is > 1, y'' is nothing or infinite, accordingly as b is $>$ or < 2.

745. 2nd Case. *The development of $f(\alpha + h)$ containing a radical of an even degree:* one of the ordinates $f(\alpha + h)$ or $f(\alpha - h)$ is in this case imaginary; the other is double, in consequence of the even radical which introduces into it the sign $\pm$. Thus, the curve extends only on one side of the ordinate β, but it has two branches.

1°. If the development be faulty in the 2nd term, a lies between 0 and 1; and the ordinate β is a tangent. Suppose that $a = \frac{m}{n}$; then, n being even, the term $\pm A\sqrt[n]{h^m}$ shows that the point (α, β) is a *Limit* of the curve in the direction of x; the curve has the form NMQ or $N'MQ'$ [fig. 41], accordingly as h requires to be taken with the sign $+$ or $-$; one of the ordinates is $>$, the other $< \beta$ or PM: also, for the points contiguous to M, the one of the values of $f''(\alpha + h)$, is positive, the other negative; which proves that one of the branches NM is convex, and the other QM concave to the axis of x.

The equations $y = k + x \pm (x - \alpha)^{\frac{3}{4}}$ and $y = k + x \pm (\alpha - x)^{\frac{3}{4}}$ give, the one QMN, the other $Q'MN'$. We shall find several examples of this sort in N°. 741.

But if the even radical affect one of the terms posterior to Ah^a, for the ordinates contiguous to that which is a tangent, β is $< f(\alpha + h)$ when A is positive; and the contrary when A is negative; so that the branches of the curve have [fig. 42] the form QMN in one case, $Q'MN'$ in the other. And it likewise appears from $f''(\alpha + h)$ being under these

circumstances of a contrary sign to A, that the curve must assume this figure, which we shall call a *Ramphoid*.

This takes place for $y = \beta + k(x - \alpha)^{\frac{1}{2}} + l(x - \alpha)^{\frac{1}{4}}$.

When h must be negative in order that $f(\alpha + h)$ may be real, the curve lies on the left of the tangential ordinate PM.

2°. When the development is not faulty till after the 2nd term, $a = 1$ and the tangent to the curve at the point (α, β) will be easy of construction. If the term Bh^b involve the even radical, it has the form $\pm B\sqrt[2]{h^m}$; one of the branches is above the tangent, the other falls below it, since this line has for its ordinate $Y = y + Ah$; and there consequently is a *Ceratoid*. We have y'' nothing or infinite, accordingly as b is $>$ or < 2.

For the equation $y = \beta + x + (x - \alpha)^{\frac{3}{2}}$, [fig. 45], the tangent is inclined at 45°.

For $2y = -1 - x + 2x(1 - x)^{\frac{5}{2}}$, we have fig. 44.

But if the exponent with the even denominator be beyond Bh^b, the sign of B is sufficient to decide which is the greater, the ordinate of the curve, or that $\beta + Ah$ of the tangent. It appears therefore that there is a *Ramphoid*. We have [fig. 46] for the equation

$$y = \beta + x + ax^2 + b\sqrt{x^5} \ldots \text{ the curve } QMN,$$
$$y = \beta + x - ax^2 + b\sqrt{x^5} \ldots\ldots\ldots\ldots\ldots\ Q'MN'.$$

746. Hence we conclude that

1°. At the limits, as they are in the direction of x or of y, y' is nothing or infinity.

2°. At the inflexions and at the ceratoids, y'' is nothing or infinity.

3°. To find the singular points, we must take the derivative $My' + N = 0$ of the equation $\phi(x, y) = 0$ of the curve; make $M = 0$, or $N = 0$; and thence deduce, by means of $\phi(x, y) = 0$, the roots which *can alone* belong to the limits.

4°. Taking in like manner the derivative of the 2nd order, or that of $y' = -\frac{M}{N}$, which gives $y'' = \frac{Q}{N}$ (adopting the 1st rule of N°. 665), then assuming $Q = 0$, or $N = 0$; these equations make known the values x and y which belong to the points of inflexion or to ceratoids.

5°. We must then investigate the development of $f(x + h)$ for each of the values of x thus obtained, and so ascertain the course of the curve on each side of the point which they determine.

6°. The ramphoids and the ceratoids may be considered as multiple points, and be submitted to the same analysis; they have a common tangent to their two branches at the point of reflection.

7°. We may also, in the discussion of the different equations, avail ourselves of the development of y in a series of the ascending or descending [N°. 707] powers of x; we shall thus easily arrive at the limits of the curve, if it allow of any; and for the infinite branches, we shall obtain their asymptotes, curvilinear or rectilinear as the case may be, &c.

We annex some examples; and many others may be met with in the Treatise of *Cramer*.

$$y = x + \sqrt[4]{(x-1)}, \qquad y = x^2 + \sqrt{(x-2)} \quad \text{[fig. 41]}.$$
$$y = x + \sqrt{(x-1)^2}, \qquad y = x^3 + \sqrt{x^3} \quad \text{[fig. 45]},$$
$$y = x^2 + \sqrt{(x-1)^5}, \qquad y = ax^2 + \sqrt{x^5} \quad \text{[fig. 46]},$$
$$y = \sqrt[3]{x^8} + ax, \qquad y = \sqrt[3]{(x-a)^{10}} + x \quad \text{[fig. 24]},$$
$$y = \sqrt[3]{(x-1)^2} \text{ [fig. 37]}, \qquad y = \beta - \sqrt[5]{x^2} \quad \text{[fig. 38]}.$$
$$y = x^2 + \sqrt[3]{(x-1)^5} \text{ [fig. 39]}, \qquad y = x^3 + x^2 - \sqrt[5]{x^7} \quad \text{[fig. 40]}.$$

SURFACES AND CURVES IN SPACE.

747. Let $z = f(x, y)$, $Z = F(X, Y)$ be the equations of two curve surfaces; that they may have a common point (x, y, z), it is necessary that, for the same ordinates $z = Z$, we have $x = X$, $y = Y$. Take in each surface another point corresponding to the abscissæ $x + h$ and $y + k$; and, for conciseness, represent the corresponding values of z [N°. 703] by

$$\begin{array}{lll} z + ph + \frac{1}{2}rh^2 + \ldots & \qquad & Z + Ph + \frac{1}{2}Rh^2 + \ldots \\ \quad + qk + shk + \ldots & & \quad + Qk + Shk + \ldots \\ \qquad + \frac{1}{2}tk^2 + \ldots & & \qquad + \frac{1}{2}Tk^2 + \ldots \end{array}$$

The distance between the two points in question will be

$$(P-p)h + (Q-q)k + \tfrac{1}{2}(R-r)h^2 + \ldots$$

If $P = p$ and $Q = q$, *i. e.* if the partial differentials of the 1st order of our functions f and F be respectively equal, the reasoning of N°. 730 will show that a third surface cannot approach the first ones so nearly as they approach each other, unless it fulfil the same conditions in respect to them; and there is then a *contact of the* 1*st order*.

For the contact of the 2nd order, it is moreover requisite that the partial differentials of the 2nd order be also equal to each other, or

$$R = r,\ S = s,\ T = t.$$

For example, every plane has for its equation [N°. 619] $Z = AX + BY + C$; its position depending on the constants A, B, C. These constants may be determined by establishing an osculation of the 1st order; and x, y, z being the co-ordinates of the point of contact, there results

$$z = Ax + By + C,\ p = A,\ q = B;$$

p and q always denoting the functions $\frac{dz}{dx}, \frac{dz}{dy}$, derived from the equation $z = f(x,y)$ of the curve; which consequently will have for its derivative $dz = pdx + qdy$.

If now A, B, C be eliminated, we find, for the tangent plane,

$$Z - z = p(X-x) + q(Y-y) \ldots (A).$$

Having thus obtained the equation of the tangent plane, it will be easy to determine every thing which regards its position. For example, the cosine of the angle which it makes with the plane xy is $\varphi = \frac{1}{\sqrt{(1 + p^2 + q^2)}}$.

Also, the normal which passes through the point (x, y, z) is perpendicular to the tangent plane; and these conditions, expressed analytically [N°. 628], give, for the equations of the normal,

$$X - x + p(Z-z) = 0,\ Y - y + q(Z-z) = 0 \ldots (B).$$

748. We shall give some examples of the use that may be made of the equations A and B.

I. All cylinders have this distinguishing property, that the plane which touches them in any point, touches them in a generatrix; and this straight line is parallel to another [N°. 620] of which are given the equations $x = az$, $y = bz$. Let this fact be expressed analytically, and we shall have it specified that the surface touched is a cylinder, without the curve that serves for the directrix being particularized; and we shall therefore have the equation for every species of cylinder. The condition of the parallelism of a plane and a straight line has been given in N°. 627; it becomes in the present instance (where $A = p$, $B = q$), $ap + bq = 1$, which is the equation required [See p. 273].

II. The tangent plane to the cone passes through the vertex. In the equation (A), substitute for X, Y, Z the co-ordinates a, b, c of this point, and the equation $z - c = p(x-a) + q(y-b)$, which expresses the property that characterizes every conical surface, whatever be its base, will be the equation of that species of surfaces [N°. 705].

III. Suppose that a straight line constantly cuts the axis of z and continues horizontal, whilst at the same time it slides along some given curve: it generates a surface called the *Conoid*, on account of its resemblance to a cone the vertex of which should be edged. What characterises these surfaces is, that a plane touches them in a horizontal generatrix; and this property we must express analytically. Making $Z = z$ in the equation (A), we have $(X-x)p + (Y-y)q = 0$; and these are the equations of a horizontal line drawn in the tangent plane. That this straight line may also cut the axis of z, it is necessary that its projection on the plane xy pass through the origin, or that $px + qy = 0$; and this is the equation of the whole species of conoids.

IV. A normal to any surface of revolution always in its course cuts the axis; and if, therefore, X, Y, Z, be eliminated between the equations (B) of the normal, and those of the axis of revolution, the resulting equation in x, y, z, which expresses the property specified, will be that of the surface of revolution, whatever be its meridian. For example, if the axis be that of z, the equations of which are $X = 0$, $Y = 0$, elimination gives $py = qx$, the equation of every surface of revolution about the axis of z [N°. 705].

When we wish to particularize the species of a cylindrical or conical surface..., we must introduce, for p and q, such functions of x and y as are determined by the nature of the curve that is given as the directrix. This subject will be farther examined subsequently [See N^os^. 879 and 880].

749. We have treated of the *maxima* of functions of two variables in N°. 720. It thence follows that if we wish to obtain the maxima or minima values of z for a curve surface, of which we have the equation $z = f(x, y)$, we must put $p = 0$, $q = 0$ (the tangent plane being consequently parallel to xy), and eliminate x, y, z between these three equations; but the co-ordinates thus obtained will not belong to points possessed of the property in question, unless they satisfy the condition (2) of p. 290, which will give us the means of distinguishing the *maximum* from the *minimum*.

750. That the tangent plane may be perpendicular to the plane yz,

its equation must reduce itself to the form $Z - z = q\,(Y - y)$ [N°. 615]; and consequently, $p = 0$. More generally, let

$$Pdx + Qdy + Rdz = 0$$

be the differential of the equation of a surface [N°. 704]; $P = 0$ is the condition which expresses that the tangent plane is perpendicular to the plane yz; and it follows therefore that the co-ordinates x, y, z of the point of contact must satisfy the equation $P = 0$, and that $\varphi\,(x, y, z) = 0$ of the surface. These two equations consequently are those of the curve which possesses the property that the tangent plane at any point of it shall be perpendicular to the plane yz; and this curve is the limit of the surface in the direction of yz. Thus, by eliminating x, we have the projection of the surface on the plane of yz. Similarly, that on the plane xy is found by eliminating z between $\varphi = 0$ and $R = 0$; the two equations $P = 0$, $Q = 0$ correspond to the maximum of z, &c.

For the sphere, for instance [N°. 614],

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2;$$

the derivative relative to z alone is $z - c = 0$; and eliminating z, we have $(x - a)^2 + (y - b)^2 = r^2$, for the equation of the circle of projection on the plane xy; as is otherwise evident.

751. The arc s of a curve in space being projected on the plane xy, let the cylinder formed by the system of the perpendiculars be developed [N°. 287, 4°]; the base will be an arc λ, the projection of the arc s. Now, we may conceive this arc s to be referred to the rectangular co-ordinates λ and z, since λ is extended in a straight line; and the length of the arc s and the area t of the cylinder will be given [N^{os}. 727 and 728] by the relations $s'^2 = 1 + z'^2$, $t' = z$, in which the derivatives are in reference to λ. If they be required to be relative to x, we shall have [N°. 694]

$$dt = zd\lambda,\ ds^2 = d\lambda^2 + dz^2.$$

But the arc λ is referred to the variables of the plane xy, so that $d\lambda^2 = dx^2 + dy^2$; and consequently

$$dt = z\sqrt{(dx^2 + dy^2)},$$

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Now, a curve in space is given by the equations of two surfaces of which it is the intersection, as $M = 0$, $F = 0$; from these deduce the differentials dy and dz in functions of x, substitute them in the preceding formulæ, and integration will then give, in the first place, the area t of

the right cylinder, which has the projection of the arc for its base, and is terminated by this arc; and, in the second place, the length of the arc when rectified.

752. Supposing the curvilinear trapezium $CBMP$ [fig. 22] to revolve about the axis Ax, let us investigate the volume v and the surface u of the solid of revolution thus generated; the equation of the arc BM being given, $y = fx$. Let $v = Fx$, $u = \phi x$; the point is to determine the functions F and ϕ. Assigning to x the increment $PP' = h$, let y, v and u become $y + k$, $v + i$, $u + l$; then

$$k = y'h + \ldots, \; i = v'h + \ldots, \; l = u'h + \ldots;$$

and we must now, in order to apply the method of limits, find magnitudes between which the increments i and l are comprised, however small h be.

1°. The rectangles $MPP'Q$, LP' generate, in their revolution about Ax, cylinders of which the volumes are $\pi y^2 h$ and $\pi (y + k)^2 h$ [N°. 308]; and the ratio of these having unity for the limit, whilst the volume i, generated by the area $MM'P'P$, is always intermediate to them, it follows that unity must also be the limit of the ratio $\dfrac{i}{\pi y^2 h}$ or $\dfrac{v' + \&c.}{\pi y^2}$; and hence

$$v' = \pi y^2.$$

2°. The chord MM' and the tangent MH describe truncated cones, the surfaces of which [N°. 290, 3°] are $\pi (2y + k).MM'$, and $\pi (2y + y'h).HM$: the ratio of MM' to MH tends continually to unity [N°. 737]; and the limit of the ratio of our two surfaces is therefore 1, which is consequently that of the ratio

$$\frac{\pi (2y + k)\, MM'}{l} = \frac{\pi (2y + k).\sqrt{(1 + y'^2 + y'y''h\ldots)}}{u' + \frac{1}{2} u''h + \ldots};$$

since the area l described by the arc MM' is intermediate to the two first, however small h be. Hence

$$u' = 2\pi y \sqrt{(1 + y'^2)} = 2\pi y s'.$$

We must substitute fx for y in these values of v' and u', and integrate; *i. e.* trace back v' and u' to the functions v and u of which they are the derivatives [N°. 811].

753. In a plane APB [fig. 47] let a trapezium $CDEF$ be described; and let $cdef$ be its projection on another plane AQB, and α the angle of these two planes: supposing that the sides CD, EF are

perpendicular to the intersection AB, we have [N°. 354] $cd = CD \times \cos\alpha$, $ef = EF \times \cos\alpha$; and hence the area of the trapezium

$$cdef = \tfrac{1}{2} GH \times (CD + EF)\cos\alpha = CDEF \times \cos\alpha.$$

This relation between our trapezium and its projection equally holds for any triangle DIF [fig. 48], since, by drawing the perpendiculars CD, EF to AB, and CE parallel to DF, we can form the parallelogram $CDEF$, the area of which is double of that of the triangle DIF. But, in the first place, every rectilinear figure is decomposable into triangles; and, in the second, by the method of limits, the same proposition may also be extended to every plane curvilinear area.

Hence, *the projection P of any plane area A on another plane is the product of this area by the cosine of the angle of the two planes*, $P = A\cos\alpha$.

Let therefore α, α', α'' be the angles which a plane area A makes with the co-ordinate planes; P, P', P'' its three projections; we have then

$$P = A\cos\alpha,\ P' = A\cos\alpha',\ P'' = A\cos\alpha'';$$

and taking the sum of the squares, there results, since $\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1$ [N°. 634, 1°],

$$A^2 = P^2 + P'^2 + P''^2.$$

Hence, *the square of any plane area is the sum of the squares of its three projections on the rectangular co-ordinate planes.*

These theorems serve for finding the extent of plane surfaces situated in space, by reducing them to others that are expressible in terms of two variables.

754. Let $z = f(x, y)$ be the equation of a curve surface; and four planes being drawn parallel, two and two, to those of xz and yz, let us investigate the volume V and the surface U of the solid $MNEF$ [fig. 49] included between these limits. Assigning to x and y the increments h and k, instead of the point $M(x,y,z)$ we shall have a corresponding point C, and the body will receive the increment of volume contained between the planes ME, SD, FM, SB; so that U and V are functions of x and y which we must now determine. When x is increased by h and y by k, V will be increased by [N°. 703].

$$\frac{dV}{dx} \quad + \frac{dV}{dy}k + \frac{d^2V}{dx^2}\cdot\frac{h^2}{2} + \frac{d^2V}{dxdy}hk + \frac{d^2V}{dy^2}\cdot\frac{k^2}{2} + \dots;$$

but supposing that we had confined ourselves to giving to x the incre-

ment h, or that of k to y, the body would have received the respective augmentations

$$MPRBF = \frac{dV}{dx}h + \frac{d^2V}{dx^2}\cdot\frac{h^2}{2} + \ldots,$$

$$PEDMQ = \frac{dV}{dy}k + \frac{d^2V}{dy^2}\cdot\frac{k^2}{2} + \ldots;$$

and hence, subtracting, we have the volume $MCRQ = \dfrac{d^2V}{dxdy}hk + \ldots$

And it will in like manner be seen that the surface

$$MC = \frac{d^2U}{dxdy}hk + \ldots$$

To apply the method of limits to these cases, we must investigate some magnitudes between which the preceding volume and surface are always comprised, however small h be: the figure $MCRSQP$ is represented separately in fig. 50.

1°. The rectangular parallelipiped $MPSs$ has hkz for its volume; that of the parallelipiped constructed on the same base, and the height of which is $SC = z + l$, is $hk(z + l)$; and the ratio $\dfrac{z}{z+l}$ of these volumes having unity for its limit, 1 is also the limit of

$$hkz : \frac{d^2V}{dxdy}hk + \ldots; \text{ whence } \frac{d^2V}{dxdy} = z.$$

Having therefore substituted for z its value $f(x, y)$, we must first integrate relatively to x, considering y as constant; and then integrate the result in respect to y alone [See N°. 812].

2°. A tangent plane Ms' being drawn at the point $M(x, y, z)$, the surface $Mr's'q'$, which is contained between the planes MR, MQ, Qs', $s'R$, is [N°. 753] the quotient of its base $PQRS$ divided by the cosine of the angle which it makes with the plane xy, viz. [N°. 634, 1°].

$$hk : \frac{1}{\sqrt{(1 + p^2 + q^2)}} = hk\sqrt{(1 + p^2 + q^2)}.$$

But it is easily seen that unity is the limit of the ratio of $\dfrac{d^2U}{dxdy}hk + \ldots$ to this quantity; and consequently

$$\frac{d^2U}{dxdy} = \sqrt{(1 + p^2 + q^2)}.$$

We must therefore differentiate the equation $z = f(x, y)$ of the

surface; from $dz = pdx + qdy$ deduce the values of p and q in functions of x and y, substitute them in this expression; and then integrate in the double manner already stated. We shall give some applications of these different formulæ in N°. 815.

755. Applying to the case of three dimensions what has been said respecting the osculations of plane curves, if $z = f(x, y)$, $Z = F(X, Y)$ be the equations of two curve surfaces, and these surfaces have a common point (x, y, z), in order to estimate the degree of separation in the parts contiguous to this point, we must change X and x into $x + h$, Y and y into $y + k$, and take the difference δ of the ordinates z. Let it still be supposed that

$$\frac{dz}{dx} = p, \frac{dz}{dy} = q, \frac{d^2z}{dx^2} = r, \frac{d^2z}{dxdy} = s, \frac{d^2z}{dy^2} = t;$$

and also that P, Q... have similar significations for the 2nd surface. It may be demonstrated then, precisely as in N°. 730, that if we have $P = p$, $Q = q$, the difference δ being of the 2nd order in h and k, no other surface, which does not fulfil these same conditions, can approach the 1st surface so nearly as the 2nd is made to do; if, besides this, we have $R = r$, $S = s$, $T = t$, the osculation will be of the 2nd order, and the two surfaces will approximate still more closely to each other in the parts contiguous to the common point; and so on.

Suppose, for instance, that one of our surfaces is a plane $Z = AX + BY + C$; it will have a contact of the 1st order with the surface $z = f(x, y)$, if the constants A, B, C be determined on these conditions, that the plane pass through the given point (x, y, z), and that we have $A = p$, $B = q$. Hence results the equation (A) of the tangent plane [N°. 747].

For the sphere, we have the equation

$$(X - a)^2 + (Y - b)^2 + (Z - c)^2 = n^2.$$

We establish a simple contact at the point (x, y, z) by making [N°. 704]

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = n^2,$$

$$(x - a) + p(z - c) = 0, (y - b) + q(z - c) = 0;$$

these three equations determine the co-ordinates of the centre, and consequently the sphere, in the case of a simple contact, when the radius n is known. Assuming, for conciseness, that $(1 + p^2 + q^2)^{-\frac{1}{2}} = \varphi$, elimination gives

$$a = x + np\phi,\ b = y + nq\phi,\ c = z - n\phi \ldots (1);$$

this sphere has the same tangent plane as the surface, and its centre is on the normal, equ. (B) p. 318. That the osculation might be of the 2nd order, it would be requisite to determine the arbitrary quantity n in such a manner as to render $R = r$, $S = s$, $T = t$: but it is obvious that these three conditions cannot be fulfilled, and that it is not, generally, the case, therefore, that every surface has an osculating sphere as every curve has an osculating circle.

756. But let the sum of the terms of the 2nd order of the series [747] be rendered the same for the sphere and our surface, or

$$r + 2s\alpha + t\alpha^2 = R + 2S\alpha + T\alpha^2,$$

α being the ratio $k : h$; we find for the derivatives of the 2nd order of the equation of the sphere relative to x and y,

$$(z-c)R + 1 + p^2 = 0,\ (z-c)S + pq = 0,\ (z-c)T + 1 + q^2 = 0;$$

whence

$$(z-c)(r + 2s\alpha + t\alpha^2) + 1 + p^2 + 2pq\alpha + (1 + q^2)\alpha^2 = 0 \ldots (2).$$

p, q, r, s, t are some functions of x and y, to be deduced from the equation $z = f(x, y)$ of the surface proposed; α is the tangent of the angle made, with the axis of x, by a straight line which touches the surface at the common point, and is drawn in an arbitrary direction. This equation makes known $z - c$ in a function of x, y and α; the equations (1) then give a, b and the radius n of curvature of the section made by a plane passing through the normal and the tangent in question. And we can therefore determine the curvature of the surface in every imaginable direction.

Having regard to those sections especially the curvature of which is the greatest, make n vary in respect to α alone, and assume $n' = 0$ [N°. 717]. According then to the equation (1), we also have $c' = 0$, supposing z, p, q to be constant; and hence the derivative of the equation (2) relative to c and α, making $c' = 0$, gives

$$\left.\begin{array}{l} (z-c)(s + t\alpha) + pq + (1 + q^2)\alpha = 0 \\ \text{whence}\quad (z-c)s\alpha + pq\alpha + (z-c)r + 1 + p^2 = 0 \end{array}\right\} \ldots (3),$$

multiplying by α and subtracting the result from (2). It is easy to eliminate α between these two equations, and we thus arrive at this relation intended to give $z - c$,

$$A(z-c)^2 + B(z-c) + \phi^{-2} = 0 \ldots (4);$$

where

$$A = tr - s^2,\ B = r(1 + q^2) + t(1 + p^2) - 2pqs.$$

We hence obtain two values of $z - c$, and (1) then gives the radii n of the greatest and least curvature of the surface at the given point (x, y, z); finally, one of the equations (3) makes known α, or the directions of these two curvatures.

Suppose that *our two lines of curvature* are drawn on the proposed surface; they are independent of the system of axes to which the surface is referred, and remain constant when the co-ordinate planes are changed. Let the tangent plane therefore be taken for that of xy; it is evident that x, y, z, p and q are then nothing, and the equations (3) become

$$c(s + t\alpha) = \alpha,\ c(s\alpha + r) = 1;$$

whence $$s\alpha^2 + \alpha(r - t) - s = 0.$$

The product of the two roots of α being -1, we hence infer that the two curves cut each other at right angles. And consequently, with the exception of the very particular cases in which the equation (4) would be satisfied of itself, *in every surface, if we take any point, there are always two planes, passing through the normal at this point, and perpendicular to each other, which give the greatest and the least curvature of the surface.* The preceding equations make known these two directions, and likewise the radii of the two curvatures.

757. A curve being given in space, by the equations of two surfaces of which it is the intersection, if x and y be successively eliminated between these equations, the surfaces will be replaced by two cylinders perpendicular to the co-ordinate planes of xz and yz, and the resulting equations $z = fx$, $z = Fy$, will be those of the projections of the curve on the above planes. A tangent to the curve is so also to the cylinder, and its projections consequently are tangents to those of the curve; so that the equations of the tangent are

$$Z - z = p(X - x),\ Z - z = q(Y - y).$$

Let $f'x$ and $F'x$ be substituted for p and q, and these equations will be determined. Eliminating x, y, z between our four equations, we shall have a relation between X, Y, Z, which is the equation of the tangent at any point whatever of the curve, *i. e.* the equation of the surface generated by the motion of a straight line which is constantly a tangent. If this surface be a plane, the curve is plane, otherwise it is of *double curvature*; and these two cases, therefore, will be easily distinguished one from the other.

At the point of contact there is an infinite number of perpendiculars to the tangent; this concourse of normals determines the *normal plane*, of which it is easy to find the equation [N°. 628]

$$Z - z + \frac{X - x}{p} + \frac{Y - y}{q} = 0.$$

758. The theory of the contacts of surfaces might be applied to the curves of double curvature, we shall not, however, enter here on this subject [See *Fonct. analyt.* N°. 141, and *l. Anal. appl.* of Monge]; but shall confine ourselves to the investigation of the *osculating plane*. Let $z = fx$, $y = \psi x$ be the equations of the curve; that of the plane which passes through the point (x, y, z) is

$$Z - z = A(X - x) + B(Y - y).$$

Let A and B be determined by establishing a contact of the 2nd order. If x be changed into $x + h$, y and z will receive, for the curve, the increments

$$l = hf' + \tfrac{1}{2}h^2f''..., \quad k = h\psi' + \tfrac{1}{2}h^2\psi''...$$

Let therefore $x + h$, $y + k$, $z + l$ be substituted for X, Y, Z in the equation of the plane: there results $l = Ah + Bk$, or

$$hf' + \tfrac{1}{2}h^2f'' + ... = (A + B\psi')h + \tfrac{1}{2}Bh^2\psi'' + ...$$

The arbitrary quantities A and B will be determined by these two conditions $A + B\psi' = f'$, $B\psi'' = f''$; and consequently *the equation of the osculating plane* is

$$\psi''(Z - f) = (f'\psi'' - f''\psi')(X - x) + f''(Y - \psi).$$

ON THE INFINITESIMAL METHOD.

759. We have already remarked [Vol. I. note, p. 244], in applying the method of limits [N°. 113] to an equation between constants and variables which allow of being diminished *ad libitum*, that when nothing more is wanted than the relation which connects the constant terms, there will be no error committed by neglecting in the calculation such of the terms as we know must, from the nature itself of the process, eventually disappear. We have had an instance of this in the method of tangents [N°. 422]. The mathematical certainty, therefore, will not be affected by these voluntary omissions, so long as we feel assured

that they do really take effect on those quantities only which, from the very nature of the operation, must disappear from the result.

We may, therefore, in every question of this sort, omit the ***indefinitely small*** terms, which geometers have, along with Leibnitz, called ***infinitesimals.*** By allowing ourselves this liberty, the calculations will be greatly abridged, since it is frequently difficult to arrive at the value of these terms; whilst the results will be equally exact. We might in fact present the theory with all the rigour of geometry, by proving that the quantities omitted are of the rank of those which ought to be suppressed. This method is valuable, not only for fixing the results in the memory, but also for its use in complicated analytical speculations; and it is of importance not to lose so powerful an aid, especially when we consider that we can always give to the process that rigour of which it is in appearance deficient.

760. The applications of these principles to the elements of Geometry are so easy, that we shall dispense with the consideration of them; the reader will have no difficulty in supplying the deficiency himself. We shall ourselves proceed to the applications to the Differential Calculus.

Let $y, z, t\ldots$ be any given functions of x; if x take the increment dx, those accruing to $y, z\ldots$ will result from the given relations which connect these variables with x, and we shall have

$$dy = Adx + Bdx^2 + \ldots, \; dz = A'dx + B'dx^2 + \ldots$$

But, whatever be the object of our operation, dy may be combined with $dz, dt\ldots$, so as to form an equation $M = 0$; and when for $dy, dz\ldots$ we substitute their values above, dx will become a common factor, and as such may be suppressed in the equation $M = 0$; so that the first coefficients $A, A'\ldots$ will alone be exempt from it. But $x, y, z\ldots$ being now considered as fixed terms, their increments $dx, dy\ldots$ may be diminished *ad libitum*, so that making $dx = 0$, the equation $M = 0$ must lose all the terms $B, B'\ldots$ We may, therefore, in the first instance dismiss these terms from the calculation, and say that $dy = Adx$, $dz = A'dx\ldots$; the other terms being neglected as *infinitesimals of the 2nd order*, to use an expression by means of which circumlocution is avoided.

The quantities are conceived to be made up of certain elementary parts, which we call *Differentials*, and denote by the letter d, as we have stated in N°. 658. These differentials, compared with the actual elements, differ from them only by *quantities that may be neglected, i. e.* by values which, were we to take them into account, the calculation would cause to vanish. The result is not affected by this species of error, in thus taking defective quantities instead of the true ones, whilst

we find our calculations and considerations simplified and the operations remarkably abridged.

$dx, dy\ldots$, the differentials of x and y, are not precisely the increments of these variables, though we treat them as such, since, instead of taking $dy = Adx + Bdx^2\ldots$, we only take $dy = Adx$; they are quantities, however, which do not differ from these increments except by parts which destroy each other in the course of the calculation, and which it is unnecessary to consider.

A is the derivative which has been denoted by y', and which we already know how to determine for every function. It is very easy, also, to obtain it anew, from the principles that we have just been explaining. The following are some examples:

Let $y = zt$, z and t being functions of x; we have

$$dy = (z + dz)(t + dt) - zt = tdz + zdt,$$

neglecting $dz.dt$, which contains only $dx^2, dx^3\ldots$

For $y = z^m$, we have $dy = (z + dz)^m - z^m = mz^{m-1}dz$, neglecting the terms in $dz^2, dz^3\ldots$ [See N°. 668].

Let $y = a^z$; then $dy = a^{z+dz} - a^z = a^z(a^{dz} - 1)$; but [p. 158] we have $a^h = 1 + kh + \ldots$; and consequently $dy = ka^z dz$, suppressing the terms in $dz^2, dz^3\ldots$

$y = \text{Log}\, z$ gives

$$dy = \text{Log}\,(z + dz) - \text{Log}\, z = \text{Log}\left(1 + \frac{dz}{z}\right);$$

whence $a^{dy} = 1 + \frac{dz}{z}$; but, $a^{dy} = 1 + kdy$; and consequently $dy = \frac{dz}{kz}$.

761. The infinitesimal method consists, as we see, in substituting in the calculation, for the actual increments that are the object of it, other quantities the error in which is of such a nature as not to affect the result. Instead of the actual variations, which would be difficult of treatment, and would render the operations very complicated, we take other quantities more simple, and which are better suited to the investigations that we have in view, and the calculations that are to be made. But that we may be at liberty to make use of these defective values, we must, first of all, be assured that no error will result thence, and that if we were to add to them their deficits, these parts added together would destroy each other.

Thus, that the method may be employed with all certainty, one indispensable condition must be fulfilled, that of the *equality of the limits*, or *the ultimate ratios*, which consists in comparing the actual magnitudes with those that we substitute for them, making them vary together, and see-

ing whether, in their progressive diminution, their ratio tends unceasingly towards unity, for *unity must be the limit of this ratio*. If an arc of the curve BM [fig. 22] have for its increment the arc MM', we may take instead of it the chord MM'; and this chord will be the differential of the arc, since as the points M, M' are brought once towards the other, the arc and the chord diminish, and their ratio tends to unity which is the limit of it. But we could not take MQ for the differential of MM', on the pretext that MM' and MQ tend to an equality, and become each nothing at the same time; for the ratio $MM' : MQ$ has not 1 for its limit. Thus, ax^2 and bx, which become nothing together, have for their ratio $\frac{ax}{b}$, the limit of which is zero, and not 1.

In comparing a circular arc with its sine, the increment of the one may be taken for that of the other. Now, $y = \sin z$ gives

$$dy = \sin(z + dz) - \sin z = \sin z . \cos dz + \sin dz . \cos z - \sin z,$$

and replacing $\sin dz$ by dz and $\cos dz$ by 1, since the ratios of these magnitudes tend to unity, we find $dy = dz . \cos z$. In the same manner might be found the differential of $\cos x$, of the arc ($\tan = x$)...

A principle, of which we must never lose sight in considerations of this sort, is that of *homogeneousness* which consists in the differentials being of the same nature with the magnitude considered, and themselves all of the same order. For the differential of a solid therefore we can only take some other solid, for that of a surface some area, &c.; a line cannot be considered as the sum of an infinite number of points, or an area as the combination of a series of lines, &c.; and moreover, *any formula must contain only terms in which the differentials are all of the same order*.

This artifice, by which differentials are treated as though they were exact, gives rise, it is true, to defective equations; but we need be under no apprehensions on this head, as it is established that the ultimate result will not be affected, so long as we have only the limits in view, which are the same for the differentials and the actual elements.

This calculation appears at first in the light of a mode of approximation, since the quantities themselves are replaced by others that are near to them; but as the calculus itself is intended only for the determination of the ultimate ratios, which are the same for both, our process acquires all the rigour of Algebra; and the language, as also the notation, are equally exact, since when we make use of the words *infinitesimal* and *differential*, we intend to apply the calculation to such problems only as depend, not on the magnitudes themselves that we have in view, but on the values of their ultimate ratios. *A differential therefore is a part of*

the difference, a part the ratio of which to this difference has unity for the limit.

In the Integral Calculus, the object of which is to trace back the derivatives to their primitive functions, the integral is considered to be the sum of the elements or the differentials, as we shall have occasion to remark in N^{os}. 802, 806 and 812.

The application of these principles to Geometry and Mechanics occurs very frequently. The following are some examples of the first nature.

762. Let $BM = s$ [fig. 22] be a curvilinear arc, x and y the co-ordinates of M, and $y = fx$ the equation of the curve. We shall suppose the tangent TM to be the prolongation of the infinitesimal element MM' of the curve; which is equivalent to saying that the chord of the arc $MM' = ds$, since it may be made to approach *ad libitum* to MH, and the angle $M'MQ$, the tangent of which is $\frac{M'Q}{MQ}$, differs from HMQ only by an infinitesimal quantity. Hence, resolving the triangle MMQ, the sides of which are dx, dy and ds, we have, as in N°. 722,

$$\tan T = \frac{dy}{dx},\ \cos T = \frac{dx}{ds},\ \sin T = \frac{dy}{ds}.$$

Since the arc $MM' = ds$ and its chord have unity for the limit of their ratio, the arc ds may be substituted for its chord, and we have the length of the hypotenuse, or $ds = \surd(dx^2 + dy^2)$.

Let t be the area $CBMP$; the indefinitely small rectangle $MPP'Q = ydx$ may be taken for dt; and consequently $dt = ydx$.

763. To apply this method to polar co-ordinates, from the pole A [fig. 25] as centre, describe the arc MQ through the point $M(r, \theta)$; we shall have

$$\frac{MQ}{mq} = \frac{AM}{Am}, \text{ or } \frac{MQ}{d\theta} = \frac{r}{1}; \text{ and consequently } MQ = rd\theta.$$

Draw AT perpendicular to AM, and the tangent TM' which, for the element $MM' = ds$, will be coincident with the arc; then the similar triangles $MM'Q$, TMA give [see p. 294]

$$\frac{MQ}{M'Q} = \frac{AT}{AM}, \text{ or } \frac{rd\theta}{dr} = \frac{AT}{r};$$

whence
$$\text{sub-tan } AT = \frac{r^2 d\theta}{dr}.$$

In the rectangular triangle TMA, we have

$$\tan TMA = \frac{AT}{AM} = \frac{rd\theta}{dr}.$$

Also,

$$MM'^2 = MQ^2 + M'Q^2 \text{ becomes } ds^2 = r^2d\theta^2 + dr^2;$$

and lastly, the area $ABM = \tau$ comprised between two radii vectores has for its differential AMM' which may be considered as equal to AMQ: but $AMQ = \frac{1}{2} AM \times MQ$; whence $d\tau = \frac{1}{2} r^2 d\theta$ [p. 297].

764. In its revolution about Ax [fig. 22], $CBMP$ generates a body the volume of which we shall represent by v and the surface by u. But, the arc MM' describes the differential of u, which is a truncated cone, and $= \frac{1}{2} MM'(\text{cir.}\, PM + \text{cir.}\, P'M')$, or $= MM' \times \text{cir.}\, PM$; and consequently $du = 2\pi y ds$.

Similarly, the area $MPP'M'$ generates the differential of the volume v, which may be considered as equal to the cylinder described by $MPP'Q = PP' \times \text{circle } PM$; and therefore $dv = \pi y^2 dx$. This agrees with N°. 752.

Let BD [fig. 49] be a curve surface, of which we have given the equation $z = f(x, y)$. When we give to x the increment dx, the volume $V = EFMN$ will increase by $MBFR = \frac{dV}{dx} dx$; and if, in this result, y be augmented by dy, the volume MB will increase by $MCSP = \frac{d^2V}{dx\,dy} dx\,dy$. Similarly, the surface $MN = U$ is augmented by $MC = \frac{d^2U}{dx\,dy} dx\,dy$. This being premised:

1°. The plane $Mrsq$ [fig. 50], parallel to the plane xy, forms the parallelipiped $MPSs$ the volume of which is $zdxdy$; and consequently $d^2V = zdxdy$, a formula which is equivalent to that of N°. 754.

2°. The tangent plane $Mr's'q'$ may be supposed to be coincident with the surface for the extent of MC; and since [N°. 753] the base PS or $dydx$ is $= MC \times \cos\alpha$, α denoting the inclination of this plane to that of xy, we have

$$MC = \frac{dx\,dy}{\cos\alpha} = dx\,dy\,\sqrt{(1 + p^2 + q^2)} \text{ [p. 318]}.$$

and consequently

$$d^2U = dx\,dy\,\sqrt{(1 + p^2 + q^2)}.$$

765. Let $M=0$ be the equation of a surface in x, y, z and the arbitrary constants α and β. If α and β have some fixed values given to them, the surface will have all its points determined in space. But suppose that in the plane xy we trace at pleasure some curve, $y=\phi x$, and that this same relation be established between β and α, or $\beta=\phi\alpha$; β being then eliminated from M, we may assign to α a series of successive values; and $M=0$ will become the equation of a multitude of curve surfaces, differing from each other only as to the values of the constants α and β. The infinite series of these surfaces forms what is called an *Envelope*.

To consider the surface, which varies by the change of α, in two immediately contiguous situations, we must differentiate M in respect to α. $M=0$ and $M'=0$ particularize, for a given value of α, the curve of intersection, or rather of contact, of the two contiguous surfaces; and to this curve we give the appellation of *Characteristic*. Let α be eliminated between these two equations, and we shall have an equation in x, y, z, without either α or β, which will belong to this curve, whatever be the position of the moveable surface; and this therefore will be the equation of the Envelope.

Moreover, if, for any characteristic, determined by a particular value of α, α be made to vary in an infinitely small degree, M and M' becoming M' and M'', we shall have a second characteristic indefinitely near to the first. For the points common to both, we have the three equations $M=0$, $M'=0$, $M''=0$, the derivatives being here relative to α alone; and making α pass through all possible degrees of magnitude, each state will give particular points of the envelope, which are those of the contact of the characteristics considered in their consecutive situations. The curve which joins these points is called the *edge of reflexion (arête de rebroussement)*; it is touched by all the characteristics, in precisely the same manner as the envelope touches all the surfaces which it envelopes in the line of these curves. The two equations of this edge are obtained by eliminating α between the three preceding equations.

Lastly, eliminating α between the equations $M=0$, $M'=0$, $M''=0$, $M'''=0$, where the derivatives are throughout relative to α, we shall see that in like manner we obtain the equation of the points of the *edge of reflexion*, which has itself a point of reflexion, or of inflexion.

766. Let the plane be taken for the moveable surface; the characteristics then will be straight lines, and the envelope will possess the property of being a *developable surface, i. e.* of being capable of being extended on a plane, without rupture or duplication, though we suppose

it to be neither flexible nor extensible. And in fact, if each element of this surface be made to turn about the straight line of section by the contiguous element, it is evident that the several elements will all be found to have been applied to a plane.

The developable surfaces may be considered as formed of plane elements of indefinite length; such, for instance, are the cone and the cylinder. Let us investigate an equation which shall belong to all these surfaces, leaving out of consideration the nature of the movement to which the variable plane is subjected. The tangent plane being coincident with a plane element, it is obvious that x, y, z may vary, without the tangent plane on this account varying. Its equation is [A, p. 318]

$$Z = pX + qY + z - px - qy;$$

and this we must differentiate in respect to x, y, z, and express that at the same time p, q and $z - px - qy$ undergo no change. Of these three conditions, the calculation shows that one is comprised in the two others, so that we have but these two equations $dp = 0$ and $dq = 0$, or rather (retaining the notation of p. 324) $r + sy' = 0$, $s + ty' = 0$; where y' depends on the direction in which the change of the point of contact takes place. Eliminating y', there finally results, for the equation of the whole species of developable surfaces, whatever may otherwise be their particular genesis, $rt - s^2 = 0$.

See *l'Analyse* of Monge, where that distinguished geometrician has given a multitude of curious applications of the infinitesimal doctrine to curve surfaces.

III. INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE.

767. The object of the Integral Calculus is to trace back the derivative functions to their primitives; and this we accomplish by means of a series of rules and transformations. To avoid subjecting our formulæ to the modifications that might be necessary, in consequence of the different changes of the independent variable [N°. 694], we shall make choice of the notation of Leibnitz. To intimate that we intend the integral of a function to be taken, we prefix to it the sign $\int$, which we

call *Sum*; thus, $y' = 4x^3$ being the derivative of $x^4 + c$, we shall write $dy = 4x^3dx$, and $y = \int 4x^3dx = x^4 + c$.

768. Let us inquire into the relation which must exist between the primitive functions fx and Fx, supposing that they have both the same derivative y'. Taylor's theorem gives

$$f(x+h) = fx + y'h + \tfrac{1}{2}y''h^2 + \ldots,$$
$$F(x+h) = Fx + y'h + \tfrac{1}{2}y''h^2 + \ldots;$$

whence $$f(x+h) - F(x+h) = fx - Fx;$$

and it follows therefore that $fx - Fx$ undergoes no change, when x is changed into $x + h$; so that $fx - Fx$ retains the same value C whatever x be, or $fx = Fx + C$. Hence, *all primitive functions which have the same derivative, differ from each other only as to the value of the constant term. And if to every integral we add an arbitrary constant, it will assume the most general form of which it is susceptible.*

769. By inverting the principal rules of the differential calculus, we shall arrive at an equal number corresponding to the integral calculus. Thus we shall readily conclude that

I. *The integral of a polynomial is the sum of the integrals of its several terms, each term retaining its sign and coefficient* [N°. 662].

II. *To integrate z^ndz, we must increase the exponent n by unity, suppress the factor dz, and divide by the exponent so increased* [N°. 668]; thus $\int Az^ndz = \dfrac{Az^{n+1}}{n+1} + C$.

Similarly $Az^{-n}dz$, or $Adz : z^n$ has for its integral

$$\frac{Az^{-n+1}}{-n+1} = -\frac{A}{(n-1)z^{n-1}}.$$

Consequently, *when the variable is in the denominator, the fraction must be taken with a contrary sign; the exponent of the variable must be diminished by unity, and the denominator be multiplied by this exponent so diminished.*

These rules apply also to all functions that can be brought under the form z^ndz. For $ax^{n-1}dx\,(b + cx^n)^m$, we observe that the differential of $b + cx^n$ is $ncx^{n-1}dx$; and since our first factor differs from this only as to the constants a and nc, we render it correspondent to the above form, by taking

$$\frac{a}{nc} \times ncx^{n-1}dx\,(b + cx^n)^m = \frac{a}{nc}\,z^m\,dz,$$

making $b + cx^n = z$. And we consequently have for the integral

$$\frac{az^{m+1}}{nc\,(m+1)} + C = \frac{a}{nc\,(m+1)}\,(b + cx^n)^{m+1} + C.$$

Similarly, $\int 6\sqrt{(4x^2 + 3)}\,xdx = \frac{1}{2}\,(4x^2 + 3)^{\frac{3}{2}} + C.$

The transformation, by which z was introduced, is by no means essential, and we shall for the future dispense with it, as it does but make the calculation tedious.

III. The preceding rule fails when $n = -1$, since we then find $\int z^{-1}dz = \infty$. This arises in fact from the integral belonging to a different species of function; we know [N°. 679] that

$$\int \frac{dz}{z} = lz + C\text{; and similarly} \int \frac{dz}{a+z} = l\,(a+z) + C.$$

Hence, *every fraction the numerator of which is the differential of the denominator has for its integral the logarithm of the denominator.*

In this case, for the convenience of calculation, we shall in future put the arbitrary constant under the form lC.

To integrate $\frac{5x^3dx}{3x^4+7}$, we observe that except as to the constant factor 5, this fraction comes under the preceding rule; and we therefore put it under the preparatory form

$$\int \frac{5}{12}\cdot\frac{12x^3dx}{3x^4+7} = \frac{5}{12}\,l\,[c\,(3x^4 + 7)].$$

IV. *Every fraction, the denominator of which is a square root, and its numerator the differential of the function affected by this root, has for its integral the double of this radical* [N°. 670]; or

$$\int \frac{dz}{\sqrt{z}} = 2\sqrt{z} + C.$$

V. One of the most important rules to be considered is that of integration *by parts*, the nature of which is as follows: it has been seen [N°. 663] that $d\,(ut) = udt + tdu$; whence, by integration,

$$ut = \int udt + \int tdu,$$

and

$$\int udt = ut - \int tdu;$$

thus, *having decomposed a proposed differential into two factors, one of which is directly integrable, we may integrate considering the other*

factor as constant; but we must then subtract the integral of the quantity obtained by differentiating this result in respect to that function alone which was previously taken as constant.

Thus, to integrate $lx.dx$, we shall first consider dx as alone variable, and we have $x.lx$; we then differentiate this result in respect to lx alone, and we obtain

$$\int lx.dx = x.lx - \int x.\frac{dx}{x} = x.lx - x + C.$$

The peculiar advantage of this rule is, that it renders the integral required dependent on some other integral, and our address in using it consists in so effecting the decomposition that this latter integral shall be less complex than the one proposed.

VI. The rule of N°. 683 gives, the radius being unity,

$$\int \frac{dz}{\sqrt{(1-z^2)}} = \text{arc}\,(\sin = z) + C,$$

$$\int \frac{-dz}{\sqrt{(1-z^2)}} = \text{arc}\,(\cos = z) + C,$$

$$\int \frac{dz}{1+z^2} = \text{arc}\,(\tan = z) + C.$$

We might also suppose the radius $= r$, and we should then have these same second sides for the values of the respective integrals

$$\int \frac{rdz}{\sqrt{(r^2-z^2)}},\ \int \frac{-rdz}{\sqrt{(r^2-z^2)}},\ \int \frac{r^2dz}{r^2+z^2}.$$

To obtain $\int \frac{mdz}{a+bz^2}$, we must divide above and below by a, whence we have

$$\frac{m}{a}.\frac{dz}{1+\frac{bz^2}{a}} = \frac{m}{a}\sqrt{\frac{a}{b}}.\frac{dt}{1+t^2},$$

assuming $\frac{bz^2}{a} = t^2$. Consequently, the radius being unity, the integral required is $\frac{m}{\sqrt{(ab)}}$ arc $(\tan = t) + C$; whence

$$\int \frac{mdz}{a+bz^2} = \frac{m}{\sqrt{(ab)}}.\text{arc}\left(\tan = z\sqrt{\frac{b}{a}}\right) + C.$$

We similarly find

$$\int \frac{mdz}{\sqrt{(a^2 - bz^2)}} = \frac{m}{\sqrt{b}} \text{ arc } \left(\sin = \frac{z}{a}\sqrt{b}\right) + C.$$

RATIONAL FRACTIONS.

770. We have given [p. 144] general methods for decomposing every rational fraction $\frac{N}{D}$ into others, the form of which shall be one of the following:

$$\frac{A}{x-a}, \frac{A}{(x-a)^n}, \frac{Ax+B}{x^2+px+q}, \frac{Ax+B}{(x^2+px+q)^n},$$

$A, B, p, q, n\ldots$ being constants, and the factors of $x^2 + px + q$ being imaginary. And we must now, therefore, give rules for tracing back these fractions to the expressions of which they are the derivatives.

We shall first observe, that if the term px be removed from the two last, by the transformation [p. 38], $x = z - \frac{1}{2}p$, and we then make $\beta^2 = q - \frac{1}{4}p^2$, a quantity which is positive by supposition, we shall simply have

$$\frac{Az+B'}{z^2+\beta^2}, \text{ and } \frac{Az+B'}{(z^2+\beta^2)^n}.$$

1st Case. The integral of $\frac{Adx}{x-a}$ is $Al(x-a) + lc$, or $Alc(x-a)$.

For example, we have seen [p. 146] that

$$\frac{dx}{a^2-x^2} = \frac{1}{2a}\left(\frac{dx}{a+x} + \frac{dx}{a-x}\right);$$

and the integral, therefore, is $\frac{1}{2a}[l(a+x) - l(a-x) + lc]$, whence

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} l \frac{c(a+x)}{a-x}.$$

Similarly

$$\int \frac{(2-4x)dx}{x^2-x-2} = \int \frac{2dx}{2-x} - \int \frac{2dx}{x+1}$$

$$= -2l(x-2) - 2l(x+1) + lc = l\frac{c}{(x^2-x-2)^2}.$$

2nd Case. The fraction $\frac{Adx}{(x-a)^n}$ has for its integral [Rule II]

$\dfrac{-A}{(n-1)(x-a)^{n-1}}$. For example [p. 146]

$$\frac{x^3+x^2+2}{x^5-2x^3+x}dx = \frac{2dx}{x} + \frac{dx}{(x-1)^2} - \frac{\frac{3}{4}dx}{x-1} - \frac{\frac{1}{2}dx}{(x+1)^2} - \frac{\frac{5}{4}dx}{x+1}$$

gives for the integral

$$2lx - \frac{1}{x-1} - \tfrac{3}{4}\,l(x-1) + \frac{1}{2(x+1)} - \tfrac{5}{4}\,l(x+1) + C.$$

3rd Case. For the fraction $\dfrac{Az+B}{z^2+\beta^2}dz$, we must separately integrate $\dfrac{Az\,dz}{z^2+\beta^2}$ and $\dfrac{Bdz}{z^2+\beta^2}$; the first by rule III, the second by VI [N°. 769]. We find

$$\int\frac{(Az+B)dz}{z^2+\beta^2} = \tfrac{1}{2}Al(z^2+\beta^2) + \frac{B}{\beta}\,\text{arc}\left(\tan = \frac{z}{\beta}\right).$$

Thus [p. 146] we decompose

$$\int\frac{xdx}{x^3-1} \text{ into } \int\frac{\frac{1}{3}dx}{x-1} - \int\frac{\frac{1}{3}(x-1)dx}{x^2+x+1}:$$

the first term $= \frac{1}{3}l(x-1)$: for the second, we make $x = z - \frac{1}{2}$, which gives $-\displaystyle\int\frac{\frac{1}{3}zdz}{z^2+\frac{3}{4}} + \int\frac{\frac{1}{2}dz}{z^2+\frac{3}{4}}$; one of these integrals is $= -\frac{1}{6}l(z^2+\frac{3}{4}) = -\frac{1}{3}l\sqrt{(x^2+x+1)}$; the other gives $\frac{1}{3}\sqrt{3}\,\text{arc}\left(\tan = \dfrac{2z}{\sqrt{3}}\right)$; and consequently

$$\int\frac{xdx}{x^3-1} = \tfrac{1}{3}\left[lc(x-1) - l\sqrt{(x^2+x+1)} + \sqrt{3}.\,\text{arc}\left(\tan = \frac{2x+1}{\sqrt{3}}\right)\right].$$

As a second example, let us take [p. 145]

$$\frac{(x^2-x+1)dx}{(x+1)(x^2+1)} = \tfrac{3}{2}.\frac{dx}{x+1} - \tfrac{1}{2}\frac{(x+1)dx}{x^2+1}:$$

the integral is $l\dfrac{c\sqrt{(x+1)^3}}{\sqrt[4]{(x^2+1)}} - \frac{1}{2}\,\text{arc}\,(\tan = x)$.

4th Case. We have now to integrate a series of fractions of the form $\dfrac{(Az+B)dz}{(z^2+\beta^2)^n}$, n being successively $= 1, 2, 3\ldots$ Each of these separates itself into two, $\dfrac{Azdz}{(z^2+\beta^2)^n}$ and $\dfrac{Bdz}{(z^2+\beta^2)^n}$. The first is immediately in-

tegrable [Rule II]*, and gives $\frac{-A}{2(n-1)(z^2+\beta^2)^{n-1}}$; only if $n=1$, we have $\frac{1}{2}Al(z^2+\beta^2)$.

771. As to the 2nd, we facilitate the integration of it by rendering it dependent on another more simple. K and L being indeterminate coefficients, we assume †

$$\int\frac{dz}{(z^2+\beta^2)^n}=\frac{Kz}{(z^2+\beta^2)^{n-1}}+\int\frac{Ldz}{(z^2+\beta^2)^{n-1}}.$$

To find the values of K and L, we differentiate this equation; then reduce to the same denominator $(z^2+\beta^2)^n$, and we have

$$1=K(z^2+\beta^2)-2K(n-1)z^2+L(z^2-\beta^2);$$

whence, comparing the corresponding terms, we derive

$$K+L=2K(n-1),\ (K+L)\beta^2=1.$$

Deducing the values of K and L from these equations, and substituting them, we finally obtain

$$\int\frac{dz}{(z^2+\beta^2)^n}=\frac{z}{2(n-1)\beta^2(z^2+\beta^2)^{n-1}}+\frac{2n-3}{2(n-1)\beta^2}\int\frac{dz}{(z^2+\beta^2)^{n-1}}.$$

The use of this equation will be easily conceived. We have a series of fractions of the form $\int\frac{dz}{(z^2+\beta^2)^n}$; we proceed first to integrate that in which n has the highest value, and our formula will replace it by two terms, the one integrated, the other of the form $\int\frac{dz}{(z^2+\beta^2)^{n-1}}$, which will unite itself with the following fraction. And we shall continue this course till we arrive at the fraction $\frac{dz}{z^2+\beta^2}$, the integral of which is known [Rule VI]. Take, for example,

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* Assuming $z^2+\beta^2=t^2$, the fraction becomes monomial, and we have

$$\int\frac{Adt}{t^{2n-1}}=\frac{-A}{2(n-1)t^{2(n-1)}}.$$

† The form of this equation is proved to be legitimate by the sequel of the calculation which serves for finding K and L. The transformation itself is pointed out by experience in analysis, whence we are able to foresee that the result can contain only terms of two sorts, those of the one sort being multiplied by z^2, those of the other constant.

$$\frac{(x^4+2x^3+3x^2+3)dx}{(x^2+1)^3} = \frac{(-2x+1)dx}{(x^2+1)^3} + \frac{(2x+1)dx}{(x^2+1)^2} + \frac{dx}{x^2+1};$$

the 1st terms of the two first fractions give [Rule II]

$$\int \frac{-2xdx}{(x^2+1)^3} = \frac{1}{2(x^2+1)^2}, \int \frac{2xdx}{(x^2+1)^2} = \frac{-1}{x^2+1}:$$

as to the 2nd terms, we have, by our formula,

$$\int \frac{dx}{(x^2+1)^3} = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\int \frac{dx}{(x^2+1)^2}.$$

This last term, joined to that of our 2nd fraction, gives $\frac{7}{4} \cdot \int \frac{dx}{(x^2+1)^2}$; and we then have, from the same formula,

$$\frac{7}{4}\int \frac{dx}{(x^2+1)^2} = \frac{7x}{8(x^2+1)} + \frac{7}{8}\int \frac{dx}{x^2+1};$$

lastly, adding to this term to be integrated along with our 3rd fraction, we find

$$\frac{15}{8}\int \frac{dx}{x^2+1} = \frac{15}{8} \text{ arc } (\tan = x).$$

It only remains now to collect the several parts, and we have, for the integral of the proposed function,

$$\frac{2+x}{4(x^2+1)^2} + \frac{7x-8}{8(x^2+1)} + \frac{15}{8} \text{ arc } (\tan = x) + C.$$

In the same manner may be found the integral of $\frac{dx}{(1+x)x^2(x^2+2)(x^2+1)^2}$.

This fraction being decomposed [p. 147], the only terms the integration of which can present any difficulty are

$$\int \frac{1}{2} \cdot \frac{x-1}{(x^2+1)^2} dx + \int \frac{1}{4} \cdot \frac{x-1}{x^2+1} dx$$

$$= c - \frac{x+1}{4(x^2+1)} + \frac{1}{8}l(x^2+1) - \frac{1}{2} \text{ arc } (\tan = x).$$

We annex two other examples [see p. 148]:

$$\int \frac{b^3dx}{x^6 - a^6} = \frac{b^3}{3a^5}\left[l\frac{(x-a)\sqrt{(x^2-ax+a^2)}}{(x+a)\sqrt{(x^2+ax+a^2)}}\right.$$

$$-\sqrt{3}\left\{\operatorname{arc}\left(\tan=\frac{2x-a}{a\sqrt{3}}\right)+\operatorname{arc}\left(\tan=\frac{2x+a}{a\sqrt{3}}\right)\right\}+C\Big].$$

$$\int\frac{x^3-6x^2+4x-1}{x^4-3x^3-3x^2+7x+6}\,dx=l\left(\frac{(x-2)\,(x+1)}{x-3}\right)+\frac{1}{x+1}+C.$$

IRRATIONAL FUNCTIONS.

772. It follows from what has been now seen, that we can integrate all rational algebraic functions, and such as can be rendered rational by means of any transformations.

Let us commence with the monomial radicals, and let our first example be

$$\frac{\sqrt[3]{x}+x\sqrt{x}+x^2}{x+\sqrt{x}}\,dx:$$

it is evident that, if we make $x=z^6$, the irrationalities will disappear, since 6 is divisible by the denominators 3 and 2 of the fractional exponents proposed. And we shall thus have to integrate

$$6dz\,.\frac{z^{14}+z^{11}+z^4}{z^3+1}=6z^{11}\,dz+6zdz-\frac{6zdz}{z^3+1},$$

which presents no difficulty.

For $\sqrt{x}.\,dx\ :\ (x-1)$, we shall assume $x=z^2$, and we shall have

$$\int\frac{2z^2dz}{z^2-1}=2\int dz+\int\frac{2dz}{z^2-1}$$

$$=2z+l(z-1)-l(z+1)=2\sqrt{x}+l\left(c\,.\,\frac{\sqrt{x}-1}{\sqrt{x}+1}\right).$$

773. Let us now take any function affected with the radical $\sqrt{(A+Bx+Cx^2)}$. Having disengaged x^2 from its coefficient C, by multiplying and dividing by $\sqrt{C}$, two cases will present themselves, accordingly as x^2 is positive or negative.*

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* X denoting a rational function of x, we have to integrate

$$\frac{Xdx}{\sqrt{(a+bx\pm x^2)}},\ \text{or}\ Xdx.\ \sqrt{(a+bx\pm x^2)}\,;$$

and these two expressions are to be treated in the manner specified above. The 2nd might also be brought under the form of the 1st, by multiplying and dividing by the

1st Case. If we have $\surd(a+bx+x^2)$, we shall assume*

$$\surd(a+bx+x^2)=z\pm x;\text{ whence } a+bx=z^2\pm 2zx,$$

$$x=\frac{z^2-a}{b\mp 2z},\ dx=\frac{bz\mp(z^2-a)}{(b\mp 2z)^2}\,.\,2dz,$$

$$\surd(a+bx+x^2)=z\pm x=\frac{bz\ \mp(z^2+a)}{b\mp 2z};$$

and thus the proposed function is rendered altogether rational.

Taking, for example, the lower signs, we find

$$\int\frac{dx}{\surd(a+bx+x^2)}=\int\frac{2dz}{2z+b}=l(2z+b)+\text{const.}$$

$$=[lc(x+\tfrac{1}{2}b+\sqrt{a+bx+x^2})].$$

Hence also

$$\int\frac{dx}{\surd(x^2\pm a^2)}=l[c(x+\sqrt{x^2\pm a^2})].$$

To integrate $dy=dx\surd(a^2+x^2)$, we make

$$\surd(a^2+x^2)=z-x,\text{ whence } dy=zdx-xdx;$$

thus, $y=-\frac{1}{2}x^2+\int zdx$; substituting for dx its value, and then integrating, we have $\int zdx=\frac{1}{4}z^2+\frac{1}{2}a^2lz$; and lastly

$$y=c+\tfrac{1}{2}x\surd(a^2+x^2)+\tfrac{1}{2}a^2l[x+\surd(a^2+x^2)].$$

For $dy=\dfrac{-dx}{\surd(1-x^2)}$ or $dy\surd-1=\dfrac{dx}{\surd(x^2-1)}$, we have

$$y\surd-1=l[x+\surd(x^2-1)]+c;$$

but $x=\cos y$, $\surd(x^2-1)=\surd-1.\sin y$; and also $c=0$, since $x=1$ must render $y=0$; thus we arrive again at the formula [N°. 591]

$$\pm y\surd-1=l(\cos y\pm\surd-1\sin y).$$

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radical: whence

$$\frac{X(a+bx\pm x^2)\,dx}{\surd(a+bx\pm x^2)}.$$

* We might in this case also make the radical $=x\pm z$; which would lead to the same values of x and dx; the radical would become $=\dfrac{\pm bz-z^2-a}{b\mp 2z}$, and the whole would be rendered rational.

774. 2nd Case. If we have $\sqrt{(a + bx - x^2)}$, the preceding method cannot be applied without introducing imaginary quantities; but we shall observe that the trinomial $a + bx - x^2$ must have its factors real, since otherwise it would be negative, whatever x were; in which case the radical being imaginary, we should, as in the preceding example, introduce $\sqrt{-1}$ as a factor, since we cannot expect to find the integral real, and the question will then be brought under the case just discussed. Let α and β, therefore, be the two real roots of $x^2 - bx - a = 0$; we shall assume

$$\sqrt{(a + bx - x^2)} = \sqrt{(x - \alpha)(\beta - x)} = (x - \alpha)z;$$

whence, squaring and suppressing the common factor $x - \alpha$, we have $\beta - x = (x-\alpha)z^2$; and x and dx consequently are rational.

Thus we find

$$\int \frac{dx}{\sqrt{(a + bx - x^2)}} = c - 2\,\text{arc}\left(\tan = \sqrt{\frac{\beta - x}{x - \alpha}}\right).$$

Similarly, for $\int \frac{dx}{\sqrt{(1-x^2)}}$, which we otherwise know to be the arc the sine of which is x, we shall make $\sqrt{(1-x^2)} = (1-x)z$; whence

$$x = \frac{z^2 - 1}{z^2 + 1},\ \sqrt{(1-x^2)} = \frac{2z}{z^2 + 1},\ dx = \frac{4zdz}{(z^2 + 1)^2};$$

$$\int \frac{dx}{\sqrt{(1 - x^2)}} = \int \frac{2dz}{z^2 + 1} = c + 2\,\text{arc}\,(\tan = z),$$

or $$\text{arc}\,(\sin = x) = -\tfrac{1}{2}\pi + 2\,\text{arc}\left[\tan = \sqrt{\left(\frac{1 + x}{1 - x}\right)}\right].$$

For $dy = dx\sqrt{(a^2 - x^2)}$, we make $\sqrt{(a^2 - x^2)} = (a - x)z$; whence

$$dy = \frac{8a^2z^2dz}{(1+z^2)^3} = \frac{-8a^2dz}{(1+z^2)^3} + \frac{8a^2dz}{(1+z^2)^2},$$

$$y = \frac{-2a^2z}{(1+z^2)^2} + \frac{a^2z}{1 + z^2} + a^2.\,\text{arc}\,(\tan = z) + C,$$

$$y = \tfrac{1}{2}x\sqrt{a^2 - x^2} + a^2.\,\text{arc}\left(\tan = \sqrt{\frac{a + x}{a - x}}\right) + C.$$

This process might be applied to the 1st case, when the roots of $x^2 + bx + a = 0$ are real.

775. The address acquired by practice will soon point out the transformations that are most advantageous. Thus, the second term under

the root might be made to disappear [N°. 504], which will reduce the expression to the form $\sqrt{(z^2 \pm a^2)}$, or $\sqrt{(a^2 \pm z^2)}$, so that the terms to be integrated will be [N°. 781]

$$\frac{z^m dz}{\sqrt{(z^2 \pm a^2)}} \text{ or } \frac{z^m dz}{\sqrt{(a^2 \pm z^2)}}.$$

In the latter case, the irrationality will disappear by assuming $\sqrt{(a^2 \pm z^2)} = a - uz$, since the square of this equation is divisible by z; and

$$z = \frac{2au}{u^2 \mp 1}, \; dz = -2adu.\frac{u^2 \pm 1}{(u^2 \mp 1)^2}.$$

Thus $\frac{dx}{\sqrt{(2bx - x^2)}}$ becomes $\frac{-dz}{\sqrt{(b^2 - z^2)}}$, making $x = b - z$; and the integral therefore is [Rule VI]

$$c + \text{arc}\left(\cos = \frac{z}{b}\right) = c + \text{arc}\left(\cos = \frac{b - x}{b}\right).$$

We might also have made the preceding transformation, which would have given

$$-\int \frac{2du}{u^2 + 1} = c' - 2\,\text{arc}\,(\tan = u).$$

In like manner, making $x = z - a$, we have

$$dy = \frac{adx}{\sqrt{(2ax + x^2)}} = \frac{adz}{\sqrt{(z^2 - a^2)}};$$

the equation therefore of the *Catenary* [see my *Mec.* N°. 91] is [p. 343]

$$y = al\,\{c[x + a + \sqrt{(2ax + x^2)}]\}.$$

BINOMIAL DIFFERENTIALS.

776. Let it be proposed to integrate $Kx^m dx(a + bx^n)^p$, m, n, p being any numbers whatever, integral or fractional, positive or negative.* We

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* m and n might easily be rendered integral by the method of N°. 772. Thus, $x^{\frac{1}{3}}dx(a + bx^{\frac{1}{2}})^p$, making $x = z^6$, becomes $6z^7dz(a + bz^3)^p$. But this transformation is not at all necessary for what is to follow.

shall assume $z=a+bx^n$; whence $x=\left(\frac{z-a}{b}\right)^{\frac{1}{n}}$; and raising each side to the power $m+1$, and differentiating, there will result

$$x=\frac{(z-a)^{\frac{m+1}{n}-1}}{n.b^{\frac{m+1}{n}}}\,dz,$$

$$Kx^m dx\,(a+bx^n)^p=\frac{K}{n.b^{\frac{m+1}{n}}}\,(z-a)^{\frac{m+1}{n}-1}.z^p dz.$$

The function can be integrated, whenever the exponent of $z-a$ is an integer. If $\frac{m+1}{n}=1$, we have to integrate $z^p dz$; if $\frac{m+1}{n}-1$ be positive and $=h$, by developing $(z-a)^h z^p dz$, we have a series of monomials; and lastly, if $\frac{m+1}{n}-1$ be negative, we have a rational fraction. Hence, *whenever the exponent of x without the binomial, increased by unity, is divisible by that of x within the binomial, the function can be integrated.*

777. This case is not the only one in which we are able to integrate; dividing the proposed binomial by x^n and multiplying without the binomial by x^{np}, we have

$$Kx^{m+np}(b+ax^{-n})^p dx;$$

and, applying the preceding theorem, it is clear that this expression will be integrable, provided that

$$\frac{m+np+1}{-n}, \text{ or rather } \frac{m+1}{n}+p=\text{an integer.}$$

Thus, when the foregoing condition is not fulfilled, we must add p to the fractional result $\frac{m+1}{n}$; and, *if the sum be integral, the function will be integrable by this method.*

778. We shall further observe, that if p be fractional (and this is the most important case, since otherwise we should but have to integrate a series of monomials), supposing q to be the denominator of p, the calculation will be more easily effected by assuming $a+bx^n=z^q$.

Let it be required, for example, to integrate $x^{-2}\,dx\,(a+x^3)^{-\frac{5}{3}}$: in this case $\frac{m+1}{n}$ is the fraction $-\frac{1}{3}$; but if we add $-\frac{5}{3}$, the sum is

-2; in order, therefore, to integrate, we must multiply and divide by $(x^3)^{-\frac{5}{3}}$ or x^{-5}, when we have

$$x^{-7}dx\,(1+ax^{-3})^{-\frac{5}{3}}.$$

We shall now assume $1+ax^{-3}=z^3$; whence $x=\left(\frac{z^3-1}{a}\right)^{-\frac{1}{3}}$; then raising to the power -6, and differentiating, we find $x^{-7}dx$; whence $-a^{\frac{1}{3}}(1-z^{-3})\,dz$, the integral of which is

$$c-\frac{1}{a^2}(z+\tfrac{1}{2}z^{-2})=c-\frac{3x^3+2a}{2a^2x\sqrt[3]{(x^3+a)^2}}.$$

Similarly, $x^3dx\,(a^2+x^2)^{\frac{1}{3}}$ will become $\frac{3}{2}\,dz\,(z^6-a^2z^3)$, making $a^2+x^2=z^3$; and hence we deduce for the integral of the function proposed $\frac{3}{56}\sqrt[3]{(a^2+x^2)^4}\,(4x^2-3a^2)+c$. We also have

$$\int\frac{adx}{(1+x^2)^{\frac{3}{2}}}=\int ax^{-3}dx\,(1+x^{-2})^{-\frac{3}{2}}=\frac{ax}{\sqrt{(1+x^2)}}+C.$$

779. When *the conditions of integrability* are not fulfilled, we seek to render the integral required dependent on some other more easy to be obtained, which is generally done by means of integration by parts [p. 336]. Always making $z=a+bx^n$, and at first treating z as constant, we shall have

$$\int x^mdx.z^p=\frac{x^{m+1}z^p}{m+1}-\frac{p}{m+1}\int z^{p-1}x^{m+1}dz;$$

whence, z being $=a+bx^n$ and $dz=nbx^{n-1}dx$,

$$\int x^mdx.z^p=\frac{x^{m+1}z^p}{m+1}-\frac{nbp}{m+1}\int x^{m+n}dx.z^{p-1}\ldots(1).$$

But $$z^p=z^{p-1}.\,z=z^{p-1}(a+bx^n);$$
and consequently

$$\int x^mdx.z^p=a\int x^mdx.z^{p-1}+b\int z^{p-1}x^{m+n}dx\ldots(2).$$

Equating the values (1) and (2), we find

$$b\,(m+1+np)\int z^{p-1}.x^{m+n}dx=x^{m+1}z^p-a\,(m+1)\int z^{p-1}x^mdx\ldots(3);$$

whence, changing $p-1$ into p, and $m+n$ into m, we have

$$\int x^mdx.z^p=\frac{x^{m-n+1}z^{p+1}-a\,(m-n+1)\int x^{m-n}z^pdx}{b\,(m+1+np)}\ldots(A).$$

Also, substituting for the last term of the equation (2) its value given by (3), we obtain

$$\int x^m dx . z^p = \frac{z^p x^{m+1} + anp \int x^m dx . z^{p-1}}{m + 1 + np} \ldots \qquad (B);$$

where $$z = a + bx^n.$$

780. We shall now show the use of these different formulæ.

1°. The equation (A) renders the integral $\int x^m dx . z^p$ dependent on $\int x^{m-n} z^p dx$, *i. e. it serves to diminish the exponent of x without the binomial by n units for each operation,* so that the integral proposed will depend on $\int x^{m-in} z^p dx$, i being a positive integer.

2°. The formula (B) serves on the other hand *to diminish the exponent p of the binomial z by* 1, 2, 3... *units.*

3°. Resolving the equations (A) and (B), in respect to the term to be integrated on the 2nd side, we obtain, $m - n$ being changed into m in (A) and $p - 1$ into p in (B),

$$\int x^m dx . z^p = \frac{x^{m+1} z^{p+1} - b(m + np + n + 1) \int x^{m+n} . z^p dx}{a(m + 1)} \ldots \quad (C),$$

$$\int x^m dx . z^p = \frac{- x^{m+1} z^{p+1} + (m + np + n + 1) \int x^m dx . z^{p+1}}{an(p + 1)} \ldots \quad (D);$$

and these formulæ serve on the contrary to increase the exponent of x without the binomial, and that of the binomial, which is of use when one or the other is negative.

4°. We shall be able therefore to determine *à priori* the law of the exponents of x in the result of a proposed integration. Thus it is easy to foresee this form:

$$\int \frac{x^5 dx}{\sqrt{(1 - x^2)}} = (Ax^4 + Bx^2 + C) \sqrt{(1 - x^2)}.$$

We can therefore, if we think proper, avoid the rather laborious task of directly employing our formulæ, by equating the differentials of these quantities, and then comparing them term by term, as in the method of indeterminate coefficients [N°. 771], which will make known A, B, C...

781. We shall here give a mode of integration which is remarkable for its simplicity and the numerous cases to which it can be applied.

Differentiating the function $x^{m-1} \sqrt{(1 - x^2)}$, we shall have

$$d[x^{n-1}\sqrt{(1-x^2)}] = (n-1)x^{n-2}\sqrt{(1-x^2)}\,dx - \frac{x^n dx}{\sqrt{(1-x^2)}};$$

multiplying and dividing the first term of this differential by $\sqrt{(1-x^2)}$, there will result

$$d[x^{n-1}\sqrt{(1-x^2)}] = (n-1)\frac{x^{n-2}dx}{\sqrt{(1-x^2)}} - \frac{nx^n dx}{\sqrt{(1-x^2)}};$$

and lastly, integrating and transposing, we have

$$\int\frac{x^n dx}{\sqrt{(1-x^2)}} = -\frac{x^{n-1}\sqrt{(1-x^2)}}{n} + \frac{n-1}{n}\int\frac{x^{n-2}dx}{\sqrt{(1-x^2)}} \dots (E).$$

Applying the same course of calculation to $x^{n-1}\sqrt{(x^2 \pm 1)}$, we find

$$\int\frac{x^n dx}{\sqrt{(x^2 \pm 1)}} = \frac{x^{n-1}\sqrt{(x^2 \pm 1)}}{n} \mp \frac{n-1}{n}\int\frac{x^{n-2}dx}{\sqrt{(x^2 \pm 1)}} \dots (F).$$

These formulæ serve to integrate every function affected with the radical $\sqrt{(A + Bx + Cx^2)}$, since it can be reduced to the form $\frac{z^n dz}{\sqrt{(a^2 \pm z^2)}}$ or $\frac{z^n dz}{\sqrt{(z^2 \pm a^2)}}$ [N°. 775]. It is easy, then, dividing above and below by a, to change the radical into $\sqrt{(1 \pm z^2)}$ or $\sqrt{(z^2 \pm 1)}$.

The expressions E and F will ultimately render the integral required dependent on

$$\int\frac{xdx}{\sqrt{(x^2 \pm 1)}} \text{ or } \int\frac{xdx}{\sqrt{(1-x^2)}} \dots\dots \text{ if } n \text{ be odd,}$$

$$\int\frac{dx}{\sqrt{(x^2 \pm 1)}} \text{ or } \int\frac{dx}{\sqrt{(1-x^2)}} \dots\dots \text{ if } n \text{ be even:}$$

the two first come under the rule IV [p. 336]; the third has been given in N°. 773; and the fourth is the arc (sin $= x$).

For example, we have

$$\int\frac{xdx}{\sqrt{(1-x^2)}} = -\sqrt{(1-x^2)} + c,$$

$$\int\frac{x^2dx}{\sqrt{(1-x^2)}} = -\frac{x}{2}\sqrt{(1-x^2)} + \tfrac{1}{2}\text{arc}(\sin = x) + c,$$

$$\int\frac{x^3dx}{\sqrt{(1-x^2)}} = -\frac{x^2+2}{3}\sqrt{(1-x^2)} + c,$$

$$\int\frac{x^4dx}{\sqrt{(1-x^2)}} = -\frac{2x^2+3}{8}x\sqrt{(1-x^2)} + \tfrac{3}{8}\text{arc}(\sin = x) + c.$$

782. If the exponent n were negative, the formulæ E and F would no longer be applicable; but we may still avail ourselves of them by making $x = z^{-1}$: we have in effect

$$\frac{dx}{x^n \sqrt{(1-x^2)}} = -\frac{z^{n-1}dz}{\sqrt{(z^2-1)}}$$

$$\frac{dx}{x^n \sqrt{(x^2 \pm 1)}} = -\frac{z^{n-1}dz}{\sqrt{(1 \pm z^2)}}.$$

We might also treat the actual case directly by a calculation similar to the preceding [N°. 781]; for, differentiating $x^{-n+1}\sqrt{(1-x^2)}$ &c., we find

$$\int \frac{dx}{x^n \sqrt{(1-x^2)}} = \frac{-\sqrt{(1-x^2)}}{(n-1)\,x^{n-1}} + \frac{n-2}{n-1}\int \frac{dx}{x^{n-2}\sqrt{(1-x^2)}} \cdots (G),$$

a formula the use of which will readily be conceived. We moreover have [N°. 773]

$$\int \frac{dx}{x\sqrt{(1-x^2)}} = c + l\left[\frac{1-\sqrt{(1-x^2)}}{x}\right].$$

In like manner we shall find

$$\int \frac{x^m dx}{\sqrt{(2ax-x^2)}} = \frac{x^{m-1}}{m}\sqrt{2ax-x^2} + \frac{(2m-1)a}{m}\int \frac{x^{m-1}dx}{\sqrt{(2ax-x^2)}} \cdots (H).$$

EXPONENTIAL FUNCTIONS.

783. It follows from the rules of differentiation [N°. 676] that

$$\int a^x dx = \frac{a^x}{la};$$

and we shall be able, therefore, to integrate two of the particular cases that exponentials can present.

1°. If $z = f(a^x)$, the function $za^x dx$, making $a^x = u$, will become $\frac{fu.du}{la}$. For example

$$\frac{a^x dx}{\sqrt{(1+a^{nx})}} = \frac{1}{la}.\frac{du}{\sqrt{(1+u^n)}}.$$

2°. Differentiating ze^x, we have $e^x dx\,(z+z')$; so that every exponential function, in which the factor of $e^x dx$ is composed of two parts,

one of which is the derivative of the other, will be easy of integration.

For example,

$$\int e^x dx\,(3x^2 + x^3 - 1) = (x^3 - 1)e^x.$$

Similarly, making $1 + x = z$, we find

$$\int \frac{e^x x dx}{(1+x)^2} = \int \frac{e^z}{e}\left(\frac{dz}{z} - \frac{dz}{z^2}\right) = \frac{e^z}{ez} = \frac{e^x}{1+x} + c.$$

784. But, in every other case, recourse must be had to integration by parts [p. 336]. Thus, for $x^n dx.a^x$, we shall first consider x^n as constant, and we shall have

$$\int x^n dx.a^x = \frac{a^x.x^n}{la} - \frac{n}{la}\int a^x x^{n-1} dx\,;$$

treating $a^x x^{n-1} dx$ in the same manner, and continuing the process step by step, we shall finally obtain

$$a^x x^n dx = a^x\left(\frac{x^n}{la} - \frac{nx^{n-1}}{l^2a} + \frac{n(n-1)x^{n-2}}{l^3a} \ldots \pm \frac{1.2.3\ldots n}{l^{n+1}a}\right) + c.$$

It is evident that the same calculation will apply to $za^x dx$, z being an integral algebraic function of x; and consequently

$$\int za^x dx = \frac{za^x}{la} - \int \frac{a^x z' dx}{la}.$$

785. But if *the exponent n be negative*, a regard to the spirit of the method which has just been employed will show, that, on the contrary, the exponent of x must now be made successively to increase.

We shall therefore integrate, considering a^x as for the present constant, and there will result

$$\int \frac{a^x dx}{x^n} = \frac{-a^x}{(n-1)x^{n-1}} + \frac{la}{n-1}\int \frac{a^x dx}{x^{n-1}}.$$

Repeating the same reasoning, the function will be reduced to the form

$$\int \frac{a^x dx}{x^n} = \frac{-a^x}{n-1}\left(\frac{1}{x^{n-1}} + \frac{la}{(n-2)x^{n-2}} + \frac{l^2a}{(n-2)(n-3)x^{n-3}} \ldots \right.$$
$$\left. + \frac{l^{n-2}a}{1.2.3\ldots(n-2)x}\right) + \frac{l^{n-1}a}{2.3\ldots(n-1)}\int \frac{a^x dx}{x}.$$

The calculation can be carried no farther than this, since it would next be necessary to make $n = 1$, which would give *infinity*, an ex-

pression which is made use of in Algebra to indicate the fact of an absurdity. The integral $\int \frac{a^x dx}{x}$ has long exercised the ingenuity of analysists, and we are compelled to consider it as a transcendental of a particular species, which cannot depend either on circular arcs, or on logarithms. In default of a more rigorous method, we employ the series [p. 157]

$$\frac{a^x}{x} = \frac{1}{x} + la + \frac{l^2a}{2}x + \frac{l^3a}{2.3}x^2 + \ldots;$$

which being multiplied by dx, and each term integrated separately, there results

$$\int \frac{a^x dx}{x} = lx + xla + \frac{x^2 l^2 a}{2.2} + \frac{x^3 l^3 a}{3.2.3} + \ldots + c.$$

786. Should n be fractional, one or other of the preceding methods will serve to reduce the exponent of x till it becomes comprised between 0 and 1 or -1; and development in series [Nos. 706, 800] will then give, approximately, the integral required.

All that has been here said will equally apply to $za^x dx$, when z is any algebraic function of x.

LOGARITHMIC FUNCTIONS.

787. Let it be proposed to integrate $zdx.l^n x$, z being any algebraic function of x.

If n be *integral and positive*, we shall integrate by parts, considering $l^n x$ at first as constant; when there will result

$$\int zdx\, l^n x = l^n x \int zdx - n \int (l^{n-1}x. \frac{dx}{x} \int zdx),$$

and since $\int zdx$ is supposed to be known from the preceding principles, the integration proposed is reduced to that of a function of the same form, in which the exponent of the logarithm is less. The same calculation applied to this function, and to the following ones successively, will effect the integration.

Thus, for $x^m l^n x.dx$, we have

$$\int x^m dx. l^n x = \frac{x^{m+1}}{m+1} l^n x - \frac{n}{m+1} \int (l^{n-1}x.x^m dx),$$

$$\int x^m dx. l^{n-1} x = \frac{x^{m+1}}{m+1} l^{n-1} x - \frac{n-1}{m+1} \int (l^{n-2}x.x^m dx),$$

and so on. Combining these successive results, we shall find

$$\int x^m l^n x dx = x^{m+1}\left(\frac{l^n x}{m+1} - \frac{nl^{n-1}x}{(m+1)^2} + \frac{n(n-1)l^{n-2}x}{(m+1)^3}\ldots\right) + c.$$

788. But if n be *integral and negative*, we shall see as before [N°. 785], that the exponent of the logarithm must on the contrary be made to increase, and that for this purpose we must first take z as constant in the integration by parts of $\int z\, dxl^n x$. Since

$$\int \frac{dx}{x}.\, l^n x = \frac{l^{n+1}x}{n+1},$$

we shall separate $zdx.l^n x$ into the two factors $zx \times \frac{dx}{x}.\, l^n x$, whence

$$\int \frac{zdx}{l^n x} = \frac{zx}{-n+1}\, l^{-n+1}x + \frac{1}{n-1}\int [l^{-n+1}x.d\,(zx)],$$

a formula which obviously accomplishes the end in view. But, the better to see the nature of the obstacles that may be met with, let us apply this formula to

$$\int \frac{x^m dx}{l^n x} = \frac{-x^{m+1}}{(n-1)\,l^{n-1}x} + \frac{m+1}{n-1}\int \frac{x^m dx}{l^{n-1}x}:$$

repeating the operation on the last term of this, &c., and then combining the several results, we shall have

$$\frac{\int x^m dx}{l^n x} = -\frac{x^{m+1}}{n-1}\left[\frac{1}{l^{n-1}x} + \frac{m+1}{n-2}.\, \frac{1}{l^{n-2}x}\right.$$

$$\left.+\frac{(m+1)^2}{(n-2)(n-3)}.\, \frac{1}{l^{n-3}x} + \ldots\right] + \frac{(m+1)^{n-1}}{1.2.3\ldots(n-1)}\int \frac{x^m dx}{lx}.$$

Here we are obliged to stop; for we cannot take $n = 1$, in our formula, without introducing infinity. If however we make

$$x^{m+1} = z, \text{ whence } (m+1)\, x^m dx = dz,$$

there results, also assuming $lz = u$,

$$\frac{x^m dx}{lx} = \frac{dz}{lz} = \frac{e^u du}{u};$$

and we therefore have again the function of N°. 785, which can only be integrated by series.

789. When n is *fractional*, as it is positive or negative, one or other of these formulæ will reduce the integral of $zdx.l^n x$ to that of a function of the same form, n being comprised between 1 and -1; after which recourse must be had to development in series [N^{os}. 706, 800].

CIRCULAR FUNCTIONS.

790. If arcs enter into a function, in order to integrate it, we shall observe that the differential of these arcs is algebraic, and that, consequently, if we put in practice integration by parts, considering these arcs in the first place as constants [see p. 336], the proposed function will be exempt from them. Thus, z being a function of x, we have

$$\int z dx . \operatorname{arc}(\sin = x) = \operatorname{arc}(\sin = x) \int z dx - \int \frac{dx . \int z dx}{\sqrt{(1-x^2)}};$$

and similarly we shall find

$$\int z dx . \operatorname{arc}(\tan = x) = \operatorname{arc}(\tan = x) \int z dx - \int \frac{dx \int z dx}{1+x^2}.$$

791. But when the functions contain trigonometrical lines, there are several modes of integrating them, which are more or less advantageous according to circumstances. We shall proceed to explain the principal of them.

1st *Method*. These functions can always be reduced to binomial differentials by making $\sin x$ or $\cos x = z$.

Thus, let

$$\sin x = z, \text{ and consequently } \cos x = \sqrt{(1-z^2)},\ dx = \frac{dz}{\sqrt{(1-z^2)}};$$

then $$\sin^m x . \cos^n x dx = z^m dz \sqrt{(1-z^2)}^{n-1}.$$

1°. If n be odd, the root disappears.

2°. If m be odd, the first condition of integrability [N°. 776] is fulfilled, since $\frac{1}{2}(m+1)$ is an integer.

3°. If m and n are even, the second condition [N°. 777] is satisfied, since $\frac{1}{2}(m+n)$ is an integer. We shall find, for example,

$$\int \sin^4 x . \cos^3 x . dx = \int z^4 dz (1-z^2) = \tfrac{1}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + c,$$

$$\int \sin^3 x . dx = \int \frac{z^3 dz}{\sqrt{(1-z^2)}} = -\tfrac{1}{3}\cos x(3-\cos^2 x) + c,$$

$$\int \sin^4 x . dx = \int \frac{z^4 dz}{\sqrt{(1-z^2)}} = -\frac{\sin^3 x + \frac{3}{2}\sin x}{4}\cos x + \frac{1.3x}{2.4} + c.$$

792. 2nd *Method.* It follows from N°. 682, that

$$\int dx.\cos kx = \frac{1}{k}\sin kx + c,\ \int dx.\sin kx = -\frac{1}{k}\cos kx + c.$$

Now we have learnt [p. 171] to develop any power of $\sin x$ and $\cos x$ in series, according to the multiples of the arc x; and we shall therefore have to integrate a series of terms of the forms above. For example,

$$\int \cos^5 x dx = \int \left(\tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x\right) dx$$

$$= \tfrac{1}{80}\sin 5x + \tfrac{5}{48}\sin 3x + \tfrac{5}{8}\sin x + c.$$

This method is frequently employed, because it is easier to obtain the numerical solutions, when the sines and cosines of the multiples of the arcs are used in preference to the powers of these lines.

793. 3rd *Method.* The formulæ K [N°. 590] will also serve to transform the sines, cosines... into exponentials, which will reduce the integrals of the former to those of the latter [N°. 783].

794. The 4th *Method* consists in integration by parts. Since $-dx\sin x$ is the differential of $\cos x$, we shall decompose the product $\sin^m x.\cos^n x dx$ into $dx\sin x\cos^n x \times \sin^{m-1}x$; and the first factor having $-\dfrac{\cos^{n+1}x}{n+1}$ for its integral, we obtain

$$\int dx\sin^m x\cos^n x = -\frac{\sin^{m-1}x}{n+1}\cos^{n+1}x + \frac{m-1}{n+1}\int \cos^{n+2}x\sin^{m-2}x dx.$$

Substituting for $\cos^{n+2}x$ its value $\cos^n x.\cos^2 x$, or $\cos^n x(1-\sin^2 x)$, and transposing, there results

$$\int dx\sin^m x.\cos^n x = -\frac{\sin^{m-1}x.\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int dx\sin^{m-2}x.\cos^n x \ldots (I).$$

Proceeding, in respect to the cosine, in the same manner that we have done for the sine, we shall have

$$\int dx\sin^m x\cos^n x = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int dx\sin^m x\cos^{n-2}x \ldots (K).$$

These formulæ reduce the exponent of the sine or the cosine; and the combined and successive use of them gives the integral *when m and n are*

positive integers. For example, we have

$$\int dx \sin^3 x \cos^2 x = -\tfrac{1}{5}\sin^2 x \cos^3 x + \tfrac{2}{5}\int dx \sin x \cos^2 x,$$

$$\int dx \sin x \cos^2 x = \tfrac{1}{3}\sin^2 x \cos x + \tfrac{1}{3}\int dx \sin x\,;$$

the last term is $= -\tfrac{1}{3}\cos x + c$; and combining these several parts, we get

$$\int dx \sin^3 x \cos^2 x = \cos x\left(-\tfrac{1}{5}\sin^2 x \cos^2 x + \tfrac{2}{15}\sin^2 x - \tfrac{2}{15}\right) + c.$$

795. But if m or n be negative, these formulæ will require some modification. The first gives, changing n into $-n$,

$$\int \frac{dx \sin^m x}{\cos^n x} = -\frac{\sin^{m-1}x}{(m-n)\cos^{n-1}x} + \frac{m-1}{m-n}\int \frac{dx \sin^{m-2}x}{\cos^n x} \dots (L),$$

which, as we see, will make the integral required dependent on that of $\dfrac{dx \sin x}{\cos^n x}$ or $\dfrac{dx}{\cos^n x}$, as m is odd or even. The first of these integrals is obtained by making $\cos x = z$, which gives $-\displaystyle\int \frac{dz}{z^n} = \frac{1}{(n-1)\cos^{n-1}x}$; the other will be given in N°. 796.

The second of our formulæ, making n negative, resolving in respect to the last term, and then changing n into $n-2$, gives

$$\int \frac{dx \sin^m x}{\cos^n x} = \frac{\sin^{m-1}x}{(n-1)\cos^{n-1}x} - \frac{m-n+2}{n-1}\int \frac{dx \sin^m x}{\cos^{n-2}x} \dots (M).$$

The integral required therefore is ultimately reduced to that of $dx \sin^m x$, or $\dfrac{dx \sin^m x}{\cos x}$, accordingly as n is even or odd. The first is about to be given, the second is so by the formula (I).

796. If we make n or m nothing in the equations I and K, we have

$$\int \sin^m x dx = \frac{-\cos x . \sin^{m-1}x}{m} + \frac{m-1}{m}\int dx \sin^{m-2}x,$$

$$\int \cos^n x dx = \frac{\sin x . \cos^{n-1}x}{n} + \frac{n-1}{n}\int dx \cos^{n-2}x,$$

and changing m into $-m+2$, n into $-n+2$, we find

$$\int \frac{dx}{\sin^m x} = \frac{-\cos x}{(m-1)\sin^{m-1}x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2}x},$$

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1}x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2}x}.$$

Instead of thus deducing these several formulæ from the two equations I and K, we might have found them directly. For this purpose nothing more would be requisite than a due regard to the nature of integration by parts, and to the end that we ought to propose to ourselves in it.

The fractions $\frac{\cos^m x dx}{\sin^n x}$ and $\frac{\sin^m x dx}{\cos^n x}$ may also be integrated in another manner: thus, the first, if m be even and $= 2h$, is equivalent to $\frac{(1-\sin^2 x)^h dx}{\sin^n x}$; and developing $(1-\sin^2 x)^h$, we have a series of terms of the form $\sin^k x dx$. If m be odd and $= 2h + 1$, we have, making $\sin x = z$,

$$\frac{\cos^{2h} x . \cos x . dx}{\sin^n x} = \frac{(1-z^2)^h dz}{z^n}.$$

797. For the case in which the exponents of the sine and the cosine are simultaneously negative, multiplying the numerator by $\cos^2 x + \sin^2 x$, we have

$$\int \frac{dx}{\sin^m x . \cos^n x} = \int \frac{dx}{\sin^{m-2} x . \cos^n x} + \int \frac{dx}{\sin^m x . \cos^{n-2} x};$$

and we shall thus arrive at fractions that are clear of $\sin x$ or $\cos x$. If $m = n$, since $\sin x \cos x = \frac{1}{2} \sin 2x$, making $2x = z$, the proposed fraction is changed into

$$\int \frac{dx}{\cos^n x \sin^n x} = 2^{n-1} \int \frac{dz}{\sin^n z}.$$

798. We integrate the following five circular functions apart, both because the calculations for them are of a simpler nature, and because our formulæ reduce all the others to them.

1°. Let the function be $\frac{dx}{\sin x}$; making $\cos x = z$, we have $-\frac{dz}{1-z^2}$, a rational fraction [p. 338]; whence

$$\int \frac{dx}{\sin x} = lc + \tfrac{1}{2} l \frac{1-\cos x}{1+\cos x};$$

and since [N°. 359] $\tan^2 \frac{1}{2} x = \frac{1-\cos x}{1+\cos x}$, we have

$$\int \frac{dx}{\sin x} = l \frac{c\sqrt{(1-\cos x)}}{\sqrt{(1+\cos x)}} = l . c \tan \tfrac{1}{2} x.$$

A similar calculation, making $\sin x = z$, gives

$$\int \frac{dx}{\cos x} = l \frac{c\sqrt{(1 + \sin x)}}{\sqrt{(1 - \sin x)}} = l.c \tan (45^\circ + \tfrac{1}{2} x).$$

3°. For $\frac{dx.\cos x}{\sin x}$, since the numerator is the differential of the denominator [Rule III, p. 336], we have

$$\int \frac{dx \cos x}{\sin x} = \int \frac{dx}{\tan x} = \int dx \cot x = l(c \sin x).$$

4°. In like manner we have

$$\int \frac{dx \sin x}{\cos x} = \int dx \tan x = \int \frac{dx}{\cot x} = l \frac{c}{\cos x}.$$

5°. Adding these two last formulæ, we find

$$\int \frac{dx}{\sin x \cos x} = l \frac{c^2 \sin x}{\cos x} = l(C \tan x).$$

ARBITRARY CONSTANTS. INTEGRATION BY SERIES.

799. Let P be the integral of a function zdx of x, or $dP = zdx$, and c be the arbitrary constant which must be added in order that it may be the most general possible [N°. 768]; we have

$$\int zdx = P + c.$$

So long as the integral calculation is the only point considered, c remains any whatever; but when we wish to apply this integral to some specific question, the constant c ceases to be arbitrary, and must satisfy some prescribed conditions. If, for example, it be required to find the area $BCPM = t$ [fig. 22], comprised between the ordinates BC, PM, the position of which corresponds to the abscissæ a and b, since [N°. 728] $dt = ydx$, we have $t = \int ydx = P + c$. But, the area $P + c$ commencing when $x = AC = a$, t must be nothing when we make $x = a$ in $P + c$, or $A + c = 0$, A being the value which the function of x denoted by P assumes, when $x = a$; we derive from this $c = -A$, and consequently the area $t = P - A$. It will then remain to substitute b for x, and the area will be comprised within the prescribed limits.

Generally, to determine the arbitrary constant, according to the conditions of the question, we must investigate the values that ought to be

assumed by the integral $t = P + c$ when $x = a$, viz. $k = A + c$, whence

$$c = k - A \text{ and } t = P + k - A,$$

without, as we see, there being any necessity to know the *origin of the integral, i. e.* to know for what value a of x it is nothing.

Every integral, the origin of which is not fixed, is called *Indefinite;* it is *Complete* only when it contains an arbitrary constant. When the limits a and b are given, we have $t = P - A$ by virtue of the first; substituting for x the second limit b, there results $t = B - A$, for the absolute numerical and constant value of $t = \int y dx$; and this we call a *definite Integral, A* and B being the values assumed by P when $x = a$ and b. If the form of this expression be observed, it will be evident that to obtain it, we have only to *make $x = a$ and $x = b$ in the indefinite integral P, and subtract the first result from the second.* This will in a little time be fully illustrated.

M. Fourier has devised a very convenient notation for expressing the definite integrals; he affects the symbol $\int$ of integration with two indices, one below, which refers to the 1st limit of the integral, the other above, for the 2nd limit: $\int_a^b$ indicates an integral taken from $x = a$ to $x = b$. Thus, $\int_{2\pi}^{\pi} \sin x dx = 1$, since the integral $-\cos x$ becomes -1 and 0 at the two limits. The expression $\int_a^x$ indicates that the integral commences at $x = a$, and extends to an indefinite value of the variable x.

800. When the function proposed is not susceptible of exact integration, recourse is had to approximations. Thus, to find $\int z dx$, we shall develop z in a series, according to the ascending or descending powers of x [N°. 706]; then multiply each term by dx, and integrate it.

We shall give but two examples.

1°. Let the integral be $\int \frac{dx}{1 + x^2}$, which is arc ($\tan = x$). Developing $(1 + x^2)^{-1}$, we have

$$\frac{dx}{1 + x^2} = dx(1 - x^2 + x^4 - x^6 + \&c.)$$

whence $\quad \text{arc}(\tan = x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

2°. For $\int \frac{dx}{\sqrt{(1-x^2)}} = \text{arc}\,(\sin = x)$, we shall develop

$$(1-x^2)^{-\frac{1}{2}} = 1 + \frac{x^2}{2} + \frac{1.3x^4}{2.4} \ldots ; \text{ whence}$$

$$\text{arc}\,(\sin = x) = x + \frac{x^3}{2.3} + \frac{3x^5}{2.4.5} + \frac{3.5x^7}{2.4.6.7} + \ldots$$

We have added no constant, since the arc now spoken of is supposed to be the least of those of which x is the sine or the tangent, and this arc is nothing when the sine and the tangent are so. The 1st of these formulæ has already served for finding [N°. 591] the ratio π of the circumference to the diameter; and the 2nd may be employed for the same purpose; for the third of the quadrant having $\frac{1}{2}$ for its sine, making $x = \frac{1}{2}$, we get

$$\tfrac{1}{6}\pi = \tfrac{1}{2} + \frac{1}{2.3.2^3} + \frac{1.3}{2.4.5.2^5} + \frac{1.3.5}{2.4.6.7.2^7} \ldots$$

The law of the series follows from the calculation itself.

801. That a series may be of any use in numerical applications, it must be convergent, and it is advisable, therefore, to have different processes for effecting integrations of this sort. The following one is due to Jean Bernoulli.

Make $h = -x$ in Taylor's formula; since then $f(x-x)$, or $f0$, is what y or fx becomes when $x = 0$, $f0$ is a constant b, and

$$b = y - y'x + \tfrac{1}{2}y''x^2 - \ldots$$

Now, the derivative y' of y being given, the integration consists in finding y: let $\int z dx$ be the integral required; then $z = y'$, $z' = y''\ldots$, and we find

$$y = \int z dx = b + zx - \tfrac{1}{2}z'x^2 + \tfrac{1}{6}z''x^3 - \ldots$$

It follows from what has been seen in N°. 701, that we can obtain limits of the sum of the terms neglected.

For example, for $\int \frac{dx}{a+x} = l(a+x)$, we have

$$b = la,\ z = \frac{1}{a+x},\ z' = \frac{-1}{(a+x)^2},\ z'' = \frac{2}{(a+x)^3} \ldots,$$

and

$$l(a+x) = la + \frac{x}{a+x} + \frac{x^2}{2(a+x)^2} + \frac{x^3}{3(a+x)^3}\cdots$$

802. Taylor's formula also gives, for $z = fx$,

$$f(x+h) - fx = zh + \tfrac{1}{2}z'h^2 + \tfrac{1}{6}z''h^3\ldots,$$

whence, making $h = b - a$,

$$f(x+b-a) - fx = z(b-a) + \tfrac{1}{2}z'(b-a)^2 + \ldots$$

If now we take $x = a$, which changes $z, z', z''\ldots$ into some constants $A, A', A''\ldots$, we obtain

$$fb - fa = A(b-a) + \tfrac{1}{2}A'(b-a)^2 + \tfrac{1}{6}A''(b-a)^3\ldots;$$

and this is the integral $\int zdx$ between the limits $x = a$ and $x = b$ [N°. 799]. That this series, however, may be applicable, it is necessary that that of Taylor be not faulty; and we must consequently examine the course of the function z from $x = a$ to $x = b$, in order to ascertain whether it become infinite for any intermediate values of this variable x.

The series can be rendered as highly convergent as may be thought fit; for the interval $b - a$ being divided into n equal parts i, so that $b - a = ni$, we can first take the integral between the limits a and $a + i$, *i. e.* substitute in the form above $a + i$ for b. In like manner we shall take the integral from $a + i$ to $a + 2i$; then from this quantity to $a + 3i,\ldots$

We shall therefore successively assume

$x = a$, which will change $z, z', z''\ldots$ into $A, A', A''\ldots,$
$x = a + i,$ .... $B, B', B''\ldots,$
$x = a + 2i,$ .... $C, C', C''\ldots,$
&c.;

whence

$$\begin{aligned}
f(a+i) - fa &= Ai + \tfrac{1}{2}A'i^2 + \tfrac{1}{6}A''i^3 + \ldots,\\
f(a+2i) - f(a+i) &= Bi + \tfrac{1}{2}B'i^2 + \tfrac{1}{6}B''i^3 + \ldots,\\
f(a+3i) - f(a+2i) &= Ci + \tfrac{1}{2}C'i^2 + \tfrac{1}{6}C''i^3 + \ldots\\
&\&c.\ldots\\
f(a+ni) - f[a+(n-1)i] &= Mi + \tfrac{1}{2}M'i^2 + \tfrac{1}{6}M''i^3 + \ldots
\end{aligned}$$

803. The sum of these equations is

$$f(a+ni) - fa = fb - fa = \int zdx =$$
$$(A+B+C\ldots+M)i + \tfrac{1}{2}(A'+B'\ldots+M')i^2 + \tfrac{1}{6}(A''+\ldots+M'')i^3\ldots;$$

and this is the value of $\int zdx$ between the limits a and b. If i be taken sufficiently small that we may confine ourselves to the 1st term, we have

$$\int zdx = Ai + Bi + Ci \ldots + Mi,$$

a series the different terms of which are the values successively assumed by the function zdx, when we make $x = a, a + i, a + 2i \ldots$ It is from this circumstance that in the infinitesimal method we consider the integral as the *sum* of an infinite number of elements, which are the consecutive values assumed by the function when the variable is made to pass through all the intermediate values between its limits; this will be farther elucidated in the sequel [N°. 806, 2°].

Consult on the approximations of definite integrals a beautiful Memoir of M. Poisson, inserted among those of the Institute, 1826. M. Cauchy has also written on the same subject, and has made some applications of it to questions of Geometry and Mechanics that are highly curious. The *Theorie de Chaleur*, of M. Fourier, contains a great number of questions which depend on definite integrals.

804. We shall say nothing on the subject of integrations of the 2nd, 3rd..., order of functions of a single variable, since they come under the rules that have been already laid down. There will in such cases be 2, 3... arbitrary constants [See N°. 831].

For example, for $\int\int \frac{(a^2 - x^2)\,dx^2}{(x^2 + a^2)^2}$, we shall integrate a first time; and since the fraction proposed resolves itself [p. 144] into $\frac{2a^2dx}{(x^2+a^2)^2} - \frac{dx}{x^2+a^2}$, the first of which gives [N°. 771] $\frac{x}{x^2+a^2} + \int \frac{dx}{x^2+a^2} + c$, it remains to repeat the integration on $\frac{xdx}{x^2+a^2} + cdx$. We consequently have

$$\int\int \frac{(a^2 - x^2)\,dx^2}{(x^2 + a^2)^2} = l\,\sqrt{(x^2 + a^2)} + cx + c'.$$

QUADRATURES AND RECTIFICATIONS.

805. The area t of a plane curve [N°. 728] is $= \int ydx$, and we have to integrate this expression between the proper limits. It is on this account that we have given the name of the *Method of Quadratures* to the processes with which we have been latterly occupied, since

by means of them are obtained the integrals of functions of a single variable. The following are some examples.

I. For the parabola AM [fig. 51], $y^2 = 2px$; and therefore

$$t = \int \sqrt{(2p)}.x^{\frac{1}{2}}dx = \tfrac{2}{3}\sqrt{(2p)}.x^{\frac{3}{2}} + c = \tfrac{2}{3}xy + c.$$

When the area is to commence from the vertex A, $x = 0$ gives $t = 0$, and consequently c is nothing; thus *the area MAM' of a segment of a parabola is two thirds of the circumscribed rectangle $M'N'NM$.*

If the area is comprised between BC and PM, making $AB = a$, $BC = b = \sqrt{(2pa)}$, t is nothing when $x = a$, whence $c = -\tfrac{2}{3}ab$, and therefore $t = \tfrac{2}{3}(xy - ab)$. The area $C'CMM'$ is two thirds of the difference of the rectangles $N'M$ and $D'C$.

For the parabolas of all degrees, $y^m = ax^n$, we have $t = \dfrac{mxy}{m+n}$; and all these curves therefore are *quadrable.*

II. For the equilateral hyperbola MN [fig. 52] between its asymptotes Ax, Ay, we have $xy = m^2$ [N°. 418]; and consequently

$$t = \int ydx = m^2 \int \frac{dx}{x} = m^2\, lx + c.$$

The area t cannot be taken from the axis Ay, for $x = 0$ would give $t = 0$ and $c = -m^2 lo = \infty$; but if the area is to commence at the ordinate BC which passes through the vertex C, since $AB = m$ [N°. 418], we have $c = -m^2 lm$, whence $t = m^2 l\dfrac{x}{m}$. It appears therefore that if $m = 1$, we have $t = lx$; and *each area, supposing it to commence from BC, is the Napierian logarithm of the abscissa.*

When the angle of the asymptotes is α, the area is [p. 296]

$t = \int ydx.\sin\alpha = \displaystyle\int \frac{\sin \alpha.dx}{x}$, making $m = 1$; and consequently $t = \text{Log}\, x$, taking for the system of log that of which the modulus is $M = \sin\alpha$ [N°. 678].

If α be a right angle, $M = 1$, which brings us back to the 1st case, and we have the Napierian logarithms; but it is obvious that by varying the angle α of the asymptotes, we may obtain all the systems for which $M < 1$. Thus when the base is 10, we have $M = 0.4342944819$; the angle which has this number for its sine, the radius being unity, is $\alpha = 25^\circ\, 44'\, 25''.47$; and this is the angle that must be formed by the asymptotes of an hyperbola the power of which is unity, in order that each area may be the tabular logarithm of the corresponding abscissa. Thus we see that it is with great impropriety that the deno-

mination of *hyperbolic Logarithms* has been given to the Napierian system, since all the systems of logarithms may find themselves represented by the areas of different hyperbolas.

III. For the circle $y^2 = a^2 - x^2$, the origin being at the centre C [fig. 53], and we have

$$t = \int \sqrt{(a^2 - x^2)}\, dx = \int \frac{a^2 dx}{\sqrt{(a^2 - x^2)}} - \int \frac{x^2 dx}{\sqrt{(a^2 - x^2)}},$$

multiplying and dividing by $\sqrt{(a^2 - x^2)}$. The last of these terms is easily integrated by parts, since $\frac{xdx}{\sqrt{(a^2 - x^2)}}$ is the differential of $-\sqrt{(a^2 - x^2)}$; and consequently

$$\int \frac{x^2 dx}{\sqrt{(a^2 - x^2)}} = -x\sqrt{(a^2 - x^2)} + \int dx \sqrt{(a^2 - x^2)} = -xy + t;$$

whence, substituting and transposing t, we shall have

$$t = \tfrac{1}{2} xy + \tfrac{1}{2} a \int \frac{adx}{\sqrt{(a^2 - x^2)}}.$$

But the formula $ds^2 = dx^2 + dy^2$, applied to the circle, gives $ds = \frac{adx}{y} = \frac{adx}{\sqrt{(a^2 - x^2)}}$; and consequently, taking the arc s between the same limits as those of the proposed integral, we finally have $t = \frac{1}{2} xy + \frac{1}{2} as + c$. Let $CA = b$, $AB = k$: doubling and integrating from $x = a$ to $x = b$, in order to obtain the area of the segment BOB', we shall have [N°. 799] $\frac{1}{2} a \times$ arc $BOB' - bk$; whence, adding the triangle CBB', there results

$$\text{sector } CBOB' = \tfrac{1}{2}\, CO \times \text{arc } BOB'.$$

IV. For the ellipse [fig. 53] $y = \frac{b}{a}\sqrt{(a^2 - x^2)}$, whence

$$t = \int \frac{b}{a} \sqrt{(a^2 - x^2)}\, dx = \frac{b}{a} \times z,$$

z denoting that part of the area of the circumscribed circle which is comprised between the limiting ordinates. The areas t and z are therefore in the constant ratio of b to a. Thus, *the area of the circle is to that of the ellipse, or the area of a segment of the circle is to that of the segment of the inscribed ellipse, which is bounded by the same ordinates, as the major axis to the minor;* and since the area of the circumscribed circle is πa^2, that of the whole ellipse is πab.

V. For the cycloid FMA [fig. 23], fixing the origin at F, so that $FS = x$, $SM = y$, we shall have [N°. 723, VI]

$$\frac{dy}{dx} = \sqrt{\left(\frac{y}{2r - y}\right)},\ t = \int y dx = \int \sqrt{(2ry - y^2)}\, dy\ ;$$

this integral is the area of the portion FKN of the generating circle; and consequently the area $FyAM = FKEF = \frac{1}{2}\pi r^2$. Since also $AE = \pi r$, the rectangle $yE = 2\pi r^2$, whence $AFE = \frac{3}{2}\pi r^2$: thus, *the area AFA' of the whole cycloid is triple of that of the generating circle.*

VI. Simpson's method for *obtaining the values of plane curvilinear areas* by approximation is deserving of notice. In the first place let us investigate the area of a small segment CEM [fig. 51] of any curve referred to the axes Ax, Ay. At the middle point K between the ordinates CB, PM, draw the ordinate KE; we may then very fairly consider the arc CEM as being that of a parabola of which the vertex L corresponds to the middle point I of the chord; and the area therefore is $CEMI = \ldots \frac{2}{3} CM.LI$. But, denoting the angle MCH formed by the chord CM with Ax by α, the rectangular triangles LEI, MCH give $LI = EI.\cos\alpha$, $CM = CH.\cos\alpha$; whence $CEMI = \frac{2}{3} EI \times CH$.

This being premised, make $BK = KP = h$, $CB = y'$, $KE = y''$, $PM = y'''$; then the area $CBPM$ is composed of the trapezium $CBPM = h\,(y' + y''')$ and of $CEMI = \frac{4}{3}h.EI$; but $EI = EK - IG - GK = \frac{1}{2}(2y'' - y' - y''')$, since $IG = \frac{1}{2} MH$: consequently, the segment $CEMI = \frac{2}{3}h\,(2y'' - y' - y''')$, and

$$\text{the small area } CEMPB = \tfrac{2}{3}h\,(\tfrac{1}{2}y' + 2y'' + \tfrac{1}{2}y''').$$

Suppose now that the plane area, the value of which we are in search of, is limited laterally by two parallels, and that we have taken the axis Ax perpendicular to these lines, as in fig. 138 of the 1st Volume. Drawing a series of lines parallel to these and equidistant from each other, the proposed area will be cut into parts the values of which will be given by our formula, h being the common distance of these parallels, and y', y'', y'''... their respective lengths. Thus, for the succeeding areas taken 2 and 2, we shall have

$$\tfrac{2}{3}h\,(\tfrac{1}{2}y''' + 2y^{IV} + \tfrac{1}{2}y^{V}),\ \tfrac{2}{3}h\,(\tfrac{1}{2}y^{V} + 2y^{VI} + \tfrac{1}{2}y^{VII}),\ \&c.\ ;$$

and adding,

$$\text{the total area} = \tfrac{2}{3}h\,(\tfrac{1}{2}y' + 2y'' + y''' + 2y^{IV} + y^{V} + 2y^{VI} \ldots \tfrac{1}{2}y^{(n)}).$$

It appears therefore that, *the proposed area being cut by a series of parallels odd in number, and at the same distance h from each other, we must take two sums, the one consisting of the lines of the even ranks, which we must double; the other of those of the odd ranks; from the total sum we must subtract the half of the extreme lines, and multiply the remainder by $\frac{2}{3}h$:* the product will be the closer approximation to

the proposed area, the nearer the parallels be to each other, or the less h be. And this is the theorem of Simpson.

806. We have some remarks to make here:

1°. If the area t be comprised between the branches BM, DK of the same curve [fig. 55], or between two different given curves, representing the ordinates PM, PE by $Y = Fx$ and $y = fx$, we have

$$BCPM = \int Ydx,\ DCPE = \int ydx, \text{ whence } BDEM = \int (Y-y)\,dx.$$

2°. According to the infinitesimal method [N^{os}. 762, 803], the area t may be considered as the sum of rectangles, such as m [fig. 55], of which dx and dy are the sides; $dxdy$ is therefore the element of the area t, and this must be integrated between the proper limits.

Similarly, suppose that in the circle C [fig. 54] we have taken any element m; its distance from the centre, or $Cm = r$, and the angle $mCx = \theta$ will fix its position. The area of this element may be represented by $dr.d\theta$, the integral of which relative to θ is θdr; and taking this from $\theta = 0$ to $\theta = 2\pi r$, we have the area of a circular band $= 2\pi rdr$, of which the breadth dr is indefinitely small. The integral of this is πr^2; and taking it from the centre C where $r = 0$ to the circumference B where $r = R =$ the radius of the circle, we have πR^2 for the area of the circle.

3°. When the area is inclosed between two curves BM, DE [fig. 55], of which we have the equations $Y = Fx$, $y = fx$, we shall integrate the element $m = dydx$ from PE to PM, *i. e.* ydx becomes $(Y - y)\,dx$, and will be a known function of x, representing the element ME comprised between two ordinates indefinitely near to each other. It will remain to integrate relatively to x between the limits AC, AP; and if the area be included within the circuit of a closed curve, $(Y - y)\,dx$ must be integrated from the least value of x to the greatest. When the area is inclosed between four branches of curves, such as BM, BI, IK, KM, it is easy to divide it, by straight lines parallel to the axes, into parts the values of which can be obtained separately by means of the preceding principles.

The opposite parabolas AF, AF' [fig. 59] have for equations $y^2 = \pm\, 2px$; the element $m = dxdy$ being integrated relatively to x, from M' to M, *i. e.* from $-\dfrac{y^2}{2p}$ to $+\dfrac{y^2}{2p}$, xdy gives $\dfrac{y^2dy}{p}$ for the area of the slice MM'. Integrating a second time from A to C, or from $y=0$, the area $F'AFC$ will be $\dfrac{y^3}{3p}$ or $\frac{2}{3}\,xy$.

4°. The ordinate y of the curve must not become infinite between the limits of the area [N°. 802].

5°. The element ydx changes its sign along with y or x, whence it follows that the area becomes negative when x and y are of contrary signs.

When the curve cuts the axis of x between the limits of the area, we must investigate each of the two parts and add, since one of them is positive and the other negative, and the sum required must be obtained without having regard to this latter sign.

For example, let the curve $KACD$ [fig. 56] have for its equation $y = x - x^3$; then, the origin being at A, $AK = AI = 1$. The area $t = \frac{1}{2}x^2 - \frac{1}{4}x^4 + c$; if it is to commence at the point B for which $AB = \sqrt{\frac{1}{2}}$, we find $c = -\frac{1}{16}$, whence $t = \frac{1}{2}x^2 - \frac{1}{4}x^4 - \frac{1}{16}$; and if it is to be terminated by ED, for which $AE = \sqrt{\frac{3}{2}}$, we find $t = 0$, which indicates only that the areas BCI, IED are equal and of contrary signs. In fact, it is easily seen that $BCI = \frac{1}{16} = -DIE$. Similarly the area taken from K to I is nothing, because $ACI = \frac{1}{4} = -KOA$.

807. To give an application of the formula [N°. 729] $\tau' = \frac{1}{2}(xy' - y)$, which serves for finding the area τ comprised between two radii vectores, let us investigate the area CMO [fig. 53] in the ellipse ODO': we have $a^2y^2 + b^2x^2 = a^2b^2$, $y' = -\dfrac{b^2x}{a^2y}$, whence

$$\tau' = -\tfrac{1}{2}\left(\frac{b^2x^2}{a^2y} + y\right) = -\frac{b^2}{2y},\ d\tau = -\frac{abdx}{2\sqrt{(a^2 - x^2)}}.$$

The area τ is measured from a fixed radius, as CO, to the radius CM, and the sign $-$ arises from x decreasing when τ increases [N°. 702].

Now the formula [N°. 727] of rectifications, applied to the circle of radius a, gives for the length of its arc s, $ds = -\dfrac{adx}{\sqrt{(a^2 - x^2)}}$; whence $d\tau = \frac{1}{2}bds$ and $\tau = \frac{1}{2}bs$, taking the arc s between the same limits as τ, viz. $x = CO$ and $x = CA$. When $b = a$, we have $\tau = \frac{1}{2}as$; thus, the circular sector $BCO = \frac{1}{2}CO \times \text{arc}\,BO$; and the elliptic sector $MCO = \frac{1}{2}b \times \text{arc}\,BO = \dfrac{b}{a} \times OCB$.

For the hyperbola MN [fig. 52], we have $xy = m^2$, whence $\tau' = -y$ and $d\tau = -ydx$, $\tau = -\int ydx$: consequently any hyperbolic sector $CAM = CBPM$.

808. When the co-ordinates are polar, [fig. 25], we have [N°. 729] $d\tau = \frac{1}{2} r^2 d\theta$. Thus, in the spiral of Archimedes [N°. 472], for which $2\pi r = a\theta$, we find $\tau = \frac{\pi}{a} \int r^2 dr = \frac{\pi}{a} \cdot \frac{r^3}{3} + c$. For the area AOI generated by an entire revolution of the radius vector AM, the integral must be taken from $r=0$ to $r=a$; and we thus obtain $AOI = \frac{1}{3}\pi a^2 =$ the third of the circle the radius of which is AI.

When the integral is to be extended beyond $\theta = 360°$, it must be taken into consideration that this second area contains the one which we have just obtained, as in N°. 806, 5°.

809. Let us now give some examples of the formula [N°. 727] of rectification, $s = \int \sqrt{(dx^2 + dy^2)}$.

I. For the parabola, $y^2 = 2px$ gives

$$ydy = pdx, \; s = \int \frac{dy}{p} \sqrt{(y^2 + p^2)};$$

and this integral is [N°. 773]

$$s = c + \frac{y}{2p} \sqrt{(p^2 + y^2)} + \tfrac{1}{2} p.l[y + \sqrt{(p^2 + y^2)}].$$

If the arc s commence at A [fig. 51], $y = 0$ gives $s = 0$; whence we deduce $c = -\frac{1}{2} plp$; and consequently

$$ACM = \frac{y\sqrt{(p^2 + y^2)}}{2p} + \tfrac{1}{2} pl\left(\frac{y + \sqrt{(p^2 + y^2)}}{p}\right).$$

II. For the second cubical parabola $y^3 = ax^2$, we have

$$s = \int dy \sqrt{\left(1 + \frac{9y}{4a}\right)} = \tfrac{8}{27} a \sqrt{\left(1 + \frac{9y}{4a}\right)^3} + c.$$

Generally, $y = ax^n$ represents the whole species of parabolas or hyperbolas, accordingly as n is a positive or negative fraction; and hence we obtain $s = \int dx \sqrt{(1 + n^2a^2x^{2n-2})}$. Whenever therefore [N°. 776] $2(n-1)$ is exactly contained in 1, or $\frac{1}{2(n-1)} + \frac{1}{2}$ is an integer, we shall have the arc s in a finite form.

III. For the circle, accordingly as the origin is at the centre or at the extremity of the diameter, we have $y^2 = r^2 - x^2$, or $y^2 = 2rx - x^2$; but in both cases, the result is $s = \int \frac{rdx}{y}$. Substituting for y its value in terms of x, it appears that the integration can be effected only by a

series [N°. 800], or by circular arcs, which brings us back to the point whence we started.

IV. For the ellipse, $a^2y^2 + b^2x^2 = a^2b^2$ gives

$$s = \int \frac{dx}{a} \sqrt{\left(\frac{a^4 - x^2(a^2 - b^2)}{a^2 - x^2}\right)} = \int dx \frac{\sqrt{(a^2 - e^2x^2)}}{\sqrt{(a^2 - x^2)}},$$

making $ae = \sqrt{(a^2 - b^2)}$, so that e denotes the ratio of the eccentricity to the semi-major axis. This expression can be integrated only by a series; and the operation must be so disposed as to render the series convergent. Thus, we might develop [N°. 485, 11] $\sqrt{(a^2 - e^2x^2)}$.

Or, otherwise, make the arc OB [fig. 53] of the circumscribed circle $= \theta$, then

$$CA = x = a \cos\theta,\ \frac{dx}{\sqrt{(a^2 - x^2)}} = -d\theta,$$

whence

$$s = -a \int d\theta \sqrt{(1 - e^2 \cos^2\theta)};$$

and we shall therefore have to integrate a series of terms of the form $A \cos^{2m}\theta . d\theta$ [N°. 796]. The arc OM will thus be made to depend, by means of a series, on the corresponding arc OB of the circumscribed circle. The rectification of the hyperbola leads to a similar calculation.

V. In the cycloid [fig. 23], the origin being at F, we have [N°. 723, VI]

$$y' = \sqrt{\frac{y}{2r - y}},\ s = \int \frac{\sqrt{2r}}{\sqrt{y}} dy = 2\sqrt{(2ry)}.$$

No constant is required when the arc s commences at F. But $\sqrt{(2ry)} = KF$; and consequently $FM =$ twice the chord KF.

810. If the co-ordinates are polar [N°. 729], we have $ds = \sqrt{(r^2d\theta^2 + dr^2)}$. Thus, the spiral of Archimedes, for which $2\pi r = a\theta$, gives

$$s = \int \frac{2\pi dr}{a} \sqrt{\left(\frac{a^2}{4\pi^2} + r^2\right)}.$$

This expression being compared with that for the arc of the parabola, it will be seen that the lengths of the arcs of these curves are equal, when r is the ordinate of the parabola, and $\frac{a}{\pi}$ the parameter.

In the logarithmic spiral [N°. 474] $\theta = lr$; we find $s = \int dr \sqrt{2} =$

$r\sqrt{2}+c$: if the arc commence at the pole, $c=0$, and we have $s=r\sqrt{2}$. Thus, though the curve do not arrive at its pole except after an infinite number of revolutions, the arc s is finite and equal to the diagonal of the square constructed on the radius vector which terminates it.

See, for the curves of double curvature, what has been said in N°. 751.

SURFACES AND VOLUMES OF BODIES.

811. The volume v and the surface u of a body of revolution about the axis of x are obtained [N°. 752] by integrating

$$v=\int \pi y^2 dx,\ u=\int 2\pi y ds=\int 2\pi y\sqrt{(dx^2+dy^2)}.$$

We annex some applications of these formulæ.

I. For the ellipse, recurring to the value of ds [N°. 809, IV], we find

$$v=\frac{\pi b^2}{a^2}\int (a^2-x^2)dx,\ u=\frac{2\pi be}{a}\int\sqrt{\left(\frac{a^2}{e^2}-x^2\right)}\,dx.$$

The 1st gives $v=\pi b^2\left(x-\dfrac{x^3}{3a^2}+c\right)$: if the vertex be one of the limits, $c=-\frac{2}{3}a$; and if z therefore be the altitude of the segment of the ellipsoid, or $x=a-z$, the volume $=\dfrac{\pi b^2z^2}{3a^2}(3a-z)$. For the whole ellipsoid, $z=2a$, and we have $v=\frac{4}{3}\pi b^2 a$.

Hence it follows, that 1°. the volume of the sphere $=\frac{4}{3}\pi a^3$: 2°. the ellipsoid of revolution is to the circumscribed sphere $::b^2:a^2$: 3°. each of these bodies is the two-thirds of its circumscribed cylinder: 4°. the spherical segment $=\pi z^2(a-\frac{1}{3}z)$.

The integral which enters into the value of u is evidently the area of that portion of the circle, concentric with the ellipse, and having the radius $\dfrac{a}{e}$, which is comprised between the same limits as the generating arc. Let z be this area, which is easily obtained, and we shall have $u=\dfrac{2\pi bez}{a}$.

If the sphere be considered, we have [N°.809, III] $ds=\dfrac{rdx}{y}$; whence $u=\int 2\pi r dx$. We easily find $2\pi rz$ for the surface of the calotte or zone of which z is the altitude; and $4\pi r^2$ for the surface of the whole sphere.

II. For the parabola $y^2 = 2ax$, we find

$$v = \int 2\pi ax.\,dx = \pi ax^2 + c,$$

$$u = \int \frac{2\pi}{a}\cdot y dy\,\sqrt{(y^2 + a^2)} = \frac{2\pi}{3a}[\sqrt{(y^2 + a^2)^3} + C]:$$

if the origin be at the vertex, $c = 0$ and $C = -a^3$. Thus we have the volume and surface of a segment of the paraboloid of revolution.

III. Let $y^m = ax^n$; we deduce from it

$$v = \int \pi \sqrt[m]{a^2}.\sqrt[m]{x^{2n}}.\,dx = \frac{m\pi x}{m + 2n}\sqrt[m]{(ax^n)^2} = \frac{m\pi xy^2}{m + 2n}.$$

This expression belongs to parabolas or hyperbolas, accordingly as n is positive or negative.

812. The volume V and the surface U of a body are given by the formulæ [N°. 754]

$$V = \int\int z dx dy,\ U = \int\int dx dy\,\sqrt{(1 + p^2 + q^2)}.$$

These double integrals are to be treated thus: having substituted for z, p and q their values in terms of x and y, deduced from the equation of the proposed surface [N°. 747], we must integrate, considering x or y as constant, accordingly as one or the other promises more simple calculations; and we must then have regard to the limits which are fixed by the question.

For example, if the surface U, which is required, is to be comprised between two planes parallel to xz, $y = a$, $y = b$, and we have integrated in respect to y, we must take the integral between the limits a and b, x being considered as constant. We shall thus have the surface MB [fig. 49] of a section of the infinitely small breadth dx, and terminated at the two planes ME, NB that have been spoken of. This 1st integral will be of the form $\varphi x.\,dx$, *i. e.* devoid of y, and only containing x. We must now integrate anew, relatively to x, from the least to the greatest value of that variable, and we shall have the area required, which has thus been considered as the sum of an infinite series of similar sections.

If the body be terminated laterally by curve surfaces, we shall have to introduce, into the first integral, functions of x, for the limit of y, proceeding in a manner analogous to that of N°. 806. This will be put in a clearer light by some examples.

For the sphere [fig. 57], we have $x^2 + y^2 + z^2 = r^2$; whence

$$p = -\frac{x}{z},\ q = -\frac{y}{z},\ \sqrt{(1 + p^2 + q^2)} = \frac{r}{z},$$

$$U = \int\int \frac{rdxdy}{\sqrt{(r^2 - x^2 - y^2)}},\ V = \int\int dxdy\,\sqrt{(r^2 - x^2 - y^2)}.$$

We shall in the first place consider y as constant, and $r^2 - y^2 = A^2$; whence

$$U = \int\int \frac{rdx}{\sqrt{(A^2 - x^2)}}\,dy,\ V = \int\int dxdy\,\sqrt{(A^2 - x^2)}.$$

A first integration gives, for the former of these, $rdy.$ arc $\left(\sin = \dfrac{x}{A}\right)$. But the plane xy cuts the sphere in a circle Cy, the equation of which is $x^2 + y^2 = r^2$, and in which the abscissa $AF = \pm\sqrt{(r^2 - y^2)} = \pm A$ is the radius of the circle formed by the cutting plane DmC. If, therefore, this integral be taken from $x = -A$ to $x = +A$, we shall have the infinitely narrow surface DmC of a band parallel to xz, and traced on the higher hemisphere.

Making, therefore, $x = -A$ and $x = +A$ in our arc above, and subtracting the 1st result from the 2nd, we shall have πrdy, because the arc, of which the sine $= 1$, is $\frac{1}{2}\pi$. Integrating now in respect to y, which has hitherto been taken as constant, we shall have πry for the 2nd integral, and the limits being $-r$ and $+r$, which are the least and greatest values of y, $2\pi r^2$ will be the surface of the higher hemisphere.

Following the same plan for the volume V [p. 349], we find

$$\int\sqrt{(A^2 - x^2)}\,dx = \tfrac{1}{2}x\sqrt{(A^2 - x^2)} + \tfrac{1}{2}A^2\text{arc}\left(\sin = \frac{x}{A}\right).$$

Taking the limits $-A$ and $+A$, as above, the 1st term disappears, and the other gives $\frac{1}{2}\pi A^2$. We must now, therefore, repeat the integration on $\frac{1}{2}\pi(r^2 - y^2)\,dy$, which represents the volume of the section $DmCE$; and we get $\frac{1}{2}\pi(r^2y - \frac{1}{3}y^3)$, which, between the limits $-r$ and $+r$, becomes $V = \frac{2}{3}\pi r^3$; and this is the volume of the hemisphere.

The element of the volume V is $dx\,dydz$: we first integrate in respect to z, from the value z of the lower surface, which bounds the body, to that z of the higher surface, *i.e.* we substitute in $zdxdy$ these two values of z in the form of functions of x and y, such as they are found to be from the equations of the two surfaces; and we thus have the parallelipiped comprised between these surfaces, and standing on the base $dxdy$. We then integrate relatively to x, in order to form the sum of all the prisms which compose a section, the breadth of which is dy, and which is comprised between two planes parallel to xz. Suppose that the volume V is bounded by a cylinder MNg [fig. 58], erected on a given base mng; the limits of our 2nd integral result from some section Pmn of the body made by a plane perpendicular to y; thus we must take the integral

from $x = Pm$ to $x = Pn$, values which are deduced in a function of y from the equation of the curve mng, the base of our cylinder. Let $x = fy$ and Fy be these values; we must substitute them successively for x in the integral, and subtract the results one from the other. It will then remain only to integrate a function of y, from the least value AB of y to the greatest AC, these values also being derived from the equation of the base fng.

Let us, for instance, investigate the volume of the right cone. Taking its axis for that of y, and the vertex for the origin, the equation is [N°. 621] $l^2y^2 = z^2 + x^2$, l being the tangent of the angle formed by the axis and the generatrix. Thus, taking it from the lower z to the higher, $zdxdy$ becomes $2\sqrt{(l^2y^2 - x^2)}\, dxdy$, since $z = \pm \sqrt{(l^2y^2 - ^2)}$. The integral of this relative to x has been given above and in p. 344, viz.

$$x\sqrt{(l^2y^2 - x^2)} + 2l^2y^2 . \text{arc}\left(\tan = \sqrt{\frac{ly + x}{ly - x}}\right) + c;$$

and since, on making $z = 0$, the equation of the cone gives $x = \pm ly$ for the limits of the body, we must in this expression change x into $-ly$ (which gives zero), and then into $+ ly$ [whence we have $2l^2y^2$. arc (tan $= \infty) = \pi l^2y^2$]; subtracting, there results πl^2y^2dy, which must be integrated from $y = 0$, or the vertex, to $y = h$, which corresponds to the base. Hence, finally, the volume of the right cone is $\frac{1}{3}\pi l^2h^3$, which agrees with the known theorem.

Similarly, if the limits of the surface are determined by a curve $FMNG$ traced out on the surface in question, we must investigate its projection fg on the plane xy [N°. 616], which will determine a right cylinder, to which exactly the same reasoning will apply. We shall therefore integrate $dxdy\ \sqrt{(1 + p^2 + q^2)}$ between the limits specified above.

To give an example, suppose that, in the plane xy, there are described two equal and opposite parabolas $FAE, F'AE'$ [fig. 59], the equations of which are $y^2 = nx$, $y^2 = -nx$, and draw the parallel FF' to the axis of x, AC being $= b$. Also, conceive a right cone with a circular base to have its vertex in the origin A and its axis along that of z, its equation being $z = k\sqrt{(x^2 + y^2)}$ [N°. 621]. It is required to find the conical surface that is included within the right cylinder erected on $AMFF'M'$.

The equation of the cone gives

$$p = \frac{kx}{\sqrt{(x^2 + y^2)}},\ q = \frac{ky}{\sqrt{(x^2 + y^2)}},\ 1 + p^2 + q^2 = 1 + k^2;$$

so that the element of the conical surface is $\sqrt{(1 + k^2)}\, dxdy$; its projection is in m. The integral relative to x is $\sqrt{(1 + k^2)}\, xdy$, which

must be taken from M' to M; and we shall thus have the area of the infinitely narrow band which is projected in MM'.

But, the equations of the parabolas give, for the abscissæ of the points M, M', the limits of the integral,

$$x = -\frac{y^2}{n},\ x = +\frac{y^2}{n};\ \text{whence } \frac{2y^2}{n}\sqrt{(1+k^2)}\,dy.$$

Operating now for y on this first integral, there results $\frac{2y^3}{3n}\sqrt{(1+k^2)}$, which must be taken from A to C, *i. e.* from $y = 0$ to $y = b$. We thus obtain, for the surface required, $\frac{2b^3}{3n}\sqrt{(1+k^2)}$.

The application of these principles to the investigation of the centres of gravity and of the moments of inertia is highly remarkable. [See my *Mec.* N^os^. 64 and 241].

IV. INTEGRATION OF EQUATIONS BETWEEN TWO VARIABLES.

SEPARATION OF THE VARIABLES. HOMOGENEOUS EQUATIONS.

813. We shall now proceed to the integration of equations of the 1st order between two variables.

Let the differential equation proposed be $Mdy + Ndx = 0$, which is of the 1st order between the two variables x and y. It is evident that if the variables be unconnected with each other, so that M do not contain x, and N be without y, the integral of the equation will be the sum of the integrals that are found by the preceding principles.

$$\int Mdy + \int Ndx = \text{constant}.$$

And the same will be the case for every equation in which we shall be able to *separate* the variables. The most simple case is that in which M is a function of x alone, and N of y; for, dividing the equation by MN, we have

$$\frac{dy}{N} + \frac{dx}{M} = 0.$$

Thus, $dx\sqrt{(1+y^2)} - xdy = 0$ gives

$$\frac{dx}{x} = \frac{dy}{\sqrt{(1+y^2)}};$$

whence [N°. 773]

$$l(cx) = l[y + \sqrt{(1 + y^2)}], \text{ and } cx = y + \sqrt{(1 + y^2)}.$$

814. If $M = XY$, $N = X_{\prime}Y_{\prime}$, X and $X_{\prime}$ being functions of x, Y and $Y_{\prime}$ functions of y, we have $XYdy + X_{\prime}Y_{\prime}dx = 0$; which gives, dividing by $XY_{\prime}$,

$$\frac{Y}{Y_{\prime}}dy + \frac{X_{\prime}}{X}dx = 0.$$

815. The separation of the variables is also possible in the equations that are *homogeneous* [N°. 322] in respect to x and y. Let m be the degree of each term Ay^kx^h, or $m = h + k$; dividing the equation by x^m, the term Ay^kx^h becomes $A\left(\frac{y}{x}\right)^k = Az^k$, making $z = \frac{y}{x}$. It appears, therefore, that M and N will become functions of z alone, so that if the equation $Mdy + Ndx = 0$ be divided by M, we shall have $dy + Zdx = 0$. But $y = xz$ gives $dy = xdz + zdx$; whence $xdz + (z + Z)\,dx = 0$; and consequently

$$\frac{dx}{x} + \frac{dz}{z + Z} = 0, \text{ and } lx + \int\frac{dz}{z + Z} = 0.$$

I. Let us take, for our first example, $(ax + by)dy + (fx + gy)dx = 0$. Dividing by $ax + by$, we shall find

$$dy + \frac{f + gz}{a + bz}dx = 0; \text{ whence } \frac{dx}{x} + \frac{(a + bz)\,dz}{bz^2 + (a + g)\,z + f} = 0,$$

an equation easily integrated; and we must then substitute $\frac{y}{x}$ for z.

Thus, $ydy + (x + 2y)\,dx = 0$, since $a = 0$, $b = f = 1$, $g = 2$, gives $\frac{dx}{x} + \frac{zdz}{z^2 + z + 1} = 0$; we add dz to the numerator of the 2nd term, which becomes $\frac{dz(1 + z)}{(1 + z)^2}$ or $\frac{dz}{1 + z}$; and we then have to integrate

$$\frac{dx}{x} + \frac{dz}{1 + z} - \frac{dz}{(1 + z)^2} = 0;$$

whence $$l(cx) + l(1 + z) + \frac{1}{1 + x} = 0,$$

or $$l.\,c(x + xz) = \pm - \frac{1}{1 + z},\ l.c(x + y) + \frac{x}{x + y} = 0.$$

II. For $ay^m dy + (x^m + by^m)\,dx = 0$, we have

$$dy + \frac{1 + bz^m}{az^m}\,dx = 0,\ \frac{dx}{x} + \frac{az^m dz}{az^{m+1} + bz^m + 1} = 0.$$

III. Let $xdy - ydx = dx\,\sqrt{(x^2 + y^2)}$: assuming $y = xz$, and dividing by x, we find

$$dy - zdx = dx\,\sqrt{(1 + z^2)},\ \text{whence}\ \frac{dx}{x} = \frac{dz}{\sqrt{(1 + z^2)}};$$

the integral of which [N°. 773] is $x = cz + c\,\sqrt{(1 + z^2)}$, or $x^2 = cy + c\,\sqrt{(x^2 + y^2)}$, which is reduced to $x^2 = 2cy + c^2$, by transposing cy and raising to the square.

IV. What is the curve of which the area $BCMP$ [fig. 51] is equal to the cube of the ordinate PM, which terminates it, divided by the abscissa, and this for each of its points, commencing from a fixed ordinate BC? From $\int ydx = \frac{y^3}{x}$, we deduce, by differentiation, $(x^2y + y^3)\,dx = 3xy^2dy$; making $y = zx$, we find [p. 337] $\frac{dx}{x} = \frac{3zdz}{1 - 2z^2}$; whence $x^4(1 - 2z^2)^3 = c$, and lastly

$$(x^2 - 2y^2)^3 = cx^2.$$

816. Every equation, therefore, that can be rendered homogeneous will be integrable. Thus, for

$$(ax + by + c)\,dy + (mx + ny + p)\,dx = 0,$$

we shall assume

$$ax + by + c = z,\ mx + ny + p = t;$$

whence $$adx + bdy = dz,\ mdx + ndy = dt;$$

and consequently

$$dy = \frac{mdz - adt}{mb - na},\ dx = \frac{bdt - nd}{mb - na};$$

and the proposed equation now becomes homogeneous,

$$zdy + tdx = 0,\ \text{or}\ (mz - nt)\,dz + (bt - az)\,dt = 0.$$

When $mb - na = 0$, this calculation is no longer possible; but in that

case $m = \frac{na}{b}$, and the proposed equation is

$$bcdy + bpdx + (ax + by)(bdy + ndx) = 0;$$

in which the variables are separated by making $ax + by = v$, substituting this value, and $dy = \frac{dv - adx}{v}$, &c.

817. Let us now consider the *linear* equation, or the one

$$dy + Pydx = Qdx,$$

in which y is of the 1st degree, and P and Q are functions of x: we shall assume $y = zt$, whence

$$zdt + tdz + Pzt\,dx = Qdx;$$

and since we may dispose arbitrarily of one of the indeterminate quantities z and t, we shall equate the coefficient of z to zero; when we have

$$dt + Ptdx = 0,\ tdz = Qdx.$$

The first of these gives $\frac{dt}{t} = -Pdx$, whence $lt = -\int Pdx = -u$, and since Pdx does not contain y, its integral u will easily be obtained. We consequently have

$$lt = -u + a, \text{ or } t = e^{-u+a} = e^a e^{-u} = Ae^{-u},$$

assuming the constant $e^a = A$. Substituting this value in $tdz = Qdx$, we have $Adz = Qe^u dx$; whence

$$Az = \int Qe^u dx + c.$$

Q and u are known functions of x, and having obtained the integral $\int Qe^u dx$, we must replace Az by its value $\frac{Ay}{t}$ or ye^u, which will finally give

$$ye^u = \int Qe^u dx + c, \text{ where } u = \int Pdx.$$

It follows from this calculation that it is unnecessary to add a constant a to the integral $\int Pdx = u$.

Take, for example, $dy + ydx = ax^3dx$: we have

$$P = 1,\ Q = ax^3,\ u = \int Pdx = x,$$

$$\int Qe^u dx = \int ax^3e^x dx = ae^x(x^3 - 3x^2 + 6x - 6);$$

and therefore $\quad y = ce^{-x} + a(x^3 - 3x^2 + 6x - 6).$

For the equation $(1 + x^2)\,dy - yxdx = adx$, we have

$$P = \frac{-x}{1+x^2},\ Q = \frac{a}{1+x^2},\ u = -\int \frac{xdx}{1+x^2} = -l\sqrt{(1+x^2)};$$

and therefore $e^u = (1 + x^2)^{-\frac{1}{2}}$ [See N°. 149, 12°];

$$\int Qe^u dx = \int \frac{adx}{(1+x^2)^{\frac{3}{2}}} = \frac{ax}{\sqrt{(1+x^2)}} + c \text{ [p. 347]};$$

and lastly $y = ax + c\sqrt{(1 + x^2)}$.

818. We shall conclude with the discussion of the equation of *Riccati*, so called because that mathematician was the first to direct his attention to it:

$$dy + by^2dx = ax^m dx.$$

1°. If $m = 0$, we have [p. 145]

$$dx = \frac{dy}{a - by^2} = \frac{1}{2\sqrt{a}}\left(\frac{dy}{\sqrt{a} + y\sqrt{b}} + \frac{dy}{\sqrt{a} - y\sqrt{b}}\right);$$

and therefore

$$2x\sqrt{(ab)} + c = l(\sqrt{a} + y\sqrt{b}) - l(\sqrt{a} - y\sqrt{b}).$$

2°. If m is not $= 0$, we assume $y = b^{-1}x^{-1} + zx^{-2}$, when we find

$$x^2dz + bz^2dx = ax^{m+4}dx;$$

this transformed equation is homogeneous when $m = -2$, and is integrated by separating the variables, if $m = -4$.

3°. In every other case, make $z = t^{-1}$, $x^{m+3} = u$; then, supposing that

$$n = -\frac{m+4}{m+3},\ b' = \frac{a}{m+3},\ a' = \frac{b}{m+3},$$

we have this equation, similar to the one proposed,

$$dt + b't^2du = a'u^n du;$$

which we can therefore treat in the manner above, and integrate when $n = -2$ or -4.

And if n be not -2 or -4, by effecting similar transformations, and repeating the same processes, we shall arrive at equations of the same forms as the one proposed, having for the variable, on the 2nd side, an exponent successively $= -\frac{m+4}{m+3}$, $-\frac{n+4}{n+3}$, $-\frac{p+4}{p+3}$, *i. e.* this

exponent is

$$= -\frac{m+4}{m+3}, \quad \frac{3m+8}{2m+5}, -\frac{5m+12}{3m+7}, -\frac{7m+16}{4m+9}\cdots$$

Let one of these fractions be 0, or -2, or -4, and the integral will be easily found; viz. we must have $m = \frac{-4i}{2i-1}$, i being any positive integer, or zero.

If we had commenced with making $y = t^{-1}$, $x^{m+1} = z$, in the proposed equation, the same calculation would have brought us to the conclusion that the integration was possible when $m = \frac{-4i}{2i+1}$: thus $m = \frac{-4i}{2i \pm 1}$ is the condition of integrability of Riccati's equation.

ON THE FACTOR PROPER FOR RENDERING A FUNCTION INTEGRABLE.

819. The equation $Mdy + Ndx = 0$ does not always result immediately from the differentiation of an equation $f(x, y) = 0$; for, we may, after this operation, have multiplied or divided the whole equation by some function, or have eliminated a constant [N°. 687] by means of $f(x, y) = 0$, or lastly have combined these equations in some arbitrary manner with each other. The proposed equation therefore may not in all cases be an *exact differential.*

Generally, let $u = f(x, y)$: the differential being $du = Mdy + Ndx$, the relation $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$ here becomes

$$\frac{dM}{dx} = \frac{dN}{dy} \cdots\cdots (1).$$

Thus, *whenever $Mdy + Ndx$ is an exact differential, the condition (1) must be fulfilled: and, conversely, if M and N satisfy the condition (1), $Mdy + Ndx$ is an exact differential which it will always be possible to integrate.*

To demonstrate this converse case, let Mdy be integrated considering x as constant, and let P be the integral, a known function of x and y, which results from $\int Mdy$, taken relatively to y alone, or $M = \frac{dP}{dy}$. Taking for the arbitrary constant a quantity X, which may contain x, we shall have $P + X$ for the integral of Mdy relatively to y; and it

is now to be proved that $P + X$ is also the integral of $Mdy + Ndx$, when the equation (1) is satisfied.

The complete differential of $P + X$ is

$$\frac{dP}{dx}dx + \frac{dP}{dy}dy + dX \text{ or } \frac{dP}{dx}dx + Mdy + dX;$$

whence it follows that $P + X$ will be the integral of $Mdy + Ndx$ (which will consequently be an exact differential), if X can be so determined that this trinomial shall be $= Mdy + Ndx$, or

$$Ndx = \frac{dP}{dx}dx + dX \text{ or } dX = \left(N - \frac{dP}{dx}\right)dx \ldots (2).$$

But, differentiating $M = \frac{dP}{dy}$ in respect to x, we find, by virtue of the assumed condition (1),

$$\frac{dM}{dx} = \frac{d^2P}{dydx} = \frac{dN}{dy}, \text{ or } \frac{dN}{dy} - \frac{d^2P}{dydx} = 0,$$

or $0 = d\left(N - \frac{dP}{dx}\right)$ relatively to y; and $N - \frac{dP}{dx}$ is therefore a function of x alone, which was the point to be demonstrated.

Thus, the integral required is $P + X$, P being that of Mdy in respect to y alone, and X the integral of the function of x given by the equation (2); and we consequently have demonstrated our converse case, and have at the same time given a process for the integration of $Mdy + Ndx$.

It is almost unnecessary to say that we might equally have commenced with integrating Ndx, y being constant, and have completed the integral with a function Y of y, &c. We shall prefer one course or the other, as seems most likely to facilitate the calculation.

I. Let it be proposed to integrate $\frac{dx}{\sqrt{(1 + x^2)}} + adx + 2bydy$: here $M = 2by$, $N = \frac{1}{\sqrt{(1 + x^2)}} + a$; and we find $P = by^2$: thus $by^2 + X$ is the integral required, since the condition (1) is fulfilled. The differential of $by^2 + X$ relative to x, compared with Ndx, gives [p. 343].

$dX = \frac{dx}{\sqrt{(1 + x^2)}} + adx$, whence $X = ax + l.c\,[x + \sqrt{(1 + x^2)}]$;
and we therefore have $by^2 + ax + l.c\,[x + \sqrt{(1 + x^2)}]$.

II. Similarly, for $\frac{a(xdx + ydy)}{\sqrt{(x^2 + y^2)}} + \frac{ydx - xdy}{x^2 + y^2} + 3by^2dy$,

$$M = \frac{ay}{\sqrt{(x^2+y^2)}} - \frac{x}{x^2+y^2} + 3by^2,\ N = \frac{ax}{\sqrt{(x^2+y^2)}} + \frac{y}{x^2+y^2}:$$

having ascertained that the equation (1) is satisfied, we shall integrate Ndx in respect to x; this will give

$$a\sqrt{(x^2+y^2)} + \text{arc}\left(\tan = \frac{x}{y}\right) + Y,$$

Y denoting a function of y; and differentiating this expression in respect to y, and comparing the result with Mdy, we shall have $dY = 3by^2dy$, whence $Y = by^3 + c$. Thus the integral is obtained in its complete form. Making $a = b = 0$, we find

$$\int \frac{ydx - xdy}{x^2+y^2} = \text{arc}\left(\tan = \frac{x}{y}\right) + c.$$

This integral, employed by M. Laplace [*Mec. cel.* t. i. p. 6], is a particular case of the one preceding [See N°. 704].

III. We shall similarly find

$$\int \frac{dx\,[x + \sqrt{(x^2+y^2)}] + ydy}{[x+\sqrt{(x^2+y^2)}]\sqrt{(x^2+y^2)}} = lc\,[x + \sqrt{(x^2+y^2)}].$$

820. When $Mdy + Ndx$ does not satisfy the condition of integrability, it may be proposed to find whether, by multiplying this expression by some function z of x and y, we can render it an exact differential. The equation $Mdy + Ndx = 0$ results from the elimination of a constant between the primitive $f(x, y, c) = 0$ and its immediate derivative. Let the two former of these equations be put under the forms $y' + K = 0$, $c = \varphi(x, y)$, which is always allowable, K representing some function of x and y. The derivative of $c = \varphi(x, y)$ being $\varphi' = Py' + Q = 0$, we have $y' + \dfrac{Q}{P} = 0$; and, since the constant c no longer enters here, this expression [N°. 687] is identical with $y' + K$, or $y' + K = \dfrac{Py' + Q}{P} = \dfrac{\varphi'}{P}$, whence we have $\varphi' = P(y' + K)$; and since the two sides of this are identical, and φ' is an exact derivative, $P(y' + K)$ must equally be one, which proves that there *is always a factor P proper for rendering integrable the function $y' + K$, as also every differential equation of the first order between x and y.*

Let us investigate this factor, which we shall represent by z.

$Mzdy + Nzdx$ cannot be an exact differential unless that

$$\frac{d(Mz)}{dx} = \frac{d(Nz)}{dy},\ \text{or}\ z\left(\frac{dM}{dx} - \frac{dN}{dy}\right) = N.\frac{dz}{dy} - M\frac{dz}{dx}\ldots\ (3).$$

This equation of partial differentials is seldom of use, on account of the difficulty of the calculations; but some remarkable properties may be deduced from it.

1°. If the integral u of $z\ (Mdy + Ndx)$ be known, the factor z will easily be found; for, comparing $\frac{du}{dx}dx + \frac{du}{dy}dy$ with $z\ (Mdy + Ndx)$, which is identical with it, we shall readily deduce z.

2°. Multiplying the equation $du = z\ (Mdy + Ndx)$ by any function of u, as ϕu, we have

$$\phi u.du = z\phi u\ (Mdy + Ndx).$$

But, the first side being an exact differential, the second, which is identical with it, must possess the same property; whence it follows that there is an infinite number of factors $z.\phi u$ proper for rendering integrable any function of x and y, and that the knowledge of one of these, as z, will suffice for obtaining an infinite number of others $z.\phi u$.

3°. If the factor z contain only one of the variables x and y, it is easily found; for let z be a function of x alone, then the equation (3) reduces itself to

$$\frac{dz}{z} = \frac{dx}{M}\left(\frac{dN}{dy} - \frac{dM}{dx}\right) \dots (4),$$

since $\frac{dz}{dy} = 0$, and $\frac{dz}{dx}$ is no longer a partial differential. The integration of this equation will give z; for the hypothesis requires that the 2nd side be independent of y; this will in fact be self-apparent if the supposition be a legitimate one

Similarly, if z be a function of y alone, we have

$$\frac{dz}{z} = \frac{dy}{N}\left(\frac{dM}{dx} - \frac{dN}{dy}\right) \dots (5);$$

and the 2nd side must be independent of x. It will be observed that, in the equations (4) and (5), the part included within the parentheses is nothing, when $Mdy + Ndx$ is an exact differential [See N^{os}. 824, 6°, and 828].

I. Take, for example, $dx + (adx + 2bydy)\sqrt{(1 + x^2)} = 0$; the condition of integrability is not fulfilled, since

$$\frac{dN}{dy} - \frac{dM}{dx} = -\frac{2byx}{\sqrt{(1 + x^2)}};$$

but this quantity, divided by M or $2by\sqrt{(1+x^2)}$, gives for the quotient this function of x, $\frac{-x}{1+x^2}$; and the equation therefore can be rendered integrable by means of a factor, which will be a function of x. The equation (4) gives

$$lz = \int \frac{-xdx}{1+x^2} = -\tfrac{1}{2}l(1+x^2) = -l\sqrt{(1+x^2)};$$

and consequently $z = \frac{1}{\sqrt{(1+x^2)}}$. The proposed equation then takes the form which has been treated of in N°. 819. 1.

II. The linear equation $dy + Pydx = Qdx$ gives $\frac{dM}{dy} - \frac{dM}{dx} = P$, so that the condition (1) is not fulfilled; but this function P, divided by $M = 1$, gives a function of x; thus $\frac{dz}{z} = Pdx$, whence

$lz = \int Pdx = u$, and $z = e^u$; and this is the factor which renders the proposed equation integrable. It thus becomes $e^u dy + e^u(Py - Q)dx = 0$; and it remains only to adopt the process of N°. 819. Integrating $e^u dy$ in respect to y, we have $e^u y + X$, the differential of which relatively to x, compared with $e^u(Py - Q)\,dx$, gives $dX = -e^u Qdx$; and consequently the integral required is, as we already know [N°. 817]

$$e^u y = \int Qe^u dx + c, \text{ where } u = \int Pdx.$$

III. Similarly, $x^3dy + \left(4x^2y - \frac{1}{\sqrt{(1-x^2)}}\right)dx = 0$ gives $lz = lx$; so that the proposed equation must be multiplied by x in order that it may be integrable: we finally find, for the integral,

$$x^4y + \sqrt{(1-x^2)} = c.$$

IV. The factor proper for rendering the homogeneous functions integrable is easily found. Let m be the degree [N°. 322] of such a function F of the variables $x, y\ldots$; if these variables be replaced by $lx, ly\ldots$, l being any number whatever, F will become l^mF; and making $l = 1 + h$, it becomes

$$(1+h)^mF = F\left(1 + mh + m.\frac{m-1}{2}h^2\ldots\right).$$

On the other hand, $x, y\ldots$ are become $x + hx$, $y + hy\ldots$, and the function F of $(x + hx)$, $(y + hy)\ldots$, developed according to the theorem of N°. 703, becomes

$$F + \frac{dF}{dx} hx + \frac{dF}{dy} hy + \frac{d^2F}{dx^2} \frac{h^2x^2}{2} + \frac{d^2F}{dx\,dy} h^2xy + \frac{d^2F}{dy^2} \frac{h^2y^2}{2} \ldots$$

Comparing the similar powers of h, in these two developments, we find

$$mF = \frac{dF}{dx} x + \frac{dF}{dy} y \ldots,$$

$$m(m-1) F = \frac{d^2F}{dx^2} x^2 + \frac{d^2F}{dxdy} 2xy + \frac{d^2F}{dy^2} y^2 + \ldots$$

821. To apply this theorem to $Mdy + Ndx$, M and N being homogeneous functions of the degree p, let us inquire whether there is a homogeneous factor z, which will render $zMdy + zNdx$ an exact differential. Let n be the degree of z: since then Nz is homogeneous and of the order $p + n$, the property above gives

$$(p + n) Nz = x \frac{d(Nz)}{dx} + y \frac{d(Nz)}{dy}:$$

but, by the supposition, $$\frac{d(Mz)}{dx} = \frac{d(Nz)}{dy};$$

and, substituting this value of the last term of the preceding expression, there results

$$(p + n)Nz = x \frac{d(Nz)}{dx} + \frac{d(Myz)}{dx} = \frac{d(Nxz + Myz)}{dx} - Nz,$$

or $$(p + n + 1) Nz = \frac{d[z(My + Nx)]}{dx},$$

an equation which is satisfied by making $z = \dfrac{1}{My + Nx}$; for then $p = -n - 1$. Consequently $\dfrac{Mdy + Ndx}{My + Nx}$ is integrable; and the integration presents no farther difficulty [N°. 819].

We find that $xdy - dx[y + \sqrt{(x^2 + y^2)}] = 0$ requires to be divided by $x\sqrt{(x^2 + y^2)}$; $\dfrac{dy}{\sqrt{(x^2 + y^2)}}$ being then integrated in respect to y, we have $l[y + \sqrt{(x^2 + y^2)}]$ [N°. 773]; adding X, differentiating in respect to x, and comparing, there results

$$dX = -dx\left(\frac{y + \sqrt{(x^2 + y^2)}}{x\sqrt{(x^2 + y^2)}} + \frac{x}{\sqrt{(x^2 + y^2)}[y + \sqrt{(x^2 + y^2)}]}\right)$$

$$= -2dx\left(\frac{x^2 + y^2 + y\sqrt{(x^2 + y^2)}}{x\sqrt{(x^2 + y^2)}[y + \sqrt{(x^2 + y^2)}]}\right) = -\frac{2dx}{x};$$

thus $X = lc - lx^2$, and the integral required is

$$cy + c\sqrt{(x^2 + y^2)} = x^2,$$

as in N°. 815, III.

822. It is sometimes found necessary to differentiate, relatively to y, functions, such as $u = \int M dx$, which are affected with the sign of integration in respect to x; and we then differentiate under the sign $\int$. In fact, since we have

$$\frac{du}{dx} = M, \ \frac{d^2u}{dy\,dx} = \frac{d^2u}{dx\,dy} = \frac{dM}{dy};$$

and integrating in respect to x, we find $\frac{du}{dy} = \int \frac{dM}{dy} dx$.

ON PARTICULAR OR SINGULAR SOLUTIONS.

823. Suppose that there be given a differential equation $V = 0$, which has for its complete integral $f(x, y, c) = 0$, c being the arbitrary constant. The immediate differential of this equation will be $Pdy + Qdx = 0$; and the proposed equation must result from the elimination of c between the two latter [N°. 687]. And so long as these two continue the same, the elimination of c between them must lead to the proposed one $V = 0$, whatever be the value assumed for c, in the two, even though c should be a function of x and y: this is manifest. Now, differentiating f in respect to x, y and c, we have

$$Pdy + Qdx + Cdc = 0,$$

which is reduced to $Pdy + Qdx = 0$, by assuming $Cdc = 0$; and therefore every value of c, which satisfies this condition, changes $f = 0$ into an equation $S = 0$, such that the differential is still $Pdy + Qdx = 0$: the elimination of c between the equations $Cdc = 0$, $f = 0$ will give back the proposed equation $V = 0$; and consequently $S = 0$ is a relation between x and y which satisfies the equation $V = 0$, and is an integral of it.

But $Cdc = 0$ gives

1°. $dc = 0$, $c =$ const., and the function f remains the same.

2°. $C = 0$ may give a constant and determinate value of c; and f then becomes *a particular integral*, which offers nothing remarkable:

this is a case included in the one preceding, a specific value being taken for c.

3°. C does not contain c, when c enters into f only in the 1st degree; and we cannot then assume $C = 0$, this equation not being capable of giving a value of c; or rather $C = 0$ gives a particular integral, which corresponds to $c = \infty$.

4°. $C = 0$, or $\frac{df}{dc} = 0$, may give for c a variable function, $c = \phi(x, y)$; and ϕ being substituted for c in $f = 0$, we shall have an equation $S = 0$, the differential of which will still be $Pdy + Qdx = 0$, on ϕ being eliminated.

Generally, S is not comprised in $f(x, y, c)$, since c cannot there receive any but constant values, whereas c is now become variable. Consequently, the equation $S = 0$, which does not contain any arbitrary constant, offers a relation between x and y, which satisfies the proposed equation $V = 0$, whilst at the same time it is not included in its general integral; and this is what is called a *singular or particular Solution*.

For example, the elimination of the constant c between the equation $y^2 - 2cy + x^2 = c^2$ and its derivative gives [N°. 687]

$$(x^2 - 2y^2)\, y'^2 - 4xyy' - x^2 = 0:$$

but if c be considered as the sole variable in the primitive equation, we shall have $c = -y$, which will change it into $x^2 + 2y^2 = 0$. It may easily be ascertained by trial that this equation satisfies our differential equation, though it is not included in its integral.

Similarly, $x^2 - 2cy - b - c^2 = 0$ has for its derivative, after the elimination of c,

$$y'^2(x^2 - b) - 2xyy' = x^2.$$

The derivative relative to c alone gives $y + c = 0$; whence $c = -y$; then $x^2 + y^2 = b$; and this is the singular solution of our derivative equation.

The equation $y = x + (c - 1)^2 \sqrt{x}$ gives $C = 2(c - 1)\sqrt{x} = 0$; whence $c = 1$, then $y = x$, a particular case of the complete integral; and which therefore is not a singular solution. This corresponds to 2°.

Lastly, the equation $y^2 + x^2 = 2cx$ gives $C = 2x = 0$, which, not containing c, gives only a particular integral relative to $c = \infty$ [See case 3°].

824. We shall here make some remarks.

1°. The singular solutions must be investigated with as much care as

the complete integrals, since they may contain the true solution of the problem, which has led to the differential equation that has been integrated.

2°. The equation $\frac{df}{dc}=0$ expresses the condition that $f(x,y,c)=0$ may have equal roots relative to c [N°. 524]. If, therefore, by means of the singular solution, we get quit of x or y, the complete integral of the resulting equation will have equal factors. Thus, in our 1st example, if we make $x^2=-2y^2$, the proposed equation becomes

$$y^2+2cy+c^2=(y+c)^2=0.$$

3°. Since the constant c is arbitrary, the complete integral $f(x,y,c)=0$ may be considered as the equation of an infinite number of curves, of which the parameter c is alone different. If, therefore, we assign to c all the values possible, these consecutive lines will cut each other, two and two, in a series of points, the system of which will form a curve that is a tangent to each of the lines. The equation $f(x,y,c)=0$ belongs to one of our curves, and also to the curve which touches them all; only c is constant in the 1st case, whatever x and y be, whilst in the 2nd, c is a variable function of the co-ordinates of the point of contact. The tangent at this point is the same for both curves; and therefore y', by which this tangent is determined, must preserve the same value, whether c be constant or variable in $f(x,y,c)=0$; whence it follows that if c be eliminated between $f=0$ and $\frac{df}{dc}=0$, the resulting equation in x and y, which is *the singular solution, belongs to the line of contact of all the curves included in the complete integral* [See N°. 765].

4°. The equation $f(x,y,c)=0$ being resolved in respect to c, let $c=\psi(x,y)$. If $\psi(x,y)$ were substituted for c in $f(x,y,c)=0$, the result would be identically nothing, as also all the derivatives relative, either to x, or to y; and we therefore have [N°. 672]

$$\frac{df}{dx}+\frac{df}{dc}\cdot\frac{dc}{dx}=0, \text{ whence } \frac{dc}{dx}=-\frac{df}{dx}:\frac{df}{dc}:$$

thus $\frac{df}{dc}=0$ gives $\frac{dc}{dx}=\infty$; and similarly $\frac{dc}{dy}=\infty$.

This property, peculiar to the singular solutions, offers another method of obtaining them.

From $x^2-2cy-c^2-b=0$, we derive

$$c=-y+\sqrt{(x^2+y^2-b)},\ \frac{dc}{dx}=\frac{x}{\sqrt{(x^2+y^2-b)}};$$

and therefore $x^2 + y^2 = b$, which renders this fraction infinite, is the singular solution.

In thus assuming $\frac{dc}{dx}$ or $\frac{dc}{dy} = \infty$, it will be requisite for us to ascertain that the relation between x and y, which thence results, combined with the proposed equation, do not give $\phi(x, y) =$ constant; for in that case we should but have a particular integral.

5°. The singular solutions owe their existence to the fact of the equation $\frac{df}{dc} = C = 0$ giving for c a variable value $c = \phi(x, y)$: but it is possible that the function ϕ may be reducible to a constant, by virtue of the complete integral $f(x, y, c) = 0$, or that f contain c under the form $(c - a)(c - \phi)$, so that $c = \phi$ will come to the same thing with $c = a$; and in that case we should have nothing more than a particular integral, as though a determinate value had been taken for c. Hence, *that $C = 0$ may give a singular solution, it is incumbent that there do not thence result for c either a constant, or even a variable function ϕ, if, when it is substituted in f, the result be the same as though a constant value had been taken for c.*

For example,

$$(x^2 + y^2 - b)(y^2 - 2cy) + (x^2 - b)c^2 = 0 \text{ gives}$$

$$C = -y(x^2 + y^2 - b) + (x^2 - b)c = 0;$$

whence $\qquad c = \dfrac{y(x^2 + y^2 - b)}{x^2 - b}$ and $y^2(y^2 + x^2 - b) = 0,$

an equation which is no more than a particular integral arising from $c = 0$.

Similarly $c^2 - (x + y)c - c + x + y = 0$ gives

$$C = 2c - x - y - 1 = 0, c = \tfrac{1}{2}(x + y + 1):$$

the proposed equation, which is equivalent to $(c - 1)(c - x - y) = 0$, becomes $(x + y - 1)^2 = 0$; and thus we have $x + y = 1$, the particular integral arising from $c = 1$, after having divided by $c - 1$.

The equation $y = x + (c - 1)^2 (c - x)^2$ gives

$$C = (c - x)(c - 1)(2c - x - 1) = 0:$$

$c = 1$ gives the particular integral $y = x$; $c = x$ gives the same thing, and not a singular solution, though c is variable.

Lastly, $c = \frac{1}{2}(x + 1)$ gives the singular solution.

6°. Let z be the multiplier which renders the equation $y' + K = 0$

an exact derivative, so that $z(y' + K) = \phi' = 0$ have for its primitive $\phi(x, y) = c$: the singular solution $S = 0$ must not be included in this equation; and consequently, if from $S = 0$, we deduce y in a function of x, $y = \psi x$, the substitution of this value of y in the function $\phi(x, y)$ must not reduce it to a constant; and therefore its derivative ϕ' must not be nothing.

It appears, therefore, that of the two expressions $y' + K$, and ϕ' or $z(y' + K)$, one must become nothing by virtue of $y = \psi x$, whilst the other must not; a circumstance which cannot take place unless z be infinite. Hence it follows, that the singular solutions render infinite all the factors proper for making the proposed differential equation integrable; or rather, the singular solutions of this equation are no other than the algebraic factors, which may be exhibited, and entirely separated from this equation by means of a suitable transformation.

[See a Memoir by M. Poisson, 13[e] *Journ. Polyt.* where it is demonstrated that we can always clear an equation of the 1st order of its particular solution, or introduce one at pleasure].

825. Suppose that $y = X$ satisfies a proposed equation $y' = F(x, y)$, X being a known function of x, and that we have

$$X' = F(x, X) \ldots (1):$$

let us look out for a mode of distinguishing whether $y = X$ is a singular solution, or a particular integral, X not containing any arbitrary constant. Let $y = \psi(x, a)$ be the complete integral of $y' = F(x, y)$, a being the arbitrary constant: if $y = X$ be a particular case of $y = \psi(x, a)$, so that $\psi(x, a)$ become X when to a we assign a value b, it follows that $\psi(x, a) - X$ must be zero for $a = b$; and consequently [N°. 500]

$$\psi(x, a) - X = (a - b)^m z,$$

m being the highest power of $a - b$, and z a function of x and a which does not become 0, or ∞, for $a = b$. Let the constant $(a - b)^m$ be represented by c; then the complete integral of $y' = F(x, y)$ will be

$$y = X + cz;$$

and if this value of y be substituted in $y' = F(x, y)$, the following relation will become identical,

$$X' + cz' = F(x, X + cz).$$

But, on the one hand, the development of z according to the ascending powers of c has [N°. 698] the form $z = K + Ac^a + Bc^b + \ldots$, the

exponents $a, b\ldots$ being increasing and positive, and $K, A, B\ldots$ functions of x; for z is not either ∞, or 0, when $c = 0$. Hence

$$X' + cz' = X' + K'c + A'c^{a+1} + \ldots$$

On the other hand, the development of $F(x, X + cz)$ must similarly be $F(x, X) + Nc^n z^n + Mc^m z^m + \ldots$ $n, m\ldots$ being increasing and positive. This series is easily obtained [N°. 706], and the numbers $n, m\ldots$ may be considered as known, as also the functions of x denoted by $N, M\ldots$ If, therefore, in this expression, we substitute for z its developed value, we have, by virtue of (1),

$$\begin{aligned} K'c + A'c^{a+1} + \ldots = \; & Nc^n(K + Ac^a + \ldots)^n \\ & + Mc^m(K + Ac^a + \ldots)^m \\ & + \&c. \end{aligned}$$

The point, therefore, to be ascertained is, whether it is possible to determine z, or rather the coefficients $A, B\ldots$ in functions of x, and the numbers $a, b\ldots$, in such a manner as to render this equation identical; for, if this be not possible, $y = X$ is a singular solution; in the contrary case, we have a particular integral.

Three cases present themselves:

1°. If $n > 1$, the term $K'c$ does not meet with a similar one by which it may be destroyed; and we shall therefore make $K' = 0$, whence $K =$ constant. We shall then assume $a + 1 = n$, $A' = NK^n$, which will determine $a = n - 1$, and $A = \int NK^n dx$; and so on for the other terms. The identity therefore will be always possible, and $y = X$ will be a particular integral.

2°. If $n = 1$, the case will be the same; for, assuming $K' = NK$, we shall have $lK = \int Ndx$: it will be easy then to arrange the two sides, and to compare the respective exponents and coefficients of the corresponding ranks; which will determine the exponents $a, b\ldots$ and the coefficients $K, A, B\ldots$

3°. And lastly, if $n < 1$, there will not be found for the term $Nc^n K^n$ any other that is similar to it, since there is no exponent of c which is < 1 on the 1st side: and since K cannot be nothing, there will be no possible mode of satisfying the identity: $y = X$ will therefore be a singular solution.

826. Since in this last case n is < 1, if, $X + cz$ being substituted for y in $F(x, y)$, the development of Taylor be faulty between the 1st and the 2nd term, *i.e.* if the derivative of $F(x, y)$ relative to y be infinite

[N°. 696, 3°], $y = X$ is a singular solution. And conversely, a value $y = X$ which satisfies $y' = F(x, y)$, and renders $\frac{dF}{dy}$ infinite, is a singular solution, since it gives to the development of $F(x, X + cz)$ the form $X' + Nc^n K^n \ldots$, n being < 1. Hence the condition $\frac{dF}{dy}$ or $\frac{dy'}{dy} = \infty$. constitutes the true characteristic of the singular solutions; and, in order that this condition may be fulfilled, we have seen [N°. 699, 3°], that, if the function F be algebraic, it must contain a radical which the hypothesis of $y = X$ causes to disappear. In the 2nd of our examples [p. 386] we have

$$y' = \frac{x[y \pm \sqrt{(x^2 + y^2 - b)}]}{x^2 - b}, \frac{dy'}{dy} = \frac{x}{x^2 - b}\left(1 \pm \frac{y}{\sqrt{(x^2 + y^2 - b)}}\right);$$

and this last fraction is rendered infinite by the singular solution $y^2 = b - x^2$.

827. *It is easy, therefore, to obtain the singular solutions without knowing the complete integral:* for, having deduced the value of $\frac{dy'}{dy}$, we shall equate it to infinity, *i. e.* if $\frac{dy'}{dy} = \frac{U}{T}$, we shall make $T = 0$, or $U = \infty$.

Considering all the factors of these equations, the results which satisfy $y' = F(x, y)$ will constitute all the singular solutions.

For $y' = a(y - n)^k$, we have $ak(y - n)^{k-1} = \infty$, which requires that k be < 1 and $y = n$; and since the proposed equation is not satisfied by $y = n$ except when k is positive, it appears that it is not susceptible of a singular solution unless k be between 0 and 1.

The complete integral is $\frac{(y - n)^{1-k}}{1 - k} = ax + c$.

It is not necessary, in order to apply our theorem, that the derivative equation should be brought under the explicit form $y' = F(x, y)$: for, let $V = 0$ be the given relation between x, y and y'; we may consider y' as a function of x and y, which this equation determines; and thus, the partial difference of y' relative to y will be given [N°. 672] by

$$\frac{dV}{dy} + \frac{dV}{dy'} \cdot \frac{dy'}{dy} = 0:$$

but, $\frac{dy'}{dy}$ is infinite when $\frac{dV}{dy'} = 0$, or $\frac{dV}{dy} = \infty$;

so that we shall in this way obtain all the singular solutions. In case the function V be algebraic, rational and integral, this latter condition will not be possible; and y' must then be eliminated between $V=0$ and $\frac{dV}{dy'}=0$. We must also take those factors only of the last equation, which are not common to $\frac{dV}{dy'}$ and $\frac{dV}{dy}$.

This method will make known those alone of the singular solutions which contain y; those which contain only x do not come within the reach of the rule: to obtain them, we must reason similarly in regard to x; and we shall thus find, besides the solutions already known into which x and y enter, those which do not depend on y.

1°. Thus, $(x^2-2y^2)\,y'^2-4xyy'-x^2=0$ gives

$$(x^2-2y^2)\,y'-2xy=0,$$

differentiating in respect to y' alone; and y' being eliminated between these equations, we find the singular solution, which is $x^2+2y^2=0$.

2°. Similarly, $xdx+ydy=dy\sqrt{(x^2+y^2-c^2)}$,

or $$x^2+2xyy'+y'^2(c^2-x^2)=0,$$

gives $$xy+y'(c^2-x^2)=0, \text{ and } x^2+y^2=c^2.$$

3°. For $ydx-xdy=ads$, where $ds=a\sqrt{(dx^2+dy^2)}$, we find

$$y^2-a^2=2xyy'+y'^2(a^2-x^2), \text{ whence } xy=y'(x^2-a^2);$$

and y' being then eliminated, we have, for the singular solution,

$$x^2+y^2=a^2.$$

4°. That of $y=xy'+Y'$, where Y' is any function of y', is obtained by eliminating y' by means of $x+\frac{dY'}{dy'}=0$.

828. Since, without knowing the complete integral of a derivative equation $V=0$, we can find the singular solutions, and the factor z, proper for rendering the proposed equation integrable, is then infinite [N°.824,6°], we can frequently, by means of analytical artifices, discover this factor z.

An example taken from the *Memoir of Trembley* [Acad. Turin, 1790—91] will suffice for giving an insight into this mode.

In the 3rd example, we have found $x^2+y^2-a^2=0$ for the singu-

lar solution; and the proposed equation, resolved in respect to y', gives

$$(a^2 - x^2)y' + xy = a\sqrt{(y^2 + x^2 - a^2)},$$

which is evidently satisfied by $x^2 - a^2 = 0$. We shall now try whether the factor z has the form $(x^2 - a^2)^m (x^2 + y^2 - a^2)^n$, m and n being indeterminate; for which purpose, we shall multiply the equation above by this function, and assume the condition (1) [N°. 819], when we shall see that it is fulfilled by taking $m = -1$ and $n = -\frac{1}{2}$; so that the factor which renders the proposed equation integrable is

$$(x^2 - a^2)^{-1} (x^2 + y^2 - a^2)^{-\frac{1}{2}}$$

EQUATIONS IN WHICH THE DIFFERENTIALS EXCEED THE FIRST DEGREE.

829. Let us investigate the integral of $F(x, y, y', y'^2 \ldots y'^m) = 0$. Since this equation cannot arise but from the elimination of a constant c between the integral equation and its immediate derivative, into which c enters in the power m, let $c = \varphi(x, y)$ be the value of this constant deduced from the integral; $\varphi'(x, y) = 0$ will contain y' in only the 1st degree, and we shall be able thence to deduce $y' = X$, X containing x and y affected with radical signs: and since, when these signs are done away with by means of involution, we must arrive at the proposed equation $F = 0$, it follows that $y' - X$ must be a factor of F.

We see, therefore, that if the proposed equation be solved in respect to y', and its factors $y' - X = 0$, $y' - X_{,} = 0, \ldots$ be integrated, these integrals will be those of the proposed equation which correspond to the different values of $c = \varphi(x, y)$. Let $P = 0$, $Q = 0$, $R = 0 \ldots$ be these integrals; their products $PQ = 0$, $PQR = 0 \ldots$, will also satisfy the proposed equation, for the derivative of the product $PQR \ldots$ being $P'QR \ldots + PQ'R \ldots + PQR' \ldots +$ &c., each term is individually nothing.

For example, $yy'^2 + 2xy' = y$ gives

$$y' = \frac{-x \pm \sqrt{(y^2 + x^2)}}{y}, \text{ whence } \frac{yy' + x}{\sqrt{(y^2 + x^2)}} = \pm 1;$$

and since the 1st side is obviously [N°. 769, IV] the derivative of $\sqrt{(y^2 + x^2)}$, we have for the integral

$$\pm \sqrt{(y^2 + x^2)} = x + c, \text{ or } y^2 = 2cx + c^2.$$

830. There are some cases in which, by means of artifices of calculation, we can avoid the necessity of solving the equations in respect to y': the two following examples are instances of this.

I. Suppose that the equation contain only x and y', and that it is easily solved in respect to x, so that we have $x = Fy'$. Since $dy = y'dx$ gives [N°. 769, V] $y = xy' - \int xdy'$, substituting for x its value Fy', we have

$$y = y'.Fy' - \int Fy'.dy';$$

and having integrated $\int Fy'.dy$, which comes under the Method of Quadratures, we must eliminate y' by means of the proposed equation $x = Fy'$.

Thus, for $(1 + y'^2)\, x = 1$, we have

$$Fy' = \frac{1}{1 + y'^2},\ y = \frac{y'}{1 + y'^2} - \int \frac{dy'}{1 + y'^2};$$

the last of these terms is $=$ arc $(\tan = y') + c$; and eliminating y, we finally find, for the integral required,

$$y = \sqrt{(x - x^2)} - \text{arc}\left(\tan = \sqrt{\frac{1 - x}{x}}\right) + c.$$

II. If the equation have the form $y = y'x + Fy'$, differentiating, we get

$$dy = y'dx + \left(x + \frac{dF}{dy'}\right) dy',\ \text{or}\ \left(x + \frac{dF}{dy'}\right) dy' = 0,$$

since $dy = y'dx$: equating each factor to 0, there results $y' = c$ and $x + \dfrac{dF}{dy'} = 0$; and it only remains to eliminate y' between the proposed equation and one or other of the two last. The latter of them gives only a singular solution [N°. 827, 4°]: the 1st leads to the complete integral $y = cx + C$, denoting by C what Fy' becomes when y' is replaced in it by c, or $C = Fc$.

Thus, $ydx - xdy = a\sqrt{(dx^2 + dy^2)}$ comes under the form

$$y = y'x + a\sqrt{(1 + y'^2)};$$

whence

$$y' = c \text{ and } x + \frac{ay'}{\sqrt{(1 + y'^2)}} = 0:$$

the 1st gives for the complete integral $y = cx + a\sqrt{(1 + c^2)}$; the 2nd leads to the singular solution $y^2 + x^2 = a^2$, when we thence deduce the value of y' in order to substitute it in the proposed equation.

ARBITRARY CONSTANTS; INTEGRATION OF DIFFERENTIAL EQUATIONS BY MEANS OF SERIES, AND THE CONSTRUCTION OF THEM.

831. Recurring to the series of Maclaurin [N°. 706], it gives

$$y = fx = f + xf' + \tfrac{1}{2}x^2 f'' + \&c,$$

in which f, f', f''... are the constant values assumed by $fx, f'x, f''x$... when we make $x = 0$. If now the given derivative equation be of the 1st order, we can thence deduce y', y'', y'''... in functions of y and x, by means of successive derivations; and since $x = 0$ corresponds to $y = f$, if these two values be substituted in y', y''..., we shall have those of f', f''..., and, consequently, every thing will be known in our series except f, which will continue arbitrary.

Similarly, if the given derivative be of the 2nd order, we can from it deduce y'', y'''... in functions of x, y and y': but $x = 0$ corresponds to $y = f$ and $y' = f'$; and these values being substituted in those of y'', y'''..., our series will become entirely known, except as to the constants f and f' which are any whatever.

And so on for the higher orders.

This mode of integration cannot be employed when, on making $x=0$, we meet with infinity in $fx, f'x, f''x$..., and the series of Maclaurin no longer subsists. If however we make $x = a$ in that of Taylor, a being any number whatever, which does not render any one of these functions infinite [N°. 695], denoting the values which they then assume by A, A', A''..., we have

$$f(a+h) = A + A'h + \tfrac{1}{2}A''h^2 + \tfrac{1}{6}A'''h^3 \ldots;$$

whence, assuming the arbitrary quantity $h = x - a$,

$$y = fx = A + A'(x-a) + \tfrac{1}{2}A''(x-a)^2 + \tfrac{1}{6}A''' \ldots;$$

and the same reasoning as above will show that every thing is here known, except the constant A, if the proposed equation be of the 1st order; except A and A', if it be of the 2nd, &c. Though we have assumed a at pleasure, this letter is not to be reckoned as an arbitrary constant; A is the value that is then assumed by y, and which supplies its place. Hence we conclude that 1°. *for any differential equation between two variables there always exists a series which is its integral; and this series can be found, barring the difficulties that may be presented by the Calculus.*

2°. *The integral always contains as many arbitrary constants as there are units in the order of the derivative.* Though founded on the theory

of series, this conclusion has all requisite rigour, since every series may be considered as the development of a finite expression $y = fx$, which must contain as many arbitrary constants as the series.

3°. *In whatever manner we have arrived at an integral, which contains the requisite number of arbitrary constants, this equation will be the primitive of the one proposed, and will necessarily include every other integral which also satisfies it with the same number of arbitrary constants.*

832. Making $h = -x$ in

$$f(x+h) = y + y'h + \tfrac{1}{2}y''h^2 + \ldots,$$
$$f'(x+h) = y' + y''h + \tfrac{1}{2}y'''h^2 + \ldots,$$
$$f''(x+h) = y'' + y'''h + \tfrac{1}{2}y^{\text{IV}}h^2 + \ldots, \text{ \&c.},$$

we have

$$(1)\ldots f = y - y'x + \tfrac{1}{2}y''x^2 - \ldots,$$
$$(2)\ldots f' = y' - y''x + \tfrac{1}{2}y'''x^2 - \ldots,$$
$$(3)\ldots f'' = y'' - y'''x + \tfrac{1}{2}y^{\text{IV}}x^2 - \ldots, \text{ \&c.}$$

Hence,

1°. If the given derivative equation be of the 1st order, we shall have y', $y''\ldots$ in functions of x and y; so that, substituting in the formula (1), we shall get the integral, f being the arbitrary constant.

2°. If the proposed equation be of the 2nd order, y'', $y'''\ldots$ will be given in terms of x, y and y'; so that, substituting them in (1) and (2) we shall have two equations between x, y and y', each containing an arbitrary constant, which will form two integral equations of the 1st order: and so on.

It is likewise evident, from the form itself of these integrals, that they are different: and thus, *every equation of the nth order has n integrals of the order* $(n-1)$. Should these latter be known, the finite integral would readily be so too, since it would but be necessary to eliminate y', y'', $y'''\ldots y^{n-1}$ between them. Hence, having a derivative equation of the 2nd order, we shall likewise have the absolute primitive; either by eliminating y' between its two derivatives of the 1st order, or by investigating a finite relation between x and y, which contains two arbitrary constants and satisfies the equation proposed. The same may be said in regard to the other orders.

We might, on the subject of the integration of equations of the higher

orders, proceed to demonstrate various theorems relative to the factors proper for rendering them integrable and to their singular solutions; but this would carry us beyond the limits which we have prescribed to ourselves, and we shall refer to the XIIth *Journ. Polyt.* lectures 13, 14 and 15, by Lagrange.

833. The theory that has now been explained is completely demonstrated; but it is not always adapted for making known the approximate integral, unless recourse be had to certain transformations which will bring the function into such a state that the preceding principles can be applied.

When the integral is not to proceed according to the integral and positive powers of x, we shall have

$$y = Ax^a + Bx^b + Cx^c + \dots \ (1);$$

and the point will be, to determine the exponents $a, b, c\dots$, and the coefficients $A, B, C\dots$. For this purpose, we shall deduce the values of $y', y''\dots$, and substitute them in the proposed derivative, which we suppose to be of the 1st order, and which must be rendered identical; having then arranged in respect to x, we must compare the powers of the corresponding terms, and also their coefficients, as in pages 163 and 176; which will determine $A, a, B, b\dots$

Thus, for $(1 + y')y = 1$, we shall have

$$(1 + Aax^{a-1} + Bbx^{b-1} + \dots)(Ax^a + Bx^b + \dots) = 1;$$

whence

$$\left.\begin{array}{llll} A^2ax^{2a-1} & + ABax^{a+b-1} & + ACax^{a+c-1} & + \dots \\ & + ABbx^{a+b-1} & + B^2bx^{2b-1} & + \dots \\ & & + ACcx^{a+c-1} & + \dots \\ -1 + Ax^a & & + Bx^b & + \dots \end{array}\right\} = 0.$$

Hence,

$$2a - 1 = 0,\ a + b - 1 = a,\ a + c - 1 = b = 2b - 1, \dots,$$

$$a = \tfrac{1}{2}, \qquad b = 1, \qquad c = \tfrac{3}{2}\dots;$$

next,

$$A^2a = 1,\ AB(a + b) + A = 0, \dots$$

$$A = \sqrt{2},\ B = -\tfrac{2}{3},\ C = \tfrac{1}{18}\sqrt{2}, \dots;$$

and $$y = x^{\frac{1}{2}}\sqrt{2} - \tfrac{2}{3}x^{\frac{2}{2}} + \tfrac{1}{18}x^{\frac{3}{2}}\sqrt{2} - \dots.$$

Could we have presumed on the law of the exponents, $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}\dots$, we

might have employed them immediately in the series (1), by which the calculations would have been simplified; or otherwise, making the transformation $z^2 = x$, we might then have applied the series of Maclaurin.

It will in like manner be seen that the equation $dy + ydx = ax^m dx$ gives

$$\frac{y}{a} = \frac{x^{m+1}}{m+1} - \frac{x^{m+2}}{(m+1)(m+2)} + \frac{x^{m+3}}{(m+1)\ldots(m+3)} - \ldots$$

834. The integral thus obtained fails in point of generality, in consequence of it being devoid of an arbitrary constant; if, however, in the proposed differential equation, we change x into $z + a$, and y into $t + b$, we shall have to develop t in powers of z; so that t be nothing when $z = 0$; and if then we substitute for z and t their values $x - a$ and $y - b$, we shall have the integral required, in which a and b will supply the place of the arbitrary constant c, since in the integral $f(x, y, c) = 0$, c may be determined in a function of a and b. It will be easy to extend these principles to the higher orders.

835. We may also approximate to the integrals by means of continued fractions. Adopting the notation of p. 113, let $y = Ax^a, Bx^b, Cx^c \ldots$ this value of y may be represented by $y = \frac{Ax^a}{1 + z}$, z denoting the rest of the continued fraction, or $z = Bx^b, Cx^c \ldots$. Substituting this value for y in the proposed differential equation, and neglecting z, *i.e.* making $y = Ax^a$, we must retain only the first terms, because x is considered as very small [note, p. 213]; and we shall thus find A and a by a comparison of the coefficients and the exponents. We shall now assume, in the proposed differential equation $y = \frac{Ax^a}{1 + z}$; and reasoning in a similar manner for the transformed equation in z, we shall make $z = Bx^b$; then, having found B and b, we shall assume $z = \frac{Bx^b}{1 + t}$ in the equation in z; and so on.

For example, $my + (1 + x)y' = 0$, making $y = Ax^a$, becomes $(m + a)Ax^a + aAx^{a-1} = 0$, which reduces itself to $aAx^{a-1} = 0$, on account of x being very small; consequently, $a = 0$, and A remains indeterminate. We then make $y = \frac{A}{1 + z}$, and we have

$$m(1 + z) = (1 + x)z';$$

whence, assuming $z = Bx^b$, we derive $m + Bx^b(m - b) = bBx^{b-1}$;

or $m = bBx^{b-1}$; and therefore $b = 1$, $B = m$. We shall then assume $z = \frac{mx}{1+t} \dots$; and we shall finally obtain this continued fraction for the integral:

$$y = A, mx, -\tfrac{1}{2}(m-1)x, \tfrac{1}{6}(m+1)x, -\tfrac{1}{6}(m-2)x, \dots$$

Since the equation proposed has for its integral $y = A(1+x)^{-m}$, we thus have the development of this function in the form of a continued fraction.

We might hence deduce the integral under the form of a series [see note p. 117].

Similarly, the equation $dx = (1 + x^2)\,dy$ gives this development of the arc in a function of the tangent

$$y = \text{arc}(\tan = x) = x, \frac{x^2}{3}, \frac{(2x)^2}{3.5}, \frac{(3x)^2}{5.7}, \frac{(4x)^2}{7.9} \dots$$

Consult on this subject the Integral Calculus of M. Lacroix, *t.* 11, N°. 668, a work of which we cannot too highly recommend the study, and in which will be found comprised every thing that is known on the doctrine of Integration.

836. When a proposed differential equation belongs to a curve, it may be of service to construct the curve without integrating, a thing which can always be accomplished, and in the manner following:

Suppose in the first place that the equation is of the first order, $F(x, y, y') = 0$; and let the constant be determined by the condition that $x = a$ gives $y = b$. We shall take [fig. 60] $AB = a$, $BC = b$, when the point C will be on the curve required; also, substituting a and b for x and y in $F = 0$, we shall thence deduce for y' a value which will fix the direction of the tangent KC at the point C. Let a point D be next taken sufficiently near to C, that we may, without any error worthy of notice, consider the straight line CD as coincident with the arc of the curve; then $AF = a'$, $FD = b'$ will be the co-ordinates of another point D of our curve; so that we may assume $x = a'$, $y = b'$ in $F = 0$, and thence deduce the corresponding value of y', and consequently the situation of the tangent IE, which will be but very slightly separated from the 1st. Continuing the same course of operation, we see that the curve will be replaced by a polygon $CDEZ$.

We might also reason in the manner following. From the equation $F=0$ and its derivative we might deduce the values of y' and y'', in functions of x and y, and substitute them in that of the radius of curvature R [N°. 733]; then, x and y being replaced by a and b, and the tangent KC,

and a perpendicular CN equal to this radius being drawn, describe from the centre N a circular arc CD; and, finally, consider the point D as lying in the curve, its co-ordinates being a' and b'. Repeating the operation, we should draw the tangent ID and the radius of curvature, DO, &c.; and the curve would thus be replaced by a system of contiguous circular arcs. It is evident too that the error would be less in this case than when we made use of tangents alone, and that in consequence the points C, D, E... might be taken more apart from each other, which would render the constructions less troublesome.

837. If the differential equation proposed be of the 2nd order, $F(x, y', y'') = 0$; having in like manner selected an arbitrary point C for one of those of the curve, we must moreover take at pleasure some straight line KC for the tangent at C; and this double condition will determine the two constants. We should then derive the value of y'', and consequently that of the radius of curvature R, in functions of x, y, y'; and since these quantities are known for the point C, we should be able to describe the circular arc CD, as before. The point D of this arc being supposed to lie in the curve, we shall describe its normal DN, by drawing a straight line to the first centre N. Thus, for the second point D, we shall know its co-ordinates, a', b', and the value of y', which results from the direction of the tangent ID at D, and we shall calculate the value of R for this point D: then taking $OD = R$, and describing the arc DE, we shall have a third point E, for which we know the co-ordinates and the direction of the tangent; and so on.

Similar reasoning will give the mode of replacing the curve by a series of osculating parabolic arcs. These principles might also be applied to the differential equations of the 3rd order; but in that case not only should we have arbitrarily to assume a point C and its tangent KC, but also the radius CN of the osculating circle at this first point, which will determine the three arbitrary constants. The curve might then be replaced by a series of parabolas, the contact of which would be of the 3rd order: and the same principle will apply to the higher orders.

Hence we conclude that, *every differential equation between two variables may be constructed by a curve, which has as many arbitrary parameters as there are units in the order of the equation:* and this agrees with the No. 831, where it has been proved that this equation always has an integral.

EQUATIONS OF THE HIGHER ORDERS, AND IN PARTICULAR THOSE OF THE SECOND.

838. In the equations of the 1st order, we have been at liberty to take for the principal variable any one we thought proper, without it being on that account requisite that the processes of integration should undergo any modifications: one advantage among others which the notation of Leibnitz affords [N°. 694]. It now however becomes indispensable to indicate, in each equation, which is the differential that has been taken as constant, and to have regard to every transformation which may on that account be necessary.

If, therefore, in a given equation, dx be required to be constant, instead of any other differential, which we may have considered as such, this equation must be modified by means of the known theory [N°. 691]. Thus, for $ds . dy = ad^2x$, or $ax'' = y'$, we have assumed as constant $ds = \sqrt{(dx^2 + dy^2)}$; consequently $a(s'x'' - x's'') = y's'^2$; and then assuming $x' = 1$, we have $x'' = 0$, $s'^2 = 1 + y'^2$, $s's'' = y'y''$,

$$y's'^2 = - as'',\ s'^3 = - ay'', \text{ or } (dx^2 + dy^2)^{\frac{3}{2}} = - adxd^2y,$$

where dx is constant. This equation, put under the form $ay'' + (1 + y'^2)^{\frac{3}{2}} = 0$, will shortly be integrated [N°. 840].

Similarly, that dx may be constant instead of ds in $(dx^2 + dy^2) \dfrac{d^2y}{dx^4} = \dfrac{1}{a} \cos \dfrac{x}{c}$, we shall observe that this equation is tantamount to

$$\frac{d^2}{ds^2} = \frac{dx^4}{ds^4} \cdot \frac{1}{a} \cos \frac{x}{c},$$

which we write $y'' = x'^4 . \dfrac{1}{a} \cos \dfrac{x}{c}$, s being throughout the principal variable; whence $\dfrac{s'y'' - s''y'}{s'^3} = \dfrac{x'^4}{s'^4} \cdot \dfrac{1}{a} \cos \dfrac{x}{c}$, no derivative being constant. Lastly, $x' = 1$ gives $s'^2 = 1 + y'^2$, $s's'' = y'y''$; then

$$y'' = \frac{1}{a} \cos \frac{x}{c}, \text{ or } dy' = \frac{dx}{a} \cos \frac{x}{c};$$

dx is constant, and b and k being the arbitrary constants,

$$y' = \frac{c}{a} \sin \frac{x}{c} + b,\ y = k + bx - \frac{c^2}{a} \cos \frac{x}{c}.$$

There is no reason why we might not take any other principal variable in preference to x, and integrate; but, in what follows, unless at least we give notice to the contrary, we shall always take dx as constant.

839. The most general equation of the 2nd order has the form $F(y'', y', y, x) = 0$; but it will be well first to examine the particular cases, in which it does not contain all the four quantities y'', y', y and x.

If but two of them enter into the equation, it may have one of these three forms,

$$F(y'', x) = 0,\ F(y'', y') = 0,\ F(y'', y) = 0.$$

When y'' is accompanied only by y' and x, or by y' and y, the equation is of one of the two forms:

$$F(y'', y', x) = 0,\ F(y'', y', y) = 0.$$

Let us first integrate these five particular cases.

1°. If we have $y'' = fx$, since $y''dx = dy'$, the proposed equation is equivalent to $dy' = fx.dx$. Let $y' = X + C$ be the integral of this equation; since then $y'dx = dy$, we have

$$dy = Xdx + Cdx;\ \text{whence } y = A + Cx + \int Xdx.$$

Suppose, for example, that $d^2y = adx^2$, or $dy' = adx$: there first results $y' = c + ax$, or $dy = cdx + axdx$; and lastly $y = A + cx + \frac{1}{2}ax^2$.

Similarly, let $d^2y = ax^n dx^2$, or $y'' = ax^n$, or lastly $dy' = ax^n dx$: we find $y = A + cx + \frac{ax^{n+2}}{(n+1)(n+2)}$. If $n = -1$, we obtain $y = A + cx + axlx$; and if $n = -2$, we have $y = A + cx - alx$.

Observe that the above calculation equally applies to

$$y^{(n)} = fx,\ \text{or } d.y^{(n-1)} = fx.dx,\ \text{whence } y^{n-1} = c + X:$$

It will remain only to operate anew on this and the succeeding equations in the same manner as on the one proposed; and the integral will have the form

$$y = A + Bx + Cx^2 \ldots + Kx^{n-1} + \Sigma fx.dx^n,$$

the sign Σ denoting n successive integrations.

840. II. If the proposed equation have the form $F(y'', y') = 0$, substituting $\frac{dy'}{dx}$ for y'', it becomes of the 1st order between y' and x; and

we thence deduce $dx = \int y'.dy'$. Moreover, since $dy = y'dx$, we have $dy = y'\int y'.dy'$. These two equations being integrated, let the results be denoted by

$$x = M + A,\ y = N + B;$$

A and B being the arbitrary constants, M and N some known functions of y'. It appears then that we have but to eliminate y' between these last equations [N°. 832], and we shall have the integral required with its two constants.

Let $ay'' + (1 + y'^2)^{\frac{3}{2}} = 0$: we find

$$-ady' = (1 + y'^2)^{\frac{3}{2}}dx;$$

whence $$dx = \frac{-ady'}{(1 + y'^2)^{\frac{3}{2}}},\ dy = \frac{-ay'dy'}{(1 + y'^2)^{\frac{3}{2}}};$$

then [N°. 777], $$x = A - \frac{ay'}{\sqrt{(1 + y'^2)}},\ y = B + \frac{a}{\sqrt{(1 + y'^2)}};$$

and lastly, $$(A - x)^2 + (B - y)^2 = a^2.$$

This integration gives the solution of the following problem: what curve is it, for which the radius of curvature is constant, or $R = a$? It appears that the circle alone is possessed of this property.

This method will apply to all the orders, provided that the equation be of the form $F[y^{(n)}, y^{(n-1)}] = 0$.

Thus, for $f(y''', y'') = 0$, we shall take

$$dy'' = y'''dx, \text{ whence } x = \int Fy''.dy'',$$

and $$y' = \int y''dx = \int(Fy''.y''dy'').$$

This integral being then substituted for y' in $dy = y'dx$, we arrive at values of x and of y expressed in y'', and containing three arbitrary constants: y'' is then eliminated between them [N°. 832].

841. III. Proceeding now to the equations of the form $y'' = Fy$, if $dy' = y''dx$ be multiplied by $y'dx = dy$, we find

$$y'dy' = y''dy \dots (A);$$

in this substituting for y'' its value Fy, we have $y'dy' = Fy.dy$; whence

$$\tfrac{1}{2}y'^2 = \tfrac{1}{2}c + \int Fy.dy,\ y' = \sqrt{(c + 2\int Fy.dy)};$$

and thence

$$x = \int \frac{dy}{y'} = \int \frac{dy}{\surd(c + 2\int Fy.dy)}.$$

For example, $a^2d^2y + ydx^2 = 0$, or $a^2y'' = -y$, becomes $a^2y'dy' = -ydy$, whence $a^2y'^2 = c^2 - y^2$; then $dx = \dfrac{ady'}{\surd(c^2 - y^2)}$; and therefore, integrating, we have

$$x = a.\text{arc}\left(\sin = \frac{y}{c}\right) + b, \text{ or } \frac{y}{c} = \sin\left(\frac{x-b}{a}\right),$$

which is equivalent to $y = c.\sin\dfrac{x}{a} + c'\cos\dfrac{x}{a}$.

Similarly, $d^2y.\surd(ay) = dx^2$ gives $\frac{1}{4}ay'^2 = C + \surd(ay)$; whence $2dx = \dfrac{dy.\sqrt[4]{a}}{\surd(c + \surd y)}$: we integrate this by making $c + \surd y = z^2$; and we finally find

$$x = \tfrac{2}{3}\sqrt[4]{a}.(\surd y - 2c)\,\surd(c + \surd y) + b.$$

This process is applicable to all equations of the form $y^{(n)} = Fy^{(n-2)}$. Thus, let $y^{\text{v}} = Fy''$; since $y^{\text{v}}dx^2 = d^2y'' = Fy''dx^2$, we must integrate twice, and we shall have $x = \varphi y''$, with two constants. Also, having substituted for dx its value in terms of y'', we can integrate $y' = \int y''dx$: and this value of dx and that of y' being substituted in $y = \int y'dx$, we shall also obtain y in terms of y''. It will then remain only to eliminate y'' by means of $x = \varphi y''$; and the result, which contains four arbitrary constants, is the complete integral required.

842. IV. If the equation have the form $F(x, y', y'') = 0$, or *do not contain* y; we reduce it to the 1st order by putting $\dfrac{dy'}{dx}$ for y'', since it will then contain only y' and x: it thus comes under the cases already discussed, and we can integrate it whenever it is separable, or homogeneous, or &c. [See p. 374].

Suppose, therefore, that this integration is accomplished, and that the integral is $\psi(x, y', c) = 0$; three cases will present themselves:

1°. When we can solve the integral in respect to y', and get $y' = fx$, we thence deduce $y = \int y'dx = \int fx.dx$.

2°. If, on the contrary, the value of x can be found in terms of y', as $x = fy'$, we have $y = \int y'dx$, and by means of integration by parts, $y = xy' - \int xdy' = xy' - \int fy'.dy'$: y' is then eliminated by means of $x = fy'$.

3°. If neither one nor the other of these methods can be employed,

we must endeavour to express x and y', by means of some transformation, in functions X and Y of a third variable z; for $x = X$ and $y' = Y$ give $y = \int y'dx = \int YdX$.

What is the curve, in which the radius of curvature R is the reciprocal of the abscissa? Let $R = \frac{a^2}{2x}$; then [N°. 733]

$$2x(1 + y'^2)^{\frac{3}{2}} = a^2y'',$$

or $2x(1 + y'^2)^{\frac{3}{2}} dx = a^2dy'$, an equation which is separable:

$$2xdx = \frac{a^2dy'}{(1 + y'^2)^{\frac{3}{2}}},\quad x^2 + c = \frac{a^2y'}{\sqrt{(1 + y'^2)}};$$

and deducing the value of y', $y = \int y'dx$ gives

$$y = \int \frac{(x^2 + c)dx}{\sqrt{[a^4 - (x^2 + c)^2]}}:$$

the line required is that formed by bending an *elastic* lamina [See N°. 898].

Had R been required to be a given function X of the abscissa x, we should have assumed $(1 + y'^2)^{\frac{3}{2}} = Xy''$; and a similar calculation would have given

$$\frac{y'}{\sqrt{(1 + y'^2)}} = \int \frac{dx}{X} = V;\ \text{whence}\ y = \int \frac{Vdx}{\sqrt{(1 - V^2)}}.$$

This is the solution of the *inverse problem of the radii of curvature.*

Let $(1 + y'^2) + xy'y'' = ay''\sqrt{(1 + y'^2)}$: this equation may be put under the form

$$dx(1 + y'^2) + xy'dy' = ady'\sqrt{(1 + y'^2)},$$

which is linear [N°. 817], and becomes integrable by dividing by $\sqrt{(1 + y'^2)}$ [See p. 383]. We find

$$x = \frac{ay' + b}{\sqrt{(1 + y'^2)}}.$$

Again, $y = y'x - \int xdy'$ becomes

$$y = y'x - a\sqrt{(1 + y'^2)} - bl[y' + \sqrt{(1 + y'^2)}] + blc$$

$$= \frac{by' - a}{\sqrt{(1 + y'^2)}} - bl\left(\frac{y' + \sqrt{(1 + y'^2)}}{c}\right);$$

and it remains only to eliminate y' from this, by means of the value of

x. We find, the operation being gone through, and having assumed, for conciseness, $z = \sqrt{(a^2 + b^2 - x^2)}$,

$$y = z + bl\,\frac{x + a}{c(b + z)}.$$

Lastly, $2(a^2y'^2 + x^2)\,y'' = xy$ gives the homogeneous equation [No. 815] $2(a^2y'^2 + x^2)\,dy' = xy'dx$, which we separate by assuming $x = y'z$; whence

$$\frac{dy'}{y'} = \frac{zdz}{2a^2 + z^2}:$$

we integrate this by logarithms, and there results

$$y' = c\sqrt{(2a^2 + z^2)} \text{ and } x = cz\sqrt{(2a^2 + z^2)};$$

whilst $y = \int y'dx$, when for y' and dx we substitute their values in z, becomes $y = \frac{1}{3}c^2z(3a^2 + z^2) + b$, and we must then eliminate z between these values of x and y.

843. V. Suppose that the equation of the 2nd order have the form $F(y'', y', y) = 0$, *i. e.* that x *do not enter*. The substitution of the value A [p. 403] of y'' will reduce the proposed equation to the 1st order between y and y'.

For example, if $y'' = f(y', y)$, we find $y'dy' = dy.f(y', y)$, which is quite simple in its form.

1°. If the integral that results is solvable in respect to y', so that $y' = fy$, we shall have $dx = \frac{dy}{y'} = \frac{dy}{fy}$, and hence we shall easily derive x in terms of y.

2°. If y can be deduced in a function of y', or $y = fy'$, $dy = y'dx$ will give

$$x = \int \frac{d.fy'}{y'} = \frac{fy'}{y'} + \int \frac{fy'}{y'^2}.\,dy';$$

and we shall then get quit of y' by means of $y = fy'$.

3°. If, lastly, we have neither of these cases, we must endeavour to express y' and y in functions of a third variable z, and $y'dx = dy$ will become $Zdx = Tdz$, &c.

The equation $y''(yy' + a) = y'(1 + y'^2)$ changes itself into

$$dy'(yy' + a) = dy\,(1 + y'^2);$$

whence [p. 377] $y = ay' + c\sqrt{(1+y'^2)}$,

$$x = \int \frac{dy}{y'} = al(by') + cl[y' + \sqrt{(1+y'^2)}];$$

and y' must then be eliminated between these equations. We find, for instance, when $c = 0$,

$$x = al\left(\frac{by}{a}\right); \text{ whence } y = Ce^{\frac{x}{a}}.$$

The equation $aby'' = \sqrt{(y^2 + a^2y'^2)}$ becomes

$$aby'dy' = dy\sqrt{(y^2 + a^2y'^2)}.$$

To integrate, we shall, for the sake of homogeneousness, assume $y' = \frac{y}{z}$, and we shall have $abzdy - abydz = z^2dy\sqrt{(z^2 + a^2)}$; the equation is separable, and having made $\sqrt{(z^2 + a^2)} = tz$, we shall thence deduce z and dz, and substitute; we find $\frac{dy}{y} = \frac{-btdt}{bt^2 - at - b}$; it will be easy now to obtain y in a function of t, as also y'; we consequently shall also have $x = \int \frac{dy}{y'}$; and we must then eliminate t.

Let $y'' + Ay' + By = 0$, A and B being constants: we have from it the homogeneous equation $y'dy' + Ay'dy + Bydy = 0$; we make $y' = yu$, whence

$$\frac{dy}{y} = \frac{-udu}{u^2 + Au + B} = -\frac{udu}{(u-a)(u-b)};$$

a and b being the roots $u^2 + Au + B = 0$; and next

$$dx = \frac{dy}{y'} = \frac{dy}{uy} = \frac{-du}{(u-a)(u-b)}.$$

Hence,

$$\frac{dy}{y} - adx = \frac{-du}{u-b}, \quad \frac{dy}{y} - bdx = \frac{-du}{u-a},$$

$$ly - ax = l\left(\frac{m}{u-b}\right), \quad ly - bx = l\left(\frac{n}{u-a}\right),$$

$$u - a = \frac{n}{y}e^{bx}, \quad u - b = \frac{m}{y}e^{ax};$$

lastly, subtracting, we obtain, for the complete integral, $y(b-a) = -me^{ax} + ne^{bx}$, which may be put under the form $y = Ce^{ax} + De^{bx}$ C and D being arbitrary constants.

If a and b are imaginary, or $a = k - h\sqrt{-1}$, $b = k + h\sqrt{-1}$, we find, substituting above,

$$y = e^{kx}(Ce^{-hx\sqrt{-1}} + De^{hx\sqrt{-1}});$$

and putting for $e^{\pm hx\sqrt{-1}}$ its value L [p. 168], we have

$$y = e^{kx}(C' \cos hx + D' \sin hx) = C''e^{kx} \cos (hx + f).$$

Lastly, if $a = b$, resuming the calculation, we have

$$\frac{dy}{y} = \frac{-udu}{(u-a)^2}, \text{ whence } y(u-a) = ce^{\frac{a}{u-a}};$$

$$\text{and } dx = \frac{dy}{y'} = \frac{dy}{yu} = \frac{-du}{(u-a)^2}; \text{ whence } u - a = \frac{1}{x+k};$$

eliminating $u - a$, we finally find

$$y = ce^{a(x+k)}(x+k) = Ce^{ax}(x+k).$$

844. The equation $y'' + Py' + Qy = 0$, P and Q being some functions of x, is integrated by means of a very simple transformation. We make

$$y = e^{\int udx}, \text{ whence } y' = ue^{\int udx}, y'' = e^{\int udx}(u^2 + u');$$

these give $$u' + (u^2 + Pu + Q) = 0,$$

the common factor $e^{\int udx}$ disappearing; and the calculation is thus reduced to the integration of the equation of the first order, $du + (u^2 + Pu + Q)\,dx = 0$. If, for example, we suppose P and Q to be constant, and a and b to be the roots of $u^2 + Pu + Q = 0$, it is evident that $u = a$ and $u = b$ also satisfy this transformed equation; and we consequently have $\int udx = ax + m$, or $= bx + n$, and

$$y = e^{ax+m} = Ce^{ax}, \text{ or } y = e^{bx+n} = De^{bx}.$$

The sum therefore of these values of y will satisfy the equation proposed; and thus its complete integral is, on account of the two arbitrary constants C and D,

$$y = Ce^{ax} + De^{bx}.$$

When the roots of $u^2 + Pu + Q = 0$ are imaginary, or $u = k \pm h\sqrt{-1}$, we have seen that this result takes the form $y = Ce^{kx}\cos(hx + f)$: and if the roots are equal, we shall have to integrate the equation $du + (u-a)^2 dx = 0$, which gives $u - a = \dfrac{1}{x+k}$; whence

$$\int udx = l(x+k) + ax + D, \; y = e^{\int udx} = Ce^{ax}(x+k).$$

And thus therefore we arrive a second time at the results obtained in the last example.

845. Let us now integrate the *linear* equation, or that of the 1st degree in y,

$$y'' + Py' + Qy = R,$$

P, Q and R being some functions of x alone. The integral of this equation is easily reduced to that of the preceding paragraph, by making the term R disappear; for which purpose make, as in N°. 817, $y = tz$; whence

$$y' = tz' + zt',\ y'' = tz'' + 2z't' + zt''.$$

Substituting and dividing the resulting equation into two others, on account of the two variables t and z, we have

$$z'' + Pz' + Qz = 0 \dots (1),$$

and
$$t'' + t'\left(P + \frac{2z'}{z}\right) = \frac{R}{z},$$

or
$$dt' + t'\left(P + \frac{2z'}{z}\right)dx = \frac{Rdx}{z} \dots (2).$$

Suppose that the first of these is integrated by N°. 844, and that we have thence derived the value of z in terms of x; the second will be a linear equation of the 1st order between t' and x, and will be easily integrated according to the plan of N°. 817.

Changing, in N°. 817, y into t', P into $P + \dfrac{2dz}{z}$, Q into $\dfrac{R}{z}$, we have

$$u = \int Pdx + 2lz,$$

$e^u = e^{\int Pdx}.e^{lz^2} = \phi.z^2$ [N°. 419, 12°], assuming, for conciseness,

$$\phi = e^{\int Pdx} \dots (3).$$

We therefore get $\qquad \phi z^2 t' = \int R\phi z dx,$

and
$$y = tz = z\int\left(\frac{dx}{\phi z^2}\int R\phi z dx\right) \dots (4).$$

The double integration contained in this result itself introduces two arbitrary constants, and consequently the complete integral of the equation proposed allows of our employing for the values of z and ϕ any functions of x which satisfy the equations (1) and (3).

Let these principles be applied to $y'' + \frac{y'}{x} - \frac{y}{x^2} = \frac{a}{x^2 - 1}$,

where $$P = \frac{1}{x},\ Q = -\frac{1}{x^2},\ R = \frac{a}{x^2 - 1}:$$

1°. The equation (1) becomes

$$z'' + \frac{z'}{x} = \frac{z}{x^2};\ \text{whence, } z \text{ being} = e^{\int udx}\ [\text{N}^\circ.\ 844],$$

$$du + \left(u^2 + \frac{u}{x} - \frac{1}{x^2}\right) dx = 0:$$

this equation is rendered homogeneous by making $u = v^{-1}$, and is then separated by assuming $x = vs$ [N°. 815]. We find

$$\frac{dv}{v} = -\frac{s^2 + s - 1}{s(s^2 - 1)} ds, \text{ whence } v = \frac{1}{s}\sqrt{\left(\frac{s+1}{s-1}\right)};$$

no constant being added. The values u^{-1} and ux of v and s being now restored, we obtain

$$u = \frac{x^2 + 1}{x(x^2 - 1)},\ \int udx = l\frac{x^2 - 1}{x},\ z = e^{\int udx} = \frac{x^2 - 1}{x}.$$

2°. On the other hand, the equation (3) gives $\phi = x$; whence we derive $\int R\phi z dx = \int a dx = ax + b$, and the equation (4) becomes

$$y = \frac{x^2 - 1}{x}\int \frac{(ax + b)\,xdx}{(x^2 - 1)^2}:$$

this last integral is equal [N°. 577] to the fourth of

$$\int\left(\frac{a - b}{(x + 1)^2} - \frac{a}{x + 1} + \frac{a + b}{(x - 1)^2} + \frac{a}{x - 1}\right) dx = -2\frac{ax+b}{x^2 - 1} + al\left(c\frac{x-1}{x+1}\right):$$

and therefore

$$y = -\frac{ax + b}{2x} + \frac{x^2 - 1}{4x}\, al\left[c\left(\frac{x - 1}{x + 1}\right)\right].$$

Similarly, $y'' - \frac{a^2 - 1}{4x^2}\, y = \frac{m}{\sqrt{x^{a+1}}}$ gives for equation (1)

$z'' - \frac{(a^2 - 1)z}{4x^2} = 0$; which is satisfied by taking $z = \sqrt{x^{a+1}}$; moreover, $\phi = 1$, and $\int R\phi z dx = \int m dx = mx + b$; and therefore

$$y = z\int \frac{(mx + b)\,dx}{x^{a+1}} = \frac{1}{\sqrt{x^{a-1}}}\left(cx^a - \frac{b}{a} - \frac{mx}{a - 1}\right).$$

846. When, y, x, dy, dx and d^2y being each reckoned as a factor, the equation is *homogeneous*, we integrate it by assuming

$$y = ux,\ dy = y'dx,\ y''x = z\ldots\ (1),$$

u, y' and z being new variables. The transformed equation, on our hypothesis of homogeneousness, will thus have x in one and the same power as a factor throughout, since y' and y'' are supposed to be of the degrees 0 and -1 [N°. 815]; and thus, the equation being cleared of the variable x by division, it will be reduced to the form $z = f(y', u)$.

But, we have $dy = y'dx = udx + xdu$, $xdy' = zdx$,

$$(2)\ldots\ \frac{dx}{x} = \frac{du}{y' - u},\ \text{or}\ \frac{dy'}{z} = \frac{du}{y' - u}\ldots\ (3);$$

putting f for z in (3), it becomes of the 1st order in y and u, and we shall integrate it; suppose that we derive from it $y' = \phi u$, and let this be substituted in (2); this equation, being separated, will have for its integral $lx = \psi u$; it will remain to eliminate u, by means of $y = ux$, and we shall have the complete integral, since the equations (3) and (2) will each have introduced an arbitrary constant.

For example, $xd^2y = dydx$, or $xy'' = y'$, gives $z = y'$, and (3) becomes $dy'(y' - u) = y'du$, whence $\frac{1}{2}y'^2 = \int(udy' + y'du) = y'u + \frac{1}{2}c$.

But, $\frac{dx}{x} = \frac{dy'}{y'}$ gives $x = ay'$; thus, y' being eliminated between these two integrals, there results $x^2 - 2axu = C$; and, eliminating u from $y = ux$, $x^2 - 2ay = C$ is the integral required.

847. Let the equation be $Ay + By' + \ldots Ky^{(n)} = 0$, the coefficients being constants: make $y = ce^{hx}$; then

$$A + Bh + Ch^2 + \ldots Kh^n = 0\ldots\ (M);$$

and consequently if, for h, we take the n roots $h, k, l\ldots$ of this equation, and for c the n constants $c, c', c''\ldots$, the proposed equation will be satisfied by each of the values $y = ce^{hx}$, as also by the sum of these quantities; and the complete integral therefore is

$$y = ce^{hx} + c'e^{kx} + c''e^{lx} + \ldots\ (N).$$

If there be any imaginary roots, they will be found in pairs, $h = a \pm b\sqrt{-1}$, and two of our terms combined will form $e^{ax}(ce^{bx\sqrt{-1}} + c'e^{-bx\sqrt{-1}})$, which we reduce [equ. L, p. 168] to

$$e^{ax}(m\cos bx + n\sin bx) = ke^{ax}\sin(bx + l).$$

848. When the equation (M) has equal roots, (N) is no more than a particular integral. Thus, let $h = k$; then the two first terms of N

reduce themselves to $(c + c')\,e^{hx}$, where $c + c'$ can be reckoned only as a single constant, and there will therefore be but $n - 1$ arbitrary constants.

1°. If the roots of (M) are all equal, the proposed equation is

$$h^n y - nh^{n-1}y' + \tfrac{1}{2}n(n-1)h^{n-2}y'' \ldots \pm y^{(n)} = 0 \ldots (P),$$

since (M) is then equivalent to $(h - h)^n = 0$. Now, let $y = ut$, whence

$$y' = ut' + t'u,\; y'' = ut'' + 2t'u' + u''t,\; y''' = ut''' + 3 \text{ \&c.} \ldots;$$

also make $t = e^{hx}$; then, since $t' = he^{hx} = ht$, $t'' = h^2 t \ldots$, $t^{(i)} = h^i t$, we find

$$y = ut,\; y' = t(hu + u'),\; y'' = t(h^2u + 2hu' + u'') \ldots$$
$$y^{(i)} = t(h^iu + ih^{i-1}u' + \tfrac{1}{2}i(i-1)h^{i-2}u'' + u^{(i)}.$$

Substitute these values in (P), and we shall have an equation, the terms of which will all destroy each other, except the last $u^{(n)}$; consequently $u^n = 0$, viz. $u = a + bx + cx^2 \ldots + fx^{n-1}$; and there results for the complete integral, in the case supposed,

$$y = ut = (a + bx + cx^2 \ldots + fx^{n-1})e^{hx}.$$

2°. When the equation (M) has m roots $= \alpha$, it has the factor $(h - \alpha)^m$, under the form $h^m + Ah^{m-1} + Bh^{m-2} \ldots + \alpha^m$. Compose the equation

$$h^m y + Ah^{m-1}y' + Bh^{m-2}y'' \ldots \pm y^{(m)} = 0:$$

it has been seen that the integral of this is $(a + bx \ldots + fx^{m-1})\,e^{\alpha x}$. On the other hand, the proposed equation is also satisfied by $y = ce^{hx}$, $c'e^{lx} \ldots$, values corresponding to the $n - m$ unequal roots of h in (M); and since, from the property of the linear equations, the sum of these solutions must also satisfy the proposed equation, the complete integral is

$$y = (a + bx + \ldots + fx^{m-1})\,e^{\alpha x} + ce^{hx} + c'e^{lx} + \ldots$$

$a, b, \ldots f, c, c' \ldots$ are the n arbitrary constants; α, h, $l \ldots$ are the roots of the equation (M).

Thus, for $y - 2y' + 2y'' - 2y''' + y^{IV} = 0$, we find

$$1 - 2h + 2h^2 - 2h^3 + h^4 = 0 = (1 - h)^2(1 + h^2);$$

whence

$$y = (a + bx)\,e^x + ce^{x\sqrt{-1}} + de^{x\sqrt{-1}},$$
$$y = e^x(a + bx) + A\cos^x + B\sin x.$$

849. The *Linear equation of all orders is*

$$Ay + By' + Cy'' \ldots + Ky^{(n)} = X.$$

Suppose that X denotes a given function of x, and that A, $B \ldots$ are constants. The integration can always be reduced to the solution of equations, by means of the following process, which we shall apply only to the 2nd order.

$$Ay + By' + Cy'' = X.$$

Let $e^{-hx}dx$ be the factor which renders this equation integrable: since $Xe^{-hx}dx$ is the differential of a function of x, such as P, the 1st side $e^{-hx}dx\,(Ay + By' + Cy'')$ is also that of a function of the form $e^{-hx}(ay + by')$. Let this last expression therefore be differentiated, and the terms of the result compared with those of the former expression; we shall have

$$-ha = A,\ -hb + a = B,\ b = C,$$

whence $$A + Bh + Ch^2 = 0,\ a = -\frac{A}{h},\ b = C.$$

The unknown constant h is one of the roots of the 1st of these equations; the two others give a and b, and the integral of the 1st order

$$ay + by' = e^{hx}(P + c).$$

We must operate afresh on this equation, or, rather, put for h the two roots h' and h'', and then eliminate y' between the two results, which will give the complete integral [N°. 832].

For the equation of the degree n, the same reasoning proves that h is a root of the equation

$$A + Bh + Ch^2 \ldots + Kh^n = 0,$$

and whatever number of these roots we know, so many integrals shall we have of the order $n - 1$, and of the form

$$ay + by' + cy'' \ldots + ky^{(n-1)} = e^{hx}(P + c);$$

between which we shall be able to eliminate any equal number of quantities $y^{(n-1)}$, $y^{(n-2)} \ldots$, which will reduce the problem so many degrees lower, or indeed make known the complete integral, if we have all the roots h [See Euler's *Int. Calculus*, Vol. II. p. 402]. We have

$$a = -\frac{A}{h},\ b = \frac{a - B}{h},\ c = \frac{b - C}{h} \ldots\ l = \frac{k - L}{h}.$$

ELIMINATION BETWEEN DIFFERENTIAL EQUATIONS.

850. When we have two equations between x, y and t, the elimination of t will lead to a relation between x and y; but in the case of differential equations, this operation will require new processes.

$$(Mx + Ny)\,dt + Pdx + Qdy = \tau dt,$$
$$(M_{,}x + N_{,}y)\,dt + P_{,}dx + Q_{,}dy = \tau_{,}dt,$$

being the most general equations between three unknown variables, eliminate dy, divide by the coefficient of dx, and do the same for dx; our equations will then be brought under the most simple form

$$(ax + by)\,dt + dx = Tdt,$$
$$(a'x + b'y)\,dt + dy = Sdt.$$

We shall here suppose that the coefficients are constant, and T, S functions of t; and the 2nd equation being multiplied by an indeterminate quantity k, and added to the 1st; we shall have

$$(a + a'k)\left(x + \frac{b + b'k}{a + a'k}y\right)dt + (dx + kdy) = (T + Sk)\,dt.$$

This being premised, it is evident that, leaving out of consideration the part $(a + a'k)\,dt$, the 2nd term will be the differential of the 1st, if we have

$$k = \frac{b + b'k}{a + a'k}, \text{ or } a'k^2 + (a - b')\,k = b.$$

Taking for k one of the roots of this equation, we shall have

$$(a + a'k)\,(x + ky)\,dt + dx + kdy = (T + Sk)\,dt,$$

or, making $x + ky = u$,

$$(a + a'k)\,udt + du = (T + Sk)\,dt.$$

It will be easy to integrate this linear equation, [N°. 817], and so derive the value of u in a function of t, or $x + ky = ft$; we must then substitute the two roots of our equation successively for k, and it will remain but to eliminate t between the results.

If the roots of k are imaginary, the exponentials are to be replaced by sines and cosines, as in N^{os}. 843, 844. And if they are equal, we shall

obtain, it is true, only one integral between x, y and t; but having deduced the value of one of these variables, and substituted in one of the equations proposed, we must repeat the integration on the equation of two variables that results.

851. If we have three equations and four variables x, y, z, and t, in order to eliminate z and t, and so obtain a relation between x and y, we shall assume

$$(ax + by + cz)\,dt + dx = Tdt,$$
$$(a'x + b'y + c'z)\,dt + dy = Sdt,$$
$$(a''x + b''y + c''z)\,dt + dz = Rdt,$$

where T, S, and R are supposed to be functions of t alone, and the other coefficients to be constant. To proceed as before, multiply the 2nd equation by k and the 3rd by l, k and l being each indeterminate; then, having added the whole, put the result under the form

$$(a + a'k + a''l)\left(x + \frac{b + b'k + b''l}{a + a'k + a''l}\,y + \frac{c + c'k + c''l}{a + a'k + a''l}\,z\right)dt$$
$$+ dx + kdy + ldz = (T + Sk + Rl)\,dt.$$

Now, it is clear that the part contained within the brackets will have $dx + kdy + ldz$ for its differential, if l and k be determined by the conditions

$$\frac{b + b'k + b''l}{a + a'k + a''l} = k, \quad \frac{c + c'k + c''l}{a + a'k + a''l} = l;$$

consequently, if we make $x + ky + lz = u$, we shall have

$$(a + a'k + a''l)\,udt + du = (T + Sk + Rl)\,dt.$$

This linear equation being integrated, the result will give u in a function of t, or $x + ky + lz = ft$; and since k, l are given by equations of the 3rd degree, the substitution of the roots in this integral will give three equations between x, y, t and z, which will serve for the elimination of t and z.

852. If we have the equations of the 2nd order

$$d^2y + (ady + bdx)\,dt + (cy + gx)\,dt^2 = Tdt^2,$$
$$d^2x + (a'dy + b'dx)\,dt + (c'y + g'x)\,dt^2 = Sdt^2,$$

we shall assume $\qquad dy = pdt, \; dx = qdt,$

whence $\qquad dp + (ap + bq + cy + gx)\,dt = Tdt,$

$$dq + (a'p + b'q + c'y + g'x)\,dt = Sdt;$$

and we shall therefore have four equations between the five variables p, q, x, y and t; which we must treat according to the process explained above. It is obvious that this mode of calculation applies generally to equations of the 1st degree and to those of all orders, whatever be their number.

PROBLEMS IN GEOMETRY.

853. When, in the equation $F(x, y, c) = 0$ of a curve, the constant c is arbitrary, and we assign to it all the values possible in succession, we have an infinite system of lines. The name of *Trajectories* is given to the curves which cut all these lines at the same angle; so that if, for example, the trajectory be *orthogonal*, and tangents be drawn to this curve and to the variable curve, at their point of intersection, these tangents will be at right angles to each other.

The following is the general mode of obtaining the equation $f(x,y)=0$ of the trajectories. Let $F(X, Y, c) = 0$ be the equation of the curve which changes its position on account of the variable parameter c. For any one value of c, this curve will take a particular position as AM [fig. 61]. Let tangents be drawn to this line and the trajectory DM at their common point M; Y' and y' will determine their inclinations to the axis of x, and the angle $T'MT$ which they form with each other has for its tangent $a = \frac{y' - Y'}{1 + Y'y'}$; whence

$$(1 + Y'y')\,a + Y' - y' = 0 \ldots (1).$$

a is here a constant or a given function; also Y and X must be replaced by y and x, because we are speaking of a point common to both curves. The reasoning then of N°. 462 shows that if c be eliminated between this equation and the one $F(y, x, c) = 0$ of the curve that is cut, and the result be integrated, we shall have the equation of the trajectory. If it be orthogonal, we simply have, instead of (1), the equation

$$1 + Y'y' = 0 \ldots (2).$$

If, for example, the curve be required which cuts at right angles a straight line that revolves about the origin, $Y = cX$ will give $Y' = c$, and the equation (2) will become $1 + cy' = 0$: eliminating c by means of $y = cx$, we find $xdx + ydy = 0$; whence $x^2 + y^2 = A^2$; and the trajectory therefore is a circle of arbitrary radius.

But if the straight line is to be cut at a given angle, of which a is the

tangent, the same calculation applied to the equation (1) gives, for the trajectory, this homogeneous differential equation [N°. 815]

$$y + ax = y'(x - ay);$$

whence $$al(c\sqrt{x^2 + y^2}) = \text{arc}\left(\tan = \frac{y}{x}\right),$$

an equation which belongs to the logarithmic spiral [N°. 473], as may easily be ascertained by transferring this relation into one between polar co-ordinates [N°. 385].

For the equation $X^m Y^n = c$, which belongs to the hyperbolas and parabolas of all orders, the same calculation gives the homogeneous equation $(nx + amy)\,y' = anx - my$. When the trajectory is to be orthogonal, $myy' = nx$ having for its integral $my^2 - nx^2 = A$, this curve is an hyperbola of the 2nd degree, or an ellipse, accordingly as the exponent n is positive or negative.

The orthogonal trajectory of the circle, which has $y^2 = 2cx - x^2$ for its equation, is another circle of which the equation is $y^2 + x^2 = Ay$. It is constructed by taking, for the centre, any point whatever of the axis of y, and, for the radius, the distance from that point to the origin.

854. When it is proposed to find a curve, such that its subtangent or tangent... shall be a given function φ of x and of y, it follows from the formulæ [N°. 722] that we shall have to integrate the equations $y = y'\varphi$, $y\sqrt{(1 + y'^2)} = y'\varphi$... It is from this circumstance that the name of the *inverse Method of tangents* has been given to that branch of the Calculus which relates to the integration of the equations of the 1st order between x and y.

The following are some examples.

What is the curve in which, at each point, the length n of the normal and the abscissa t of the foot of that line have to each other a given relation $n = Ft$?

Since [N°. 722] we have $t = x + yy'$ and $n = y\sqrt{(1 + y'^2)}$, it is clear that the proposed problem reduces itself to integrating the equation

$$y\sqrt{(1 + y'^2)} = F(x + yy').$$

If n and t be required to be the co-ordinates of a parabola, of which $2p$ is the parameter, it follows that $n^2 = 2pt$, whence

$$y^2(1 + y'^2) = 2p(x + yy').$$

To integrate this equation, resolve it in respect to yy', then divide

throughout by the radical, and we shall have

$$\frac{p - yy'}{\sqrt{(p^2 + 2px - y^2)}} + 1 = 0;$$

the 1st term of this is evidently the derivative of $\sqrt{(p^2 + 2px - y^2)}$; and therefore $\sqrt{(p^2 + 2px - y^2)} = a - x$.

Squaring, and putting c in place of the arbitrary constant $a + p$, we obtain

$$y^2 + x^2 - 2cx + c^2 - 2pc = 0.$$

The curve required, therefore, is a circle, of which the centre has its locus any where in the axis of x, and the radius is the mean proportional between $2p$ and the distance from the centre to the origin. This is, in fact, what is otherwise evident.

But, besides this unlimited number of circles which satisfy the problem, it also has a parabola for one of its solutions; for, on recurring to the processes of N^{os}. 823 and 827, we shall find the singular equation $y^2 = 2px + p^2$. It is a point easily verified [as we have seen N°. 824, 3°], that this parabola results from the continual intersection of all the successive circles comprised in the general solution.

855. To find a curve such, that the perpendiculars let fall from two fixed points on all its tangents shall form a constant rectangle $= k$. Take for the axis of x the line which joins the two points, one of them being the origin, and the other at the distance $2a$: then the N°. 374 gives the expressions for the distances from these two points to the tangent, which has for its equation $Y - y = y'(X - x)$, and we find

$$(2ay' + y - y'x)(y - y'x) = k(1 + y'^2) \dots (1).$$

This equation is integrated by first differentiating it; y'' results as a common factor, and we find $y'' = 0$, and

$$-x(2ay' + y - y'x) + (y - y'x)(2a - x) = 2ky' \dots (2):$$

the 1st gives $y' = c$, which changes the proposed equation into

$$(2ac + y - cx)(y - cx) = k(1 + c^2);$$

these are the equations of two straight lines; and we might readily assure ourselves that they do actually correspond to the problem. The number of straight lines comprised in pairs in this relation is also infinite. As to the equation (2), if we deduce from it the value of y', and substitute it in (1), changing x into $x + a$, we have

$$y^2(a^2 + k) + x^2 = k(a^2 + k);$$

and we, therefore, find an ellipse which has for its foci the given fixed points, and for its semi-axes $\sqrt{(k+a^2)}$ and $\sqrt{k}$. This curve is a singular solution of the problem, and results from the successive intersections of the straight lines comprised in the complete integral.

The following questions will serve for additional practice:

To find a curve such, that all the perpendiculars let fall from a given point on its tangents shall be equal.

What is the curve, in which the lines, drawn to two fixed points from any point in its course, are equally inclined to the tangent?

INTEGRATION OF EQUATIONS WHICH CONTAIN THREE VARIABLES.

TOTAL DIFFERENTIAL EQUATIONS.

856. Since the equation $dz = pdx + qdy$ results from the sum of the derivatives [N°. 704] of $z = f(x, y)$, taken relatively to x and y considered as independent variables, we thence conclude that the functions of x and y represented by p and q must be such, that [N°. 703]

$$\frac{dp}{dy} = \frac{dq}{dx} \dots (1).$$

If a proposed equation satisfy this condition, we shall integrate the exact differential $pdx + qdy$, by the process of N°. 819; and the result will be the value of z or $f(x, y)$. Thus, from example 1, p. 380, we see that the integral of

$$dz = \frac{dx}{\sqrt{(1+x^2)}} + adx + 2bydy,$$

is

$$z = by^2 + ax + lc\,(x + \sqrt{1+x^2}).$$

857. If the differential equation proposed be implicit,

$$Pdx + Qdy + Rdz = 0;$$

P, Q, and R being functions of x, y and z, we shall be able to bring it under the form $dz = pdx + qdy$, by assuming $p = -\frac{P}{R}$, $q = -\frac{Q}{R}$. In ascertaining whether the condition (1) is fulfilled, since p contains z

which is a function of x and y, to obtain the first side of the equation (1), we must not confine ourselves to considering x as constant in p, and y as variable; we must also make z vary in respect to y; whence [N°. 704] we have $\frac{dp}{dy} + q \cdot \frac{dp}{dz}$, q being $= \frac{dz}{dy}$. The same is to be said of q relatively to x, and we therefore have, in lieu of the condition (1),

$$\frac{dp}{dy} + q\,\frac{dp}{dz} = \frac{dq}{dx} + p\,\frac{dq}{dz}.$$

Replacing p and q by their values, this gives

$$P\frac{dR}{dy} - R\frac{dP}{dy} + R\frac{dQ}{dx} - Q\frac{dR}{dx} + Q\frac{dP}{dz} - P\frac{dQ}{dz} = 0 \ldots (2),$$

an equation which expresses that z is a function of two independent variables, with which it is connected by a single equation.

858. Let F be the factor which renders the equation $Pdx + Qdy + Rdz = 0$ the exact differential of $f(x, y, z) = 0$. It follows, then, from the principles explained [p. 270], that if we make x constant, or $dx = 0$, the equation $FQdy + FRdz = 0$ must be an exact differential between y and z: and the same may be said for $dy = 0$ and $dz = 0$; whence we deduce

$$\frac{d.FR}{dy} = \frac{d.FQ}{dz},\quad \frac{d.FP}{dz} = \frac{d.FR}{dx},\quad \frac{d.FQ}{dx} = \frac{d.FP}{dy},$$

or

$$\left.\begin{aligned} F\left\{\frac{dR}{dy} - \frac{dQ}{dz}\right\} &= Q\frac{dF}{dz} - R\frac{dF}{dy} \\ F\left\{\frac{dP}{dz} - \frac{dR}{dx}\right\} &= R\frac{dF}{dx} - P\frac{dF}{dz} \\ F\left\{\frac{dQ}{dx} - \frac{dP}{dy}\right\} &= P\frac{dF}{dy} - Q\frac{dF}{dx} \end{aligned}\right\} \ldots (3).$$

If, now, these equations be respectively multiplied by P, Q and R, and the results be added together, the 2nd sides will destroy each other, so that the common factor F disappearing, we shall be brought back to the relation (2); we have no chance therefore of rendering the proposed equation integrable by means of a factor F, except when the condition (2) is satisfied. Thus, every equation between two variables is integrable, at least by approximation, whereas this is not the case for equations of three or more variables.

859. If the differentials exceed the first degree, the circumstances of

the case are these: Whatever be the integral required, it is obvious that, by differentiating it, we might bring it under the form $Pdx + Qdy + Rdz = 0$, to which the differential proposed must be reducible; if therefore we resolve the proposed equation in respect to dz, dx and dy must not remain involved under the radical sign; and it consequently is not integrable, except when it is decomposable into rational factors. For

$$Adx^2 + Bdy^2 + Cdz^2 + Ddxdy + Edxdz + Fdydz = 0,$$

the radical comprised in the value of dz is

$$\sqrt{[(E^2 - 4AC)\,dx^2 + 2(EF - 2DC)\,dxdy + (F^2 - 4BC)\,dy^2]}:$$

which being submitted to the known condition [N°. 138], we find

$$(EF - 2DC)^2 - (E^2 - 4AC)(F^2 - 4BC) = 0 \ldots (4).$$

If this equation be satisfied, we shall have to integrate two equations of the form $Pdx + Qdy + Rdz = 0$, of which the one proposed is the product.

860. To integrate $Pdx + Qdy + Rdz = 0$, when the condition (2) is fulfilled, we must consider one of the variables as z, constant; and then integrate $Pdx + Qdy = 0$. Let $f(x, y, z, Z) = 0$ be the integral, Z being the arbitrary constant which may contain z: we shall take the complete differential of this equation and compare it with the one proposed; there must thence result for dZ an expression independent of x and y, a function of z and Z alone; and a subsequent integration will determine Z. This process has its origin in the first principles of the differentiation of equations [N°. 704].

I. Let $dx(y + z) + dy(x + z) + dz(x + y) = 0$: making $dz = 0$, we have $dx(y + z) + dy(x + z) = 0$, the integral of which [N°. 813] is $(x + z)(y + z) = Z$. Differentiating this, and comparing the result with the differential proposed, we shall have $dZ = 2zdz$, whence $Z = z^2 + c$. And the integral required is therefore $xz + yz + xy = c$.

II. Before we proceed to treat the equation $zdx + xdy + ydz = 0$, we shall submit it to the condition (2); and since $x + y + z$ is not $= 0$, we see that the equation is not integrable. Had we carried into effect the plan of integration marked out, we should have found that Z was not free from x and y.

III. For $[x(x - a) + y(y - b)]\,dz = (z - c)\,(xdx + ydy)$, the same is the case, unless a and b be each nothing. In this case, we

have $(x^2 + y^2)dz = (z - c)(xdx + ydy)$; we make $dz = 0$ and integrate; whence $x^2 + y^2 = Z^2$. Differentiating this and comparing the result with the equation proposed, we find $Zdz = (z - c)dZ$; whence $Z = A(z - c)$: thus the integral is $x^2 + y^2 = A^2(z - c)^2$.

IV. Again, let the equation proposed be

$$(y^2 + yz + z^2)dx + (x^2 + xz + z^2)dy + (x^2 + xy + y^2)dz = 0:$$

making $dz = 0$, we have to integrate

$$\frac{dx}{x^2 + xz + z^2} + \frac{dy}{y^2 + yz + z^2} = 0; \text{ whence}$$

$$\frac{2}{z\sqrt{3}}\left[\text{arc}\left(\tan = \frac{z + 2x}{z\sqrt{3}}\right) + \text{arc}\left(\tan = \frac{z + 2y}{z\sqrt{3}}\right)\right] = fz,$$

or*
$$\text{arc}\left(\tan = \frac{(x + y + z)z\sqrt{3}}{z^2 - zx - zy - 2xy}\right) = \tfrac{1}{2}z\sqrt{3}.fz.$$

Since this arc is a function of z, its tangent is so also, and, making the denominator $= \varphi$, we may assume

$$\frac{(x + y + z)z}{z^2 - zx - zy - 2xy} = \frac{(x + y + z)z}{\varphi} = Z \ldots (a).$$

Differentiate this equation, get quit of the denominator φ^2, and compare the result with the proposed equation multiplied by $2z$; since the terms in dx and dy are the same on each side, we equate the other terms to each other, viz.

$$2(x^2 + xy + y^2)zdz = -2(z^2x + z^2y + 2xyz + x^2y + y^2x)dz - \varphi^2 dZ,$$

$$2(x^2z + 3xyz + y^2z + z^2x + z^2y + x^2y + y^2x)dz + \varphi^2.dZ = 0.$$

Putting for φ its value derived from (a), and suppressing the common factor $x + y + z$, we have

$$2(xy + yz + xz)\,Z^2dz + (x + y + z)\,z^2dZ = 0;$$

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* We frequently meet with formulæ in which arcs that are given by their tangents are to be added together. Suppose that we had arc $(\tan = \alpha)$ + arc $(\tan = \beta)$; if m and n denote these two arcs, we have $\alpha = \tan m$, $\beta = \tan n$, and the point is to find an expression for the arc $m + n$. Now [*K*, N°. 359], we have

$$\tan(m + n) = \frac{\alpha + \beta}{1 - \alpha\beta}; \text{ whence } m + n = \text{arc}\left(\tan = \frac{\alpha + \beta}{1 - \alpha\beta}\right).$$

The equation above has been reduced in this manner.

and this is the equation that is to give Z in terms of z, and which must not contain x or y. We deduce from (a)

$$xz + yz = \frac{z^2Z - z^2 - 2xyZ}{Z + 1};$$

substituting, we find that $2Z(z^2 - xy)$ is a common factor, and we have $Z(Z-1)dz + zdZ = 0$; and consequently

$$\frac{dz}{z} = \frac{dZ}{Z} - \frac{dZ}{Z-1}, \quad z = \frac{cZ}{Z-1}, \quad Z = \frac{z}{z-c}.$$

With this value of Z, the equation (a) is the integral required, which may be written $xy + xz + yz = c(x + y + z)$.

861. If the condition (2) be not satisfied, on following the course that has just been explained, dZ can no longer be found expressed in z and Z alone. F being the factor which renders $Pdx + Qdy$ integrable, and $u + Z$ the integral of $FPdx + FQdy$, compare the differential of $u + Z = 0$ with $FPdx + FQdy + FRdz = 0$; we find then

$$u + Z = 0, \quad \frac{du}{dz} + \frac{dZ}{dz} = FR \ldots (5).$$

x, y and z enter into (5), so that we cannot from it deduce Z, nor the required integral, in the same manner as when the condition of integrability is fulfilled. All that we can infer is that $u + Z = 0$ will satisfy the proposed equation and be its integral, if Z can be determined in a function of z, $Z = \phi z$, and so that at the same time the relation (5) be established.

It has been seen [N°. 704] that, in the differentiation of equations, we tacitly suppose the variables x and y to be dependent, by virtue of an arbitrary relation which connects them one with the other: in the case before us, we cannot integrate without establishing this dependence. It is evident that, if we put $Z = \phi z$, the system of our two equations

$$u + \phi z = 0, \quad \frac{du}{dz} + \phi' z = FR \ldots (6)$$

satisfies the equation proposed, whatever be the form of the function ϕ.

The equations which do not satisfy the condition of integrability were formerly termed *Absurd*; and it was established as a principle that they had no meaning, and that a problem susceptible of solution could never lead to this species of relations, which it was maintained were no other than imaginary. Monge, in giving the preceding theory, proved that this idea was erroneous.

If, therefore, we have to investigate a curve surface that shall fulfil certain conditions, which, expressed analytically, lead to a differential equation between the co-ordinates x, y and z, the points of space which satisfy the problem are, in the present case, not those of a surface, but those of a curve of double curvature, since the equation cannot exist except by separating itself into two others, similarly to what we have frequently met with elsewhere [Nos. 112, 533, 576]. x, y and z enter into (5), so that we cannot from it deduce Z, nor the required integral, in the same manner as when the condition of integrability is fulfilled. Likewise, since φ is arbitrary, it is not merely a single curve which corresponds to the problem, but an infinite number of curves subject to a common law.

Thus, for $zdx + xdy + ydz = 0$, we find

$$F = x^{-1},\ R = y,\ y + zlx = u,$$

whence $$y + zlx + \varphi z = 0,\ lx + \varphi' z = yx^{-1},$$

for the equations (6), the system of which satisfies the one proposed, whatever be the function φ.

In the example III of No. 860, we have

$$R = -x(x-a) - y(y-b),\ F = (z-c)^{-1};\ \text{and therefore}$$

$$x^2 + y^2 + 2\varphi z = 0,\ (z-c)\varphi' z + x(x-a) + y(y-b) = 0.$$

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

862. Let the equation be $dz = pdx + qdy$; p and q being the partial differentials of z, in respect to x and y respectively. We have already given the mode of tracing back this equation to its integral $z = f(x,y)$; but let it now be proposed to find $z = f(x, y)$, from the knowledge only of one of the coefficients p and q, or of a relation between them.

Take first the case in which q does not enter into the relation, viz. $F(p, x, y, z) = 0$. We know that the variables x and y are independent in $z = f(x, y)$, it being possible for one to vary without the other [No. 704]; and as q does not enter into the equation proposed, it belongs to the case in which x and z have alone varied, since, though y varied, the given relation $F = 0$ would continue the same. We must, therefore, make $p = \frac{dz}{dx}$ in the proposed equation, and we shall have to integrate an

equation between the variables x and z, y being supposed constant. The constant to be then added to the integral must be *an arbitrary function* of y, which we shall represent by φy.

Hence, *to integrate the equation* $F(p, x, y, z) = 0$, *we must eliminate* p *by means* $dz = pdx$, *integrate considering* y *as constant, and add* φy.

We might reason in the same manner in respect to q for the integration of the equation $F(q, x, y, z) = 0$.

For example, $p = 3x^2$ has for its integral $z = x^3 + \varphi y$.

That of $x = p\sqrt{(x^2 + y^2)}$ is $z = \sqrt{(x^2 + y^2)} + \varphi y$.

For $p\sqrt{(a^2 - y^2 - x^2)} = a$, we find

$$z = a.\,\text{arc}\left(\sin = \frac{x}{\sqrt{(a^2 - y^2)}}\right) + \varphi y.$$

Let $qxy + az = 0$: we integrate $xydz + azdy = 0$, x being constant, and we have $l(z^x y^a) = lc$; whence $z^x.y^a = \varphi x$.

Lastly, let $p(y^2 + x^2) = y^2 + z^2$: hence results the homogeneous equation $(y^2 + x^2)dz = (y^2 + z^2)dx$; and next

$$\text{arc}\left(\tan = \frac{z}{y}\right) - \text{arc}\left(\tan = \frac{x}{y}\right) = c,$$

or [note, p. 422]

$$\text{arc}\left(\tan = \frac{y(z-x)}{y^2 + xz}\right) = c,\ \frac{z - x}{y^2 + xz} = \varphi y.$$

863. Take the *general linear equation of the* 1*st order*

$$Pp + Qq = V,$$

P, Q, V being given functions of x, y, z. Eliminating p from $dz = pdx + qdy$, we shall have

$$Pdz - Vdx = q(Pdy - Qdx) \ldots (1),$$

an equation which must be satisfied in the most general manner, q being any whatever, since, from the equation proposed, q continues indeterminate. When the variables x, y, z are separated in this equation, each side in particular may be rendered integrable.

Let $\varpi = \alpha$, $\varrho = \beta$, be the integrals of the respective equations

$$Pdz - Vdx = 0,\ Pdy - Qdx = 0 \ldots (2);$$

the equation (1) then is equivalent to $\mu d\varpi = \mu' q d\varrho$, μ and μ' being the factors which render the equations (2) integrable; and that this equa-

tion may itself be so, it is incumbent that $\frac{\mu'}{\mu} q$ be a function of ϱ, viz. $\varpi = \varphi\varrho$, φ denoting a function altogether arbitrary.

When x, y, z are intermixed in the equations (2), if $\varpi = \alpha$, and $\varrho = \beta$, be functions which satisfy them, the proposed equation still has for its integral $\varpi = \varphi\varrho$, this point it remains for us to demonstrate.

To ascertain whether the proposed equation be satisfied by any equation $\varpi = \varphi\varrho$, it follows that, having got its differential under the form $dz = pdx + qdy$, we must see whether the values that we thus find for p and q, being substituted, give $Pp + Qq = V$. Now the differentials of $\varpi = \alpha$, $\varrho = \beta$ being

$$d\varpi = Adx + Bdy + Cdz = 0,\ d\varrho = adx + bdy + cdz = 0,$$

we find for the differential of $\varpi - \varphi\varrho = 0$, relative

$$\text{to } z \text{ and } x \ldots\ (C - c.\varphi'\varrho)p + A - a\varphi'\varrho = 0,$$
$$\text{to } z \text{ and } y \ldots\ (C - c\varphi'\varrho)q + B - b\varphi'\varrho = 0.$$

Deducing p and q from this, for the purpose of substituting them in $Pp + Qq = V$, we find that the equation $\varpi = \varphi\varrho$ satisfies the one proposed, if we have

$$AP + BQ + CV = \varphi'\varrho \times (aP + bQ + cV);$$

but if we admit that the functions ϖ and ϱ have been so selected as to satisfy the equations (2), we may from these equations deduce dz and dx for the purpose of substituting in $d\varpi = 0$ and $d\varrho = 0$; when it appears that the equations which express the condition that determines ϖ and ϱ are

$$AP + BQ + CV = 0,\ aP + bQ + cV = 0;$$

thus, $\varpi = \varphi\varrho$ satisfies the equation proposed, the function φ being arbitrary, and $\varpi = \varphi\varrho$ is the integral required.

If dx be eliminated between the equations (2), there results $Qdz - Vdy = 0$: and any two, therefore, of the following equations contain the third, and may be employed indifferently:

$$Pdz - Vdx = 0,\ Pdy - Qdx = 0,\ Qdz - Vdy = 0 \ldots (3).$$

Hence we conclude, that *the integration of the equation of partial differences of the 1st order $Pp + Qq = V$ reduces itself to satisfying two of the equations (3) by some functions $\varpi = \alpha$, $\varrho = \beta$, and assuming $\varpi = \varphi\varrho$, φ denoting an arbitrary function;* α and β are constants, which do not enter into the integral, the function φ containing as many constants as are necessary. If we make $\varphi\varrho$ = constant, we have ϖ = con-

stant, which also satisfies the equation proposed; so that $\varpi = a$ and $\varrho = \beta$ are particular integrals of it.

864. Let us now inquire into what takes place in different cases.

1°. If V be nothing, one of our equations (3) is $dz = 0$, $z = \alpha = \varpi$; there will only remain therefore, in the 2nd, the two variables x and y; the integral $\varrho = \beta$ will then be obtained [chap. IV], and $z = \varphi\varrho$ will be the integral of $Pp + Qq = 0$.

For example, $py = qx$ gives $P = y$, $Q = -x$, $ydy + xdx = 0$, whence $\varrho = x^2 + y^2$, and $z = \varphi(x^2 + y^2)$, the equation of surfaces of revolution about the axis of z [N^{os}. 622 and 705].

For $px + qy = 0$, we find $xdy - ydx = 0$, $ly = lax$, $y = ax$, $\frac{y}{x} = \varrho$;

and lastly $z = \varphi\left(\frac{y}{x}\right)$; which is the equation of the conoids [N°. 748].

Similarly, if $q = pP$, P not containing z, the integral is

$$z = \varphi\varrho,\ \varrho = \int F(dx + Pdy),$$

F being the factor which renders $dx + Pdy$ integrable.

2°. When it occurs that two of the equations (3) contain but two variables and their differentials, the integration easily gives ϖ and ϱ.

Let the equation proposed be $px + qy = nz$; then $xdz = nzdx$, $xdy = ydx$; whence $z = \alpha x^n$, $y = \beta x$; hence we deduce α and β, the values of ϖ and ϱ, and consequently the integral required $z = x^n\varphi\left(\frac{y}{x}\right)$. We see that φ is homogeneous and arbitrary; and since the equation proposed is the enunciation of the theorem regarding homogeneous functions [p. 383], we thus arrive again at the demonstration for the case of two variables.

For $px^2 + qy^2 = z^2$, we have $x^2dz = z^2dx$, $x^2dy = y^2dx$; whence $z^{-1} - x^{-1} = \varpi$, $y^{-1} - x^{-1} = \varrho$; and the integral therefore is

$$\frac{1}{z} - \frac{1}{x} = \varphi\left(\frac{1}{y} - \frac{1}{x}\right), \text{ or } \frac{x-z}{xz} = \varphi\left(\frac{x-y}{xy}\right).$$

X and V being functions of x alone, the equation $q = pX + V$ gives $Xdz + Vdx = 0$, $Xdy + dx = 0$, and

$$z = -\int\frac{Vdx}{X} + \varphi\left(y + \int\frac{dx}{X}\right).$$

3°. When one only of the equations (3) is between two variables, having integrated it, we eliminate, by means of this result $\varpi = \alpha$, one

of the variables from the 2nd or 3rd of our equations, then integrate, and we have $\varrho = \beta$; we replace ϖ for α in ϱ, and we have $\varpi = \varphi\varrho$, or $\varrho = \varphi\varpi$.

Let the equation be $qxy - px^2 = y^2$: then $x^2dz + y^2dx = 0$, $x^2dy + xydx = 0$; the latter of these gives $xy = \beta$; substituting in the other βx^{-1} for y, there results $dz + \beta^2x^{-4}dx = 0$; whence $z = \frac{1}{3}\beta^2x^{-3} + \alpha$; in this replacing xy for β, we have $z - \frac{1}{3}y^2x^{-1} = \alpha = \varpi$; and consequently the integral required is $3xz = y^2 + 3x\varphi(xy)$.

For $px + qy = n\sqrt{(x^2 + y^2)}$, we have

$$xdz = ndx\sqrt{(x^2 + y^2)},\ xdy = ydx;$$

the 2nd gives $y = \beta x$; eliminating y from the 1st, it becomes

$$dz = n\sqrt{(1 + \beta^2)}\,dx;\ \text{whence } z - nx\sqrt{(1 + \beta^2)} = \alpha,$$

then $$\varpi = z - n\sqrt{(a^2 + y^2)},\ \varrho = yx^{-1},$$

and lastly $$z = n\sqrt{(x^2 + y^2)} + \varphi\left(\frac{y}{x}\right).$$

865. But when x, y and z are intermixed in the equations (3), it is no longer possible to integrate each in particular, for y cannot be supposed to be constant in the 1st, nor z in the 2nd... We are then obliged to have recourse to particular analytical artifices. Thus we frequently accomplish the integration, by substituting for p or q, in the following equations, the value derived from the one proposed; these equations result from $dz = pdx + qdy$, treated according to the method of integration by parts:

$$z = px + \int(qdy - xdp)\ldots \qquad (4),$$
$$z = qy + \int(pdx - ydq)\ldots \qquad (5),$$
$$z = px + qy - \int(xdp + ydq)\ldots \quad (6).$$

If, for example, p be a given function of q, as $p = Q$, the relation (6) becomes

$$z = Qx + qy - \int(xQ' + y)\,dq;$$

whence it follows that the factor of dq must not contain either x, or y,

$$xQ' + y = \varphi'q,\ z = Qx + qy - \varphi q;$$

the function φ is arbitrary. The integral results from the elimination of q between these two equations, when this function φ has been determined [N°. 879].

866. After having substituted in $dz = pdx + qdy$ the value of p,

or that of q, derived from the equation proposed, we have a differential equation between the four variables x, y, z and q or p. Suppose that this equation be reducible to the form of an exact differential, by taking as constant p or q, or a function θ of this letter; and let $f(x, y, z, \theta)=c$ be the integral on this hypothesis of θ being constant. It is obvious that if this equation be differentiated, the result will be that whence it was derived, not only if θ and c continue constant, but also if θ and c are variable, provided that we have $\frac{df}{d\theta}d\theta - dc = 0$. Thus, that θ may assume its state of being any variable function in the differential equation, and that at the same time the equation $f = c$ may always be the integral, we have but to suppose that c is an arbitrary function of θ, such that we have at once

$$f(x, y, z, \theta) = \varphi\theta, \frac{df}{d\theta} = \varphi'\theta.$$

In the case in which the proposed equation is an exact differential, θ *being assumed as constant, we shall integrate on this hypothesis, when we shall have the* 1*st of the preceding equations, which we shall then differentiate relatively to θ alone, in order to form the 2nd;* the system of these two equations will satisfy the one proposed, φ being an arbitrary function. When we have determined φ [N°. 879], it will remain to eliminate θ between them, and we shall have the integral required.

It follows from what has been seen in N°. 765, that if the 1st equation be considered as belonging to a curve surface of which θ is a variable parameter, these two equations are those of the characteristic; the investigation of this curve is obviously equivalent to the integration of the equation proposed.

Let the equation $z = pq$ be given: we find

$$dz = \frac{zdx}{q} + qdy, \; dy = \frac{qdz - zdx}{q^2} = \frac{(\theta + x)\,dz - zdx}{(\theta + x)^2},$$

assuming $q = \theta + x$; for θ constant, the integral is

$$y = \frac{z}{x + \theta} + \varphi\theta; \text{ whence } \frac{z}{(x + \theta)^2} = \varphi'\theta,$$

differentiating relatively to θ alone. The system of these two equations is the integral of the one proposed $z = pq$.

867. The integration is frequently facilitated by introducing an indeterminate quantity θ, which allows of the proposed equation being separated into two.

Let $f(p, x) = F(q, y)$: make $f(p, x) = \theta$, whence $F(q, y) = \theta$; resolve these equations in respect to p and q, and we shall have

$$p = \psi(x, \theta),\ q = \chi(y, \theta),\ dz = \psi dx + \chi dy.$$

Integrating on the supposition of θ being constant, according to what has been just stated;

$$z + \phi\theta = \int \psi dx + \int \chi dy:$$

it will remain to differentiate relatively to θ alone, and, having determined the function ϕ, to eliminate θ between these equations.

For example, for the equation $a^2pq = x^2y^2$, we have

$$\frac{ap}{x^2} = \frac{y^2}{aq} = \theta,\ p = \frac{x^2\theta}{a} = \psi,\ q = \frac{y^2}{a\theta} = \chi;$$

and therefore

$$3az + \phi\theta = x^3\theta + \frac{y^3}{\theta},\ \phi'\theta = x^3 - \frac{y^3}{\theta^2}.$$

868. When the equation $Pp + Qy = V$ is homogeneous in x, y, z, we make $x = tz$, $y = uz$; P, Q, V become changed into $P_/z^n$, $Q_/z^n$, $V_/z^n$ [No. 815], and the equations (3) give

$$(P_/ - tV_/)\,dz = zV_/dt,\ (Q_/ - uV_/)\,dz = zV_/du;$$

whence $$(P_/ - tV_/)\,du = (Q_/ - uV_/)\,dt.$$

The integral of this equation in t and u being found, we shall make use of it for eliminating either t or u from one of the preceding, which we can then integrate; lastly, eliminating u and t by $x = tz$, $y = uz$, we have the solutions ϖ and ϱ of the equations (3), and consequently the integral $\varpi = \phi\varrho$.

For the equation $pxz + qyz = x^2$, we find

$$(1 - t^2)\,dz = ztdt,\ u(1 - t^2)\,dz = zt^2du;$$

whence $$udt = tdu,\ t = \alpha u,\ z\sqrt{(1 - t^2)} = \beta;$$

and lastly, $$x = \alpha y,\ \sqrt{(z^2 - x^2)} = \beta,\ z^2 = x^2 + \phi\left(\frac{x}{y}\right).$$

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

869. Besides the coefficients p and q of the 1st order, the equations may contain

$$\frac{dz^2}{dx^2} = r,\ \frac{d^2z}{dxdy} = \frac{d^2z}{dydx} = s,\ \frac{d^2z}{dy^2} = t \ldots\ (A);$$

whence $$dp = rdx + sdy,\ dq = sdx + tdy \ldots\ (B),$$

$$d^2z = dpdx + dqdy = rdx^2 + 2sdxdy + tdy^2.$$

The object is to integrate the equation $f(x, y, z, p, q, r, s, t) = 0$. We shall in the first place observe that y must be considered constant in the equation of the form $r = Pp + Q$, which is equivalent to $\frac{d^2z}{dx^2} = P\frac{dz}{dx} + Q$, P and Q being functions of x, y and z; for the partial differentials q, s and t, which refer to the variation of y, do not enter here [N°. 862]; and we have in this case to integrate an equation of the ordinary differentials of the 2nd order between x and z; but instead of the constant usually added, we must take an arbitrary function φy.

For example, if z do not enter into P and Q, substituting $\frac{dz}{dx}$ for p, we have $\frac{dp}{dx} = Pp + Q$; the function $(Pp + Q)\,dx$ is linear between the variables p and x; y is likewise constant; and the integral therefore, making $u = \int Pdx$, is [N°. 817]

$$p = \frac{dz}{dx} = e^u\left(\int e^{-u}Qdx + \varphi y\right);$$

repeating the integration, and adding a new arbitrary function ψy, we have the integral required.

When $P = 0$, we have $p = \int Qdx + \varphi y$; whence

$$z = \int dx \int Qdx + x\varphi y + \psi y.$$

For the equation $xyr = (n-1)\,py + a$, since, in this example,

$$P = \frac{n-1}{x},\ Q = \frac{a}{xy},$$

we obtain

$$\frac{dz}{dx} = \frac{-a}{(n-1)\,y} + x^{n-1}\varphi y,\ z = \frac{-ax}{(n-1)\,y} + \frac{x^n}{n}\varphi y + \psi y.$$

Lastly, if $xr = (n-1)\,p$, we have $nz = x^n\varphi y + \psi y$.

870. To integrate $t = Pq + Q$, or $\frac{d^2z}{dy^2} = P\frac{dz}{dy} + Q$, we must take x constant, and add φx and ψx.

Let $at = xy$: we have in the first place $q = \frac{dz}{dy} = \frac{y^2x}{2a} + \phi x$; and then

$$6az = y^3x + y\phi x + \psi x.$$

871. The integral of $s = M$, or $\frac{d^2z}{dxdy} = M$, is included in the theory of cubatures [N°. 812];

$$z = \int dx \int Mdy + \phi x + \psi y.$$

Thus, $s = ax + by$ gives

$$z = \tfrac{1}{2}\, xy\,(ax + by) + \phi x + \psi y.$$

872. Let $s = Mp + N$, M and N being known in x and y, or

$$\frac{d^2z}{dxdy} = \frac{dp}{dy} = Mp + N;$$

p and y are here the only variables, and we fall in with a linear equation [N°. 817]; whence, making $u = \int Mdy$, we have

$$p = \frac{dz}{dx} = e^u(\phi' x + \int e^{-u}.Ndy):$$

integrating then, in respect to x, there results

$$z = \int (e^u dx \int e^{-u}Ndy) + \int e^u\phi' xdx + \psi y.$$

For example, for $sxy = bpx + ay$, we find

$$p = \frac{-ay}{(b-1)x} + y^b\phi' x,\ z = \frac{aylx}{1-b} + y^b.\phi x + \psi y.$$

873. Let us take *the linear equation of the 2nd order,*

$$Rr + Ss + Tt = V, \text{ or } R\frac{d^2z}{dx^2} + S\frac{d^2z}{dxdy} + T\frac{d^2z}{dy^2} = V,$$

R, S, T, V, being given in x, y, z, p and q. Eliminating r and t by means of the equations (B), which serve as definitions of these functions, we have

$$Rdpdy + Tdqdx - Vdxdy = s\,(Rdy^2 - Sdxdy + Tdx^2).$$

Suppose that we know two functions ϖ and ϱ, which reduce each side respectively to nothing, or that we have $\varpi = \alpha$, $\varrho = \beta$, with

$$Rdy^2 + Tdx^2 = Sdxdy,$$

$$Rdpdy + Tdqdx = Vdxdy.$$

The point is to prove that here, as in N°. 863, $\varpi = \varphi\varrho$ will satisfy the equation proposed, whatever be the function φ, ϖ and ϱ containing x, y, z, p and q. To demonstrate this, let these equations be in the first place reduced to the 1st order, by assuming $dy = \Omega dx$; there results

$$R\Omega^2 - S\Omega + T = 0 \ldots (1),$$

$$dy = \Omega dx,\ R\Omega dp + Tdq = V\Omega dx \ldots (2).$$

The 1st of these equations gives for Ω two values in x, y, z, p, q; and we suppose that $\varpi = \alpha$, $\varrho = \beta$ have been determined so as to satisfy the equations (2). Express therefore the complete differentials $d\varpi = 0$, $d\varrho = 0$, under the form

$$Adx + Bdy + Cdz + Ddp + Edq = 0,\ adx + bdy \ldots edq = 0;$$

put $pdx + qdy$ for dz, Ωdx for dy; and lastly, for dq its value derived from (2); we shall have then two sorts of terms in each equation, those of the one sort factors of dx, those of the other factors of dp; and these being separately equated to zero (inasmuch as the proposed equation $Rr + Ss \ldots = V$ can determine but one of the quantities r, s, t, in a function of the others, and of x, y, z, p and q, dx and dp remain independent), there results

$$A + \Omega B + (p + q\Omega)\, C + \frac{EV\Omega}{T} = 0,\ D = \frac{ER\Omega}{T},$$

$$a + \Omega b + (p + q\Omega)\, c + \frac{eV\Omega}{T} = 0,\ d = \frac{eR\Omega}{T}.$$

These equations express the condition that ϖ and ϱ satisfy the conditions (2). This being premised, to ascertain whether in fact the equation $\varpi = \varphi\varrho$ do satisfy the one proposed, take its complete differential $d\varpi = \varphi'\varrho . d\varrho$, or

$$Adx + Bdy \ldots Edq = \varphi'\varrho\, (adx + bdy \ldots edq);$$

substitute for A, D, a, d, their values derived from the four preceding equations, and $pdx + qdy$ for dz; then, collecting the terms which have for coefficients B, C, E, b, c, e, we see that they have for a factor one or other of the quantities

$$R\Omega dp + Tdq - V\Omega dx,\ dy - \Omega dx,$$

which are each nothing by virtue of the equations (2), and this whatever the function φ be; consequently *$\varpi = \varphi\varrho$ is the first integral of the equation proposed, denoting an arbitrary function of x and y.*

It appears therefore that our attention must be directed to the equations (2); and besides these two relations, we also have $dz = pdx + qdy$,

which however make but three equations between the five variables x, y, z, p and q. It may happen therefore that the equation which is obtained between three of these quantities, by means of elimination, does not fulfil the condition of integrability [N°. 857]; in which case it cannot have arisen from a single equation. We should thus be led to an integration that cannot be effected, without our being at liberty on that account to conclude that it is not possible, and that the differential equation proposed does not result from a single primitive.

Hence we shall conclude that, to integrate the linear equation of the 2nd order $Rr + Ss + Tt = V$, we must assume the equations (1) and (2); the first will make known Ω, and the substitution of its value in (2) will give two equations which must be satisfied by integrals $\varpi = \alpha$, $\varrho = \beta$: we shall then make $\varpi = \varphi\varrho$, and it will remain to integrate this equation of the 1st order.

Since the equation (1) is of the 2nd degree, there will result from it two values of Ω; that must be taken in preference which is the better adapted for the ulterior calculations.

874. Take, for example, $q^2r - 2pqs + p^2t = 0$: $R = q^2$, $S = -2pq$, $T = p^2$, $V = 0$, give, for the equation (1), $q^2\Omega^2 + 2pq\Omega + p^2 = 0$; whence $q\Omega + p = 0$; and eliminating Ω from equations (2),

$$pdx + qdy = 0,\ qdp = pdq.$$

The latter of these gives $p = \beta q$; the other is equivalent to $dz = 0$, $z = \alpha$; and $\beta = \varphi\alpha$, or $p = q\varphi z$, remains for a fresh integration.

Applying the method of N°. 863, it gives

$$dz = 0,\ dy = -dx.\varphi z;\ \text{whence } z = \alpha,\ y + x\varphi\alpha = \beta;$$

assuming $\beta = \psi\alpha$, there finally results, for the integral required, φ and ψ being the two arbitrary functions, $y + x\varphi z = \psi z$.

The equation $rx^2 + 2xys + y^2t = 0$, in which $R = x^2$, $S = 2xy\ldots$, gives $\Omega x = y$, and the equations (2) become $ydx = xdy$, $xdp + ydq = 0$. The 1st gives $y = \alpha x$; eliminating y from the 2nd, it becomes $dp + \alpha dq = 0$; whence $p + \alpha q = \beta$; lastly, $\beta = \varphi\alpha$ gives, for the first integral, $px + qy = x\varphi\left(\frac{y}{x}\right)$.

The equations (2) of N°. 863 are here $dz = dx.\varphi$, $xdy = ydx$; we derive from the latter of these $y = \alpha x$; eliminating y from the other, $dz = dx.\varphi\alpha$, $z = x\varphi\alpha + \beta$; lastly, $\beta = \psi\alpha$ gives $z = x\varphi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$.

In the example following we have made $p + q = m$,

$$r(1 + qm) + s(q - p)m = t(1 + pm).$$

The equation (1) is

$$(1 + qm)\,\Omega^2 - (q - p)\,m\Omega = 1 + pm;$$

whence $\Omega = 1$. As to the other root of Ω, since it would lead to a first integral, containing $p + q$ under the sign φ, and also without this sign, it cannot be employed. The equations (2) are

$$dy = dx,\ (1 + qm)\,dp = (1 + pm)\,dq.$$

We have from the 1st, $x - y = \alpha$; to integrate the 2nd, making $p - q = n$, and $p + q = m$, we deduce the values of p, q, dp, dq, and we have this separable equation $mndm = (2 + m^2)\,dn$, which gives $n = \beta\sqrt{(2 + m^2)}$; and consequently, $\beta = \varphi\alpha$ becomes

$$n = \sqrt{2 + m^2}.\varphi'(x - y),\ p = q + \sqrt{2 + (p + q)^2}.\varphi'(x - y).$$

To integrate this equation, put for p and q their values in terms of m and n in $dz = pdx + qdy$; there results

$$2dz = (m + n)\,dx + (m - n)\,dy$$
$$= m\,(dx + dy) + (dx - dy)\sqrt{(2 + m^2)}.\varphi'(x - y).$$

Integrating, on the supposition that m is constant, by the method of N°. 866, we find that the integral in question is represented by the system of the two equations

$$2z + \psi m = m\,(x + y) + \sqrt{(2 + m^2)}\,\varphi(x - y),$$

$$\psi' m = (x + y) + \frac{m}{\sqrt{(2 + m^2)}}\,\varphi(x - y).$$

875. The complexity of these calculations is frequently an obstacle to their succeeding; but in the case in which the coefficients R, S, T are constant, and V is a function of x and y, the equation (1) gives for Ω two numerical values, such as m and n: and the relations (2), which $\varpi = \alpha$, $\varrho = \beta$ must satisfy, being integrated give, for the root m,

$$y = mx + \alpha,\ \text{and}\ Rmp + Tq = m\int Vdx:$$

In V we must substitute, for y, its value $mx + \alpha$; and $\int Vdx$ will then depend solely on quadratures. The integration being accomplished, we must add a constant β, and replace α by its value $y - mx$. We shall thus have the required equations $\varpi = \alpha$, $\varrho = \beta$, so that $\varrho = \varphi'\varpi = \varphi'(y - mx)$ will become

$$Rmp + Tq = m\int Vdx + \varphi'(y - mx).$$

We might reason in the same manner for the 2nd root n of Ω, or rather change m into n in the preceding expression; but it will be

sufficient to treat one of these two cases, since the other leads to the same result. We select that which seems the better adapted for the calculation. The integration must now be repeated; for which purpose, resume our 1st integral, and thence deduce the value of p in order to substitute it in $dz = pdx + qdy$: observing that from the nature of the two roots m and n of Ω, we have $Rmn = T$, we find, including in ϕ' the constant divisor m,

$$Rdz - dx \int Vdx - dx\phi'(y - mx) = Rq(dy - ndx).$$

To integrate this equation [N°. 863], we shall equate each side separately to zero; whence

$$y = nx + c,\ Rz - \int dx \int Vdx - \int dx.\phi'(y - mx) = b.$$

Some observations will now be necessary.

1°. We must put $nx + c$ for y in the 2nd equation, and integrate in respect to x; then replace $y - nx$ for c in the result.

2°. An important distinction is to be noticed in the double integrals $\int dx \int Vdx$, since we have first put $mx + \alpha$ for y in V, and $y - mx$ for α in the result; whereas we must make $y = nx + c$ in $dx \int Vdx$, and restore $y - nx$ for c.

3°. $\int dx.\phi'(y - mx)$ becomes $\int dx.\phi'[x(n - m) + c]$, or $\dfrac{\phi}{n - m}$, or rather $\phi[(n - m)x + c]$, including the constant $n - m$ in ϕ; thus, we have $\phi(y - mx)$.

4°. The constant b is some function ψ of c, or $b = \psi(y - nx)$: and therefore

$$Rz = \int dx \int Vdx + \phi(y - mx) + \psi(y - nx).$$

For example, for $r - s - 2t = ky^{-1}$, we have

$\Omega^2 + \Omega = 2$, whence $m = 1,\ n = -2,\ y = x + \alpha,\ y' = \alpha' - 2x$,

and consequently

$$\int Vdx = \int \frac{kdx}{x + \alpha} = kl(x + \alpha) = kly,$$

$$\int dx \int Vdx = \int kdx\, ly = \int kdx\, l(\alpha' - 2x).$$

This integral is easily obtained [N°. 787, or 769, V]; it becomes

$-kx - kyl\sqrt{y}$, replacing $2x + y$ for α'; and thus

$$z + k(x + yl\sqrt{y}) = \phi(y - x) + \psi(y + 2x).$$

For $r = b^2t$ or $\frac{d^2z}{dx^2} = b^2\frac{d^2z}{dy^2}$, which is the equation of vibrating chords [See my *Mec.* N°.310], we have $R = 1$, $T = -b^2$, $S = 0 = V$; whence $\Omega^2 = b^2$, $m = b = -n$, $y = bx + \alpha$, or $y = \alpha' - bx$; lastly, $\int dx \int Vdx = 0$; and consequently

$$z = \phi(y - bx) + \psi(y + bx).$$

We shall refer the reader, for more ample details on this subject, to the *Integral Calculus* of M. Lacroix.

876. We sometimes integrate by following the method of N°. 867, which consists in separating the proposed equation into two others by means of an indeterminate quantity θ. For example, the equation $rt = s^2$, that of developable surfaces [N°. 766], gives $\frac{r}{s} = \theta = \frac{s}{t}$,

whence $\qquad r = s\theta,\ s = t\theta,\ rdx + sdy = \theta(sdx + tdy)$,

or $dp = \theta\, dq$ [B, N°. 869]. This equation is not integrable unless θ be a function of q; and consequently $p = \phi q$ is the 1st integral. The equation $dz = pdx + qdy$ becomes $dz = dx\phi q + qdy$; and supposing q constant, there results, by the method of N°. 866,

$$z = x\phi q + qy + \psi q,\ x\phi' q + y + \psi' q = 0.$$

All the developable surfaces are comprised in the system of these two equations; to determine any one of them in particular, the functions ϕ and ψ must be found, and q be eliminated between the two resulting equations.

INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS BY SERIES.

877. As to approximate integrals, take for example the 2nd order, and let an equation be given between $r, s, t \dots x$: fix on one of the variables, as x, and assume the formula of Maclaurin [N°. 706],

$$z = f + xf' + \tfrac{1}{2}x^2f'' + \tfrac{1}{6}x^3f''' + \dots,$$

where $f, f', f'' \dots$ are certain functions of y, the values of the integral

$z=f(x,y)$ and its derivatives relative to x, when x is made$=0$.* Deduce from the proposed equation $r=F(t,s,p,q,x,y)$; it is clear, then, that if x be changed into zero, and therefore z into f, p into f', and lastly, s or $\frac{dp}{dy}$ into $\frac{df'}{dy}$, there will enter into the value of r only f,f', and their derivatives relative to y, since q becomes $\frac{df}{dy}$, and $t=\frac{d^2f}{dy^2}$: but r will thus become f''. Similarly, the derivative of r, relative to x, will give f''', by means of the same functions f and f' which remain any whatever; and so on for $f^{\text{IV}}, f^{\text{V}}\ldots$, so that the series will contain two arbitrary functions of y.

For the 3rd order, the same reasoning proves that the series above is the integral, and contains three arbitrary functions f, f', f''. And generally, *every partial differential equation of the order n has an integral which contains n arbitrary functions.*

878. Lagrange has also proposed to approximate to integrals by the method of indeterminate coefficients. We are to assume

$$z=\phi+x\psi+x^2\chi+x^3\varpi+x^4\omega\ldots;$$

and taking the proper differentials, substituting in the equation proposed, and equating to each other the terms into which x enters in the same degree, we shall have different equations which will serve to determine the functions of y, those only excepted that ought to remain arbitrary.

For instance, for $r=q$, we find

$$\frac{d^2z}{dx^2}=r=2\chi+6x\varpi+12x^2\omega\ldots,$$

$$\frac{dz}{dy}=q=\phi'+x\psi'+x^2\chi'\ldots;$$

whence, substituting in $r=q$ and comparing, we find

$$z=\phi+x\psi+\tfrac{1}{2}x^2\phi'+\tfrac{1}{6}x^3\psi'+\tfrac{1}{24}x^4\phi''\ldots$$

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* If the function $f(x,y)$ be of such a nature as to give infinity for any of the values of $f, f', f''\ldots$, we must, as in N°. 831, change x into $x-a$; a being, in the proposed equation, a constant that we assume at pleasure, so as to preclude our meeting with derivatives that are infinite in the processes which are about to be pointed out.

Similarly, for the equation $\frac{d^2z}{dx^2}+\frac{d^2z}{dy^2}+\frac{d^2z}{dt^2}=0$, we find

$$z=\varphi+x\psi-\frac{x^2}{2}\left(\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dt^2}\right)-\frac{x^3}{2.3}\left(\frac{d^2\psi}{dy^2}+\frac{d^2\psi}{dt^2}\right)\ldots$$

ARBITRARY FUNCTIONS.

879. We may apply to the arbitrary functions $\varphi, \psi\ldots$ of the partial differential equations, the remarks that were made [N°. 799] respecting the constants introduced in the ordinary integrations. So long as our object is merely to integrate, *i. e.* to compose an expression which, being submitted to the rules of the Differential Calculus, will satisfy the equation proposed, these functions $\varphi, \psi\ldots$ are in fact any whatever. But if the results are to be applied to questions of Geometry, Mechanics, &c., these functions may cease to be arbitrary. A few examples will render this more intelligible.

It has been seen [N^{os}. 620, 705] that the equation of the cylindrical surfaces is

$$y-bz=\varphi(x-az), \text{ or } ap+bq=1;$$

the 1st being the integral of the 2nd, and the form of the function φ depending on the curve which serves for the directrix. If, now, the base of the cylinder on the plane xy be given by its equation $y=fx$, it follows that φ must be such, that this base shall be comprised among the points of space designated by the equation $y-bz=\varphi(x-az)$; and if therefore we make $z=0$, the equations $y=\varphi x$, $y=fx$ will be identical. The functions φ and f are consequently of the same form, *i. e.* if, in $y=fx$, we change y into $y-bz$, and x into $x-az$, the equation which we obtain will be that of the particular cylinder referred to [N°. 748, I].

Generally, let $M=0$, $N=0$ be the equations of the directrix: we shall assume $x-az=u$, and eliminating between these three equations, we shall from them deduce the values of x, y, z, and consequently that of $y-bz$, or φu, in functions of u, *i. e.* we shall have the mode in which φu is composed of u. It will but remain then to put $x-az$ for u, in $y-bz=\varphi u$, and we shall have the equation of the particular cylindrical surface referred to.

Similarly, the surfaces of revolution about the axis of z have for their equation $py=qx$, the integral of which is $x^2+y^2=\varphi z$ [N^{os}. 622, 705]; and the function φ continues indeterminate so long as the generatrix of

the surface is any whatever. If, however, this curve be given by its equations $M=0$, $N=0$, in all its positions it will be on the surface; and x, y, z will be common to both: assume $z=u$, eliminate x, y and z between these three equations, then substitute their values in $x^2 + y^2 = \phi u$, and we shall know how the function ϕ is composed of u; z therefore being restored for u, and $x^2 + y^2$ for ϕu, we shall have ϕ particularized, so that the equation will belong exclusively to the surface proposed.

And if the body be generated by the revolution of a moveable surface, which is invariably connected with the axis of z, and for which, by considering it in one of its positions, we have the equation $M = 0$, differentiating this, we shall arrive at the expressions for p and q in terms of x, y and z; which being substituted in $py - qx = 0$, we shall have the equation $N = 0$ of the curve of contact of the generating body with the surface generated, since the tangent planes are common to both. We thus, therefore, have the equations of a curve which may be considered in the light of a generatrix, and the case becomes the same with the preceding.

The conoid has for its equation $px + qy = 0$, the integral of which is $y = x\phi z$ [p. 319]. Making $z = u$, deduce x, y, z in terms of u, by means of the equations $M = 0$, $N = 0$ of the directrix; then, put for x and y their values in $y = x\phi u$, and we shall know how ϕu is composed of u. Lastly, replacing u by z, we shall have ϕz, and the particular equation $y = x\phi z$ of the conoid proposed.

When the directrix is a circle traced in a plane parallel to yz, and its equations are $x = a$, $y^2 + z^2 = b^2$, we find $a^2y^2 + z^2x^2 = b^2x^2$.

The equation of the cones is $z - c = p(x - a) + q(y - b)$, the integral of which is $\dfrac{y-b}{z-c} = \phi\left(\dfrac{x-a}{z-c}\right)$ [Nos. 621, 705]. That the base may be a circle traced on the plane xy and with its centre at the origin, we have

$$z = 0,\ x^2 + y^2 = r^2, \text{ and } x - a = u(z - c),$$

whence $$z = 0,\ x = a - cu,\ y = \sqrt{[r^2 - (a - cu)^2]}.$$

These values being substituted for x, y and z in $y - b = (z - c)\phi u$, we have $c\phi u = b - \sqrt{[r^2 - (a - cu)^2]}$; and finally, replacing for u and ϕu their values, we have for the equation of the cone, as in p. 206,

$$(cy - bz)^2 + (az - cx)^2 = r^2(z - c)^2.$$

880. These examples will sufficiently show how the arbitrary functions are to be determined, when we wish to apply the general processes to particular cases. In general, let $K = \phi(L)$ be an integral contain-

ing an arbitrary function φ, K and L being given functions of x, y and z: the condition specified establishes that the equation becomes $F(x, y, z) = 0$, when we suppose $f(x, y, z) = 0$. This condition is equivalent, in Geometry, to demanding that the surface under investigation, the equation of which is $K = \varphi L$, pass through the given curve, the equations of which are $F = 0, f = 0$. To satisfy this condition, we shall assume $L = u$; from these three equations deduce x, y, and z in terms of u; then, substituting these values in K, we shall have for our result a function $K_{\prime}$, which will be $= \varphi u$ expressed in u, and whence the composition of this function φ will be determined. Finally, let L be replaced for u, in $K = K_{\prime}$, and we shall have the integral required.

Should we have two arbitrary functions to determine, it will be requisite that two conditions be given; when a calculation similar to the preceding will make known these functions.

881. But if the nature of the question (and this will be found to be the case in a great number of problems in Physics and high Geometry) do not allow of the arbitrary functions being determined, they remain any whatever, and the properties which we have discovered, without these functions being particularized, exist generally. To draw our illustrations from Geometry, if there be found a term of the form φx, and we proceed to describe on the plane xy the line which has for its equation $y = \varphi x$, its several ordinates y will be the values of the function φ, so that this curve may not only be any whatever, but may also be struck off at once by a free and irregular movement: the curve too may be *Discontinuous*, *i. e.* formed of different branches placed together, or *Discontiguous*, *i. e.* formed of parts isolated and detached from each other. Euler was the first to establish these principles with certainty, in opposition, in fact, to the opinion of D'Alembert, who may be regarded as the inventor of the calculus of partial differences; a calculus, the resources of which are immense, its applications of unbounded utility, and which, as we have seen, furnishes us with the means of submitting irregular functions to mathematical analysis.

VI. CALCULUS OF VARIATIONS.

882. The *Isoperimetrical* Problems had been already solved by different geometricians before the discovery of the Calculus of Variations; but the processes of which they made use did not constitute a regular

system, each of these problems being solved only by a method peculiar to itself, and by artifices of analysis that were frequently involved in great intricacy. It was reserved for the celebrated Lagrange to reduce all the solutions under one uniform method, of which we shall now give the substance.

A function $Z = F(x, y, y', y'' \ldots)$ being given, where y' $y''\ldots$ denote the derivatives of y considered as a function of x, $y = \phi x$, it may be proposed to render Z possessed of different properties (as that of being a maximum, or any other), either by assigning to the variables x, y, certain *numerical values*, or by establishing relations between these variables, and *connecting them by equations*. When the equation $y = \phi x$ is given, we deduce from it y, y', $y''\ldots$ in functions of x, and substituting, Z becomes $= fx$. It can then be assigned, by the known rules of the differential Calculus, what are the values of x which render fx *a maximum or a minimum;* and we thus determine what are the points of a given curve, for which the proposed function Z is greater or less than for any other point of the same curve.

But if the equation $y = \phi x$ be not given, then, different forms being successively taken for ϕx, the function $Z = fx$ will also have different values in x; and it may be proposed to assign to ϕx such a form as shall render Z greater or less than it is for any other form of ϕx, *for the same numerical value of x, whatever it may be.* This latter species of problem belongs to the *Calculus of Variations.* The calculus itself is far from being confined to the theory of *maxima and minima;* but we shall rest satisfied with discussing this branch, it being amply sufficient for a complete understanding of the different rules. We must at the same time keep in mind that, in what follows, *the variables x and y are not independent;* it is only that the equation $y = \phi x$, by which they are connected with each other, is unknown, and that, if we suppose it to be given, it is solely with the view of facilitating the solution of the problem: x must be regarded as an indeterminate quantity which remains the same for the different forms $y = \phi x$; and the forms of ϕ, ϕ', $\phi''\ldots$ are therefore variable, whilst x is constant.

883. In $Z = F(x, y, y', y'' \ldots)$ put $y + k$ for y, $y' + k'$ for $y' \ldots$, k being an arbitrary function of x, and k', $k''\ldots$ its derivatives. Then, Z will become

$$Z_{,} = F(x, y + k, y' + k', y'' + k'' \ldots);$$

and since Taylor's theorem holds good [N°. 703], whether the quantities x, y, k be dependent or independent, we have

$$Z_{\prime} = Z + \left(k.\frac{dZ}{dy} + k'\frac{dZ}{dy'} + k''\frac{dZ}{dy''} + \&c.\right) + \&c.;$$

and $x, y, y', y''\ldots$ may be considered as so many independent variables, so long as our object is but to arrive at this development.

This being premised, the nature of the question requires that the equation $y = \phi x$ be determined in such a manner that, for the same value of x, we have throughout $Z_{\prime} > Z$, or $Z_{\prime} < Z$: and reasoning as in the theory of the ordinary *maxima and minima* [N°. 717], it will be apparent that the terms of the 1st order must be nothing, and that we have

$$k.\frac{dZ}{dy} + k'.\frac{dZ}{dy'} + k''.\frac{dZ}{dy''} + \&c. = 0.$$

Since k is arbitrary for each value of x, and that it is not incumbent that its value, or its form, remain the same, when x varies or is constant, $k', k''\ldots$ are equally arbitrary with k. For, for any value $x = X$, we may assume $k = a + b(x - X) + \frac{1}{2}c(x - X)^2 + \&c.$, $X, a, b, c\ldots$ being taken at pleasure; and since this equation and its differentials are to stand good, whatever x be, they must subsist when $x = X$, which gives $k = a$, $k' = b$, $k'' = c\ldots$ Our equation $Z_{\prime} = Z + \ldots$ cannot be satisfied therefore, having regard to the independence of $a, b, c\ldots$, unless each term be individually nothing. Thus, the equation separates itself into as many others as it contains terms, and we have

$$\frac{dZ}{dy} = 0, \frac{dZ}{dy'} = 0, \frac{dZ}{dy''} = 0\ldots, \frac{dZ}{dy^{(n)}} = 0,$$

n being the highest order of y in Z. These several equations must all agree with each other, and subsist simultaneously, whatever x be. If this agreement really take place, there will be a *maximum* or *minimum*, and the relation which thence results between y and x will be the required equation, $y = \phi x$, which will have the property of rendering Z greater or less than can be effected by any other relation between x and y. The *maximum* will be distinguished from the *minimum*, as in the ordinary theories, by the signs of the terms of the 2nd order of Z, [See p. 289].

If however these equations all give different relations between x and y, the problem will be impossible in the state of generality that has been given to it; and if it happen that some only of these equations accord with each other, then the function Z will have *maxima* and *minima*, relative to certain of the quantities $y, y', y''\ldots$, without having them absolute and common to all these quantities. The equations between which there is the accordance will give the relations which esta-

blish the relative *maxima* and *minima*. And if we wish to render Z a *maximum* or *minimum* in respect only to one of the quantities $y, y', y''...$, since in that case we have but to satisfy one equation, the problem will always be possible.

884. It follows from the preceding considerations, that

1°. The quantities x and y are dependent on each other, and yet are to be made to vary as though they were independent, this being no more than a process of calculation for arriving at the result.

2°. These variations are not infinitely small; and if we employ the Differential Calculus for the purpose of obtaining them, it is but as an expeditious mode of arriving at the second term of the development, the only one which is here necessary.

Let us now apply these general ideas to some examples:

1°. The equation of a curve being $y = \varphi x$, take on the axis of x two abscissæ m and n, and draw through their extremities two indefinite parallels to the axis of y: if then at any point of the curve a tangent be drawn, it will cut our parallels in points which have for ordinates [N°. 722] $l = y + y'(m - x)$ and $h = y + y'(n - x)$. Supposing the form of φ to be given, every thing in these expressions is known; but if it be not so, it may be asked what is the curve which possesses the property of having, for every point of contact, the product of these two ordinates less than for any other curve. In the present instance we have $Z = l \times h$, or

$$Z = [y + (m - x)y'][y + (n - x)y'].$$

According to the enunciation of the problem, the curves *which pass through any the same point* (x, y), have tangents of different directions, and that which we are in quest of must have a tangent such, that the condition $Z =$ *a maximum* be fulfilled. We must therefore consider x and y as constant in $dZ = 0$; whence

$$\frac{dZ}{dy'} = 0, \frac{2y'}{y} = \frac{2x - m - n}{(x - m)(x - n)} = \frac{1}{x - m} + \frac{1}{x - n};$$

and then by integration,

$$y^2 = C(x - m)(x - n).$$

The curve is an ellipse or an hyperbola, accordingly as C is negative or positive, the vertices being given by $x = m$ and $= n$: in the first case, the product $l.h$, or Z, is a *maximum*, since y'' has the the sign $-$; in the second, Z is a *minimum*, or rather a negative *maximum*: this

product moreover is constant, viz. $lh = -\frac{1}{4}C(m-n)^2$, the square of the semi-axis minor; as will be found by substituting in Z the values of y and y'.

II. What is the curve for which, at each of its points, the square of the sub-normal together with the abscissa is a *minimum*? We have $(yy' + x)^2 = Z$, whence we deduce two equations, which agree with each other, on making $(yy' + x) = 0$, and consequently $x^2 + y^2 = r^2$. Hence the several circles described from the origin as centre, and they alone, satisfy the question.

885. The theory that has now been explained is of no great extent; but it serves as a preliminary development, and is of use for the understanding of the much more interesting problem which it remains for us to solve. The object is to apply all the preceding reasons to a function of the form $\int Z$: the sign $\int$ indicating that the function Z is of a differential form, and that after having integrated between the specified limits, we wish it to be possessed of the forementioned properties. The difficulty therefore which we here meet with arises from the necessity that there is of solving the problem without first effecting the integration; for it is obvious that in general it is impossible to accomplish it.

When a body moves, we may compare with each other, either the different points of the body in one of its positions, or the loci successively occupied by a specified point in succeeding instants of time.

In the first case, the body is considered as fixed, and the sign d will refer to the changes in the co-ordinates of its surface; in the second, we have to express, by a new sign, variations altogether independent of the former, and we shall for this purpose make use of δ. When a curve is considered as immoveable, or indeed as variable, but taken in some one of its positions, dx, dy... indicate a comparison between its co-ordinates; but, in order to take into account the different loci occupied by one and the same point of a curve, which varies its form according to some law, we shall write δx, δy..., which denote the increments considered in this point of view, and are functions of x, y... Similarly, dx becoming $d(x + \delta x)$, it will be augmented by $d\delta x$; d^2x will be augmented by $d^2\delta x$, &c.

It is to be observed that the variations indicated by the sign δ are finite and altogether independent of those denoted by the characteristic d: and the operations therefore to which these signs refer being equally independent of each other, it is immaterial, as to the result, in what order they are executed. So that $\delta.dx$ and $d.\delta x$ are identical with each other, as also $d^2.\delta x$ and $\delta.d^2x$..., and $\int \delta U$ and $\delta \int U$.

Our object now is to establish such relations between x and y, that $\int Z$ shall be a *maximum* or a *minimum between certain specified limits.*

In order to render the calculations more symmetrical, we shall here suppose no variable to be constant; we shall also introduce but three variables, this being sufficient for the understanding of the theory, and it being easy to generalize the results. For brevity's sake, let $dx, d^2x\ldots$, $dy, d^2y\ldots$, &c., be replaced by $x_{,}, x_{,,}\ldots, y_{,}, y_{,,}\ldots$, &c., so that

$$Z = F(x, x_{,}, x_{,,}\ldots, y, y_{,}, y_{,,}\ldots, z, z_{,}, z_{,,}\ldots):$$

then x, y, z receiving the arbitrary and finite increments $\delta x, \delta y, \delta z$, dx or $x_{,}$ becomes $d(x + \delta x) = dx + \delta dx$, or $x_{,} + \delta x_{,}$; similarly $x_{,,}$ is increased by $\delta x_{,,}$, and so on for the others; so that, $Z_{,}$ being developed by the theorem of Taylor, and the result integrated, $\int Z$ becomes

$$\int Z_{,} = \int Z + \int\left(\frac{dZ}{dx}.\delta x + \frac{dZ}{dy}.\delta y + \frac{dZ}{dz}.\delta z + \frac{dZ}{dx_{,}}.\delta x_{,}\right.$$

$$\left. + \frac{dZ}{dy_{,}}\delta y_{,} + \frac{dZ}{dz_{,}}\delta z_{,} + \frac{dZ}{dx_{,,}}\delta x_{,,} + \frac{dZ}{dy_{,,}}\delta y_{,,} + \frac{dZ}{dz_{,,}}.\delta z_{,,} + \ldots\right) + \int \&c.:$$

And the condition of the *maximum* or *minimum* requires, as has been already seen, that the integral of the terms of the 1st order be nothing between the limits specified, *whatever* $\delta x, \delta y$, *and* δz *be.* Now let the differential of the known function Z be taken, considering $x, x_{,}, x_{,,}\ldots$, $y, y_{,}, y_{,,}\ldots$, as so many independent variables; we shall have then

$$dZ = mdx + ndx_{,} + pdx_{,,}\ldots + Mdy + Ndy_{,}\ldots + \mu dz + \nu dz_{,}\ldots,$$

$m, n\ldots, M, N\ldots, \mu, \nu\ldots$, being the coefficients of the partial differences of Z in respect to $x, x_{,}\ldots, y, y_{,}\ldots, z, z_{,}\ldots$, treated as so many variables; and which therefore are known functions for each proposed value of Z. This process of differentiation being in like manner carried into effect by the sign δ, we have

$$\left.\begin{aligned}\delta Z = m.\delta x + n.\delta dx + p.\delta d^2x + q.\delta d^3x + \ldots\\ + M.\delta y + N.\delta dy + P.\delta d^2y + Q.\delta d^3y + \ldots\\ + \mu.\delta z + \nu.\delta dz + \pi.\delta d^2z + \chi.\delta d^3z + \ldots\end{aligned}\right\}(A).$$

But this quantity, which is known and is limited in the number of its terms, is precisely that which comes under the sign $\int$, in the terms of the 1st order of our development: so that the condition of the *maximum* or the *minimum* required is, that $\int\delta Z = 0$, between the limits specified, whatever be the variations $\delta x, \delta y, \delta z$. It must be observed that here, as before, the differential calculus is employed but as an easy mode of ob-

taining the assemblage of terms which it is necessary to equate to zero; so that the variations are still finite and any whatever.

It has been already stated that $d.\delta x$ may be introduced in lieu of δdx; and thus the 1st line of the above expression is equivalent to

$$m.\delta x + n.d\delta x + p.d^2\delta x + q.d^3\delta x + \&c.:$$

m, n... contain differentials, so that the want of homogeneity is but apparent. The point now is to integrate: but, in the first place, it will be evident from the course of the calculation that we ought to detach from the sign $\int$ as many as possible of the terms which contain $d\delta$. To accomplish this, we employ the formula of integration by parts [p. 336]:

$$\begin{aligned}
&\int n.d\delta x = n.\delta x \quad - \int dn.\delta x,\\
&\int p.d^2\delta x = p.d\delta x - dp.\delta x \quad + \int d^2p.\delta x,\\
&\int q.d^3\delta x = q.d^2\delta x - dq.d\delta x - \int d^3q.\delta x, \ \&c.
\end{aligned}$$

Combining these results, we arrive at this series the law of which will easily be recognised:

$$\int(m - dn + d^2p - d^3q + d^4r - \ldots)\delta x$$
$$+ (n - dp + d^2q - d^3r\ldots)\delta x + (p - dq + d^2r\ldots)d\delta x + (q - dr\ldots)d^2\delta x$$
$$+ \&c.$$

The integral of (A), or $\int.\delta Z = 0$, becomes therefore

$$(B)\ldots\int[(m - dn + d^2p\ldots)\delta x + (M - dN + d^2P\ldots)\delta y + (\mu - d\nu\ldots)\delta z\ldots] = 0,$$

$$(C)\ldots\left\{\begin{aligned}
&(n - dp + d^2q\ldots)\delta x + (N - dP + d^2Q\ldots)\delta y + (\nu - d\varpi + \ldots)\delta z\\
&+ (p - dq + d^2r\ldots)d\delta x + (P - dQ\ldots(d\delta y + (\varpi - d\chi\ldots)d\delta z\\
&+ (q - dr\ldots)d^2\delta x + \&c.\ldots + K = 0,
\end{aligned}\right.$$

K being the arbitrary constant. We have divided our equation into two, because the terms which remain under the sign $\int$ cannot be integrated, except by giving to δx, δy, δz particular values, which is contrary to the hypothesis, so that $\int\delta Z$ cannot become $= 0$, unless these terms be in themselves nothing; and in fact if the nature of the question do not establish any relation between δx, δy and δz, the independence of these variations requires that the equation (B) separate itself into three others,

$$\left.\begin{aligned}
0 &= m - dn + d^2p - d^3q + d^4r - \ldots\\
0 &= M - dN - d^2P - d^3Q + d^4R - \ldots\\
0 &= \mu \ - d\nu \ + d^2\varpi - d^3\chi + d^4\varrho - \ldots
\end{aligned}\right\}\ldots(D).$$

886. Hence, to find the relations between x, y and z, which render

$\int Z$ a *maximum*, we must take the differential of the given function Z, considering x, y, z, dx, dy, dz, d^2x... as so many independent variables, the increments being designated by the use of the letter δ; which is what is termed *taking the variation of* Z. The result being compared with the equation (A), we shall thence deduce the values of m, M, μ, n, N..., in terms of x, y, z and their differentials expressed by d; which values must then be substituted in the equations C and D. The 1st of these depends on the limits between which the *maximum* ought to exist; the equations (D) constitute the relations required; they are of the differential form between x, y, z; and, the case of absurdity excepted, they cannot form distinct conditions, since from them are to be determined numerical values for the variables. If the question proposed belong to Geometry, these equations are those of the curve or of the surface which possesses the property required.

887. Since the integration, when carried into effect, is to be taken between certain specified limits, the terms which remain and compose the equation (C) must also be in correspondence to these limits. This equation is now become of the form $K + L = 0$, L being a function of $x, y, z, \delta x, \delta y, \delta z$... Let the numerical values of these variables be marked by one accent for the 1st limit, and by two for the 2nd: since then the integral is to be taken between these limits, we must mark the different terms of L, which compose the equation (C), first with one, then with two accents; subtract the 1st result from the 2nd, and equate this difference to zero [N°. 799]; so that the equation $L_{\prime\prime} - L_{\prime} = 0$ will no longer contain variables, since x, δx... will have assumed the values $x_{\prime}, \delta x_{\prime}, \ldots, x_{\prime\prime}, \delta x_{\prime\prime}, \ldots$, assigned by the limits of the integration. It must be kept in mind that these accents refer to the limits of the integral, and do not denote derivatives.

Four cases now present themselves:

1°. *If the limits are given and fixed,** *i. e.* if the extreme values of x, y and z are constant, since $\delta x_{\prime}$, $d\delta x_{\prime}$, &c., $\delta x_{\prime\prime}$, $d\delta x_{\prime\prime}$, &c., are nothing, the several terms of $L_{\prime}$ and $L_{\prime\prime}$ are $= 0$, and the equation (C) is satisfied of itself. The constants which the integration introduces into the equations (D) are then determined from the conditions consequent on the limits.

* This case is equivalent, in Geometry, to that of investigating a curve which, besides that it must have the property of the *maximum* or *minimum* required, must also pass through two given points. The equations (D) are those of the curve in question; and the constants are determined from the condition that this curve pass through the two points mentioned.

2°. *If the limits are arbitrary and independent,* then each of the coefficients of $\delta x_{,}$, $\delta x_{,,}$..., in the equation (C), is individually nothing.

3°. *If there exist equations of conditions for the limits,** i.e. if the nature of the question connect with each other, by means of equations, certain of the quantities $x_{,}$, $y_{,}$, $z_{,}$, $x_{,,}$, $y_{,,}$, $z_{,,}$, we shall make use of the differentials of these equations for obtaining certain of the variations $\delta x_{,}$, $\delta y_{,}$, $\delta z_{,}$, $\delta x_{,,}$..., in functions of the others; substitution being made in $L_{,,} - L_{,} = 0$, these variations will be found reduced to the lowest number possible: those remaining being absolutely independent, the equation will become separated into several others, by equating their coefficients to zero.

Instead of this course, we might adopt the following one, which is the more elegant. Let $u = 0$, $v = 0$... be the given equations of conditions; we shall multiply their variations δu, δv... by indeterminate quantities λ, λ'..., which will give $\lambda\delta u + \lambda'\delta v$.., a known function of $\delta x_{,}$, $\delta x_{,,}$, $\delta y_{,}$...; and this sum being added to $L_{,,} - L_{,}$, we shall have

$$L_{,,} - L_{,} + \lambda\delta u + \lambda'\delta v + ... = 0... (E).$$

We shall treat the several variations $\delta x_{,}$, $\delta x_{,,}$..., as independent, and having equated their coefficients to zero, we shall between these equations eliminate the indeterminate quantities λ, λ'...; when we shall arrive at the same result as by the preceding method; for the operations performed are such as are entirely allowable, and we thus obtain the same number of final equations. This process is tantamount to the method of elimination given in the Algebra [N°. 111].

It is to be observed that we are not at liberty to conclude, from the equations $u = 0$, $v = 0$..., that at the limits we have $du = 0$, $dv = 0$; these conditions are independent, and may very probably not be coexistent. If, however, this were the case,† $du = 0$, $dv = 0$... would have to be regarded as new conditions, and besides the term $\lambda\delta u$, it would be necessary also to include $\lambda'\delta du$...

4°. We shall say nothing as to the case in which one of the limits is

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* This signifies, in Geometry, that the curve required is to be terminated at points which are no longer fixed, but which must be situated on two curves or two surfaces that are given.

† If the question under consideration be one of Geometry, the curve required must, in this case, have at its limit a contact of a certain order with the curve or the surface, the equation of which is $u = 0$.

fixed and the other subjected to certain condition, or is, in fact, altogether arbitrary,* since it is included in the three preceding cases.

888. It may also happen that the nature of the question subjects the variations δx, δy and δz to certain conditions given by equations $\epsilon = 0$, $\theta = 0$..., and this independently of the limits; as, for example, when the curve required is to be traced on a given curve surface. In this case the equation (B) will not separate itself into three, and the equations (D) will no longer have existence. We must in the first place reduce the variations, as above, to the lowest number possible in the formula (B) by means of the equations of condition, and equate the coefficients of the remaining variations to zero; or, what comes to the same thing, add to (B) the terms $\lambda\delta\epsilon + \lambda'\delta\theta + \ldots$; divide this equation into others by considering $\delta x, \delta y, dz$ as independent; and, finally, eliminate the indeterminate quantities λ, λ'...

It is observable that, in the particular cases, it is frequently preferable to carry into effect, on the given function Z, the whole of the calculations which have led to the equations (B) and (C), instead of comparing each particular case with the general formulæ just given.

Such are the general principles of the calculus of variations: let us apply them to some examples:

889. *What is the plane curve CMK* [fig. 24], *of which the length MK, comprised between two radii vectores AM and AK, is the least possible?* We have [N^os. 763, 729] $s = \int\sqrt{(r^2d\theta^2 + dr^2)} = Z$, and the point is, to find the relation $r = \varphi\theta$, which renders Z a *minimum*. The variation is

$$\delta Z = \frac{rd\theta^2.\delta r + r^2d\theta.\delta d\theta + dr.\delta dr}{\sqrt{(r^2d\theta^2 + dr^2)}};$$

comparing this with the equation (A), where we shall suppose $x = r$, $y = \theta$, we have $m = \frac{rd\theta^2}{ds}$, $n = \frac{dr}{ds}$, $M=0, N=\frac{r^2d\theta}{ds}$, $0=p=P=\varpi=\ldots$; and the equations (D) are

$$\frac{rd\theta^2}{ds} = d\left(\frac{dr}{ds}\right), \frac{r^2d\theta}{ds} = c.$$

* In this case, one of the extremities of the curve required is compelled to pass through a fixed point, whilst the other may either be any whatever, or be situated on a curve or surface that is given.

Eliminating $d\theta$, then ds, between these equations and $ds^2 = r^2d\theta^2 + dr^2$, it will be found that they agree; so that it is sufficient to integrate one of them. But the perpendicular AI, let fall from the origin A on any tangent TM, is

$$AI = AM \times \sin AMT = r \sin \beta,$$

which is equivalent [p. 331] to

$$AI = \frac{r \tan \beta}{\sqrt{(1 + \tan^2\beta)}}, \text{ or } \frac{r^2d\theta}{\sqrt{(r^2d\theta^2 + dr^2)}} = \frac{r^2d\theta}{ds} = c;$$

and since this perpendicular is here constant, the line required is a straight line. The limits M and K being indeterminate, it has not been necessary to employ the equation (C).

890. *To find the shortest line between two given points, or two given curves.*

The length s of the line is $\int Z = \int \sqrt{(dx^2 + dy^2 + dz^2)}$ No. 751; and this quantity is to be rendered a *minimum*: we have

$$\delta Z = \frac{dx}{ds}\delta dx + \frac{dy}{ds}\delta dy + \frac{dz}{ds}\delta dx;$$

and comparing with the formula (A), we find

$$m = 0,\ M = 0,\ \mu = 0,\ n = \frac{dx}{ds},\ N\frac{dy}{ds},\ \nu = \frac{dz}{ds};$$

the other coefficients $P, p, \varpi\ldots$ are nothing. The equations (D), therefore, here become

$$d\left(\frac{dx}{ds}\right) = \quad d\left(\frac{dy}{ds}\right) = 0,\ d\left(\frac{dz}{ds}\right) = 0;$$

whence we infer $dx = ads$, $dy = bds$ and $dz = cds$; and squaring and adding, we obtain $a^2 + b^2 + c^2 = 1$, a condition which the constants a, b, c must fulfil in order that these equations may be compatible with each other. By division, we find

$$\frac{dy}{dx} = \frac{b}{a},\ \frac{dz}{dx} = \frac{c}{a};\ \text{whence } bx = ay + a',\ cx = az + b';$$

the projections therefore of the line required are straight lines; and consequently the line itself is straight.

To determine its position we must know the five constants a, b, c, a' and b'. If the object be to find the shortest distance between two given fixed points [fig. 62], $A(x_, y_, z_)$, $C(x_{,,} y_{,,} z_{,,})$, it is clear that $\delta x_,$, $\delta x_{,,}$, $\delta y_, \ldots$ are nothing, and that the equation (C) is established of itself. Subjecting our two equations to the condition of being satisfied when $x_,$ $x_{,,}$ $y_, \ldots$, are substituted for x, y, z, we shall obtain four equations, which, with $a^2 + b^2 + c^2 = 1$, will determine our five constants.

Suppose that the 2nd limit be a fixed point C in the plane xy, and the 1st a curve AB, also situated in this plane; the equation $bx = ay + a'$ will then be sufficient.

Let $y_, = fx_,$ be the equation of AB; we deduce $\delta y_, = A\delta x_,$; the equation (C) becomes $L = \frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y$; and since the 2nd limit C is fixed, it will be sufficient if we combine together the equations $\delta y_, = A\delta x_,$, and $dx_, \delta x_, + dy_, \delta y_, = 0$. Eliminating $\delta y_,$, we obtain $dx_, + Ady_, = 0$.

We might also have multiplied the equation of condition $\delta y_, - A\delta x_, = 0$ by the indeterminate λ, and added to $L_,$ which would have given

$$\frac{dx_,}{ds_,}\delta x_, + \frac{dy_,}{ds_,}\delta y_, + \lambda\delta y_, - \lambda A\delta x_, = 0,$$

whence

$$\frac{dx_,}{ds_,} - \lambda A = 0, \quad \frac{dy_,}{ds_,} + \lambda = 0;$$

and eliminating λ, we similarly obtain $dx_, + Ady_, = 0$. But since the point $A(x_, y_,)$ is on our straight line AC, we also have $bdx_, = ady_,$; whence $a = -bA$, and $\frac{dy}{dx} = -\frac{1}{A} = \frac{b}{a}$; which shows that the straight line AC is a normal [N°. 722] to the proposed curve AB. The constant a' is determined from the consideration of the 2nd limit which is fixed and given.

It would be easy to apply the preceding reasoning to the case of three dimensions, and we should arrive at the same consequences; it may therefore be concluded that, generally, the shortest distance AC [fig. 63], between two curves AB, CD, is the straight line AC which is a normal to both of them.

If the line required is to be traced on a curve surface, of which $u = 0$ is the equation, then the equation (B) will no longer resolve itself into three, unless we add to it the term $\lambda\delta u$; when $\delta x, \delta y, \delta z$ may be regarded as independent, and we shall find the relations

$$d\frac{dx}{ds} + \lambda\frac{du}{dx} = 0, \quad d\frac{dy}{ds} + \lambda\frac{du}{dy} = 0, \quad d\frac{dz}{ds} + \lambda\frac{du}{dz} = 0.$$

Eliminating λ, we have the two equations

$$\frac{du}{dz}\, d\left(\frac{dx}{ds}\right) = \left(\frac{du}{dx}\right) d\left(\frac{dz}{ds}\right), \left(\frac{du}{dy}\right) d\left(\frac{dz}{ds}\right) = \left(\frac{du}{dz}\right) d\left(\frac{dy}{ds}\right),$$

which are those of the curve required.

Take for example the least distance $A'C'$ measured along a sphere which has its centre at the origin: then

$$u = x^2 + y^2 + z^2 - r^2 = 0, \frac{du}{dx} = 2x, \frac{du}{dy} = 2y, \frac{du}{dz} = 2z;$$

our equations therefore become, taking ds constant,

$$zd^2x = xd^2z, zd^2y = yd^2z, \text{ whence } yd^2x = xd^2y;$$

and integrating, we have

$$zdx - xdz = ads, zdy - ydz = bds, ydx - xdy = cds.$$

The 1st of these equations being multiplied by $-y$, the 2nd by x, the 3rd by z, and the results added, we find $ay = bx + cz$, the equation of a plane which passes through the origin of the co-ordinates. Thus, the curve required is the great circle $A'C'$ [fig. 63], which passes through the two given points A' and C', or which is a normal to the two curves $A'B$ and $C'D$, which serve as limits, and are given on the spherical surface.

891. When a body moves in a fluid, it experiences a resistance which, supposing all other circumstances to be the same, depends on its form: if this body be one of revolution and move in the direction of its axis, it is proved in Mechanics that the resistance is the least possible, when the equation of the generating curve fulfils the condition

$$\int \frac{ydy^3}{dx^2 + dy^2} = \textit{minimum}, \text{ whence } Z = \frac{yy'^3dx}{1 + y'^2}.$$

Let us determine this generating curve of the *solid of least resistance.* Taking the variation, we find

$$m = 0, n = \frac{-2ydy^3dx}{(dx^2 + dy^2)^2} = \frac{-2yy'^3}{(1 + y'^2)^2}, p = 0, \ldots,$$

$$M = \frac{dy^3}{dx^2 + dy^2} = \frac{y'^3dx}{1 + y'^2}, N = \frac{yy'^2(3 + y'^2)}{(1 + y'^2)^2} \ldots$$

The second of the equations (D) is $M - dN = 0$; and it follows from the above equations, and $M = dN$, that

$$d\left(\frac{y'^3y}{1 + y'^2}\right) = M\frac{dy}{dx} + Ndy' = y'dN + Ndy'.$$

Thus, integrating, we have

$$a + \frac{y'^3 y}{1+y'^2} = Ny' = \frac{yy'^3(3+y'^2)}{(1+y'^2)^2}, \text{ and therefore } a(1+y'^2)^2 = 2yy'^3.$$

Observe that the 1st of the equations (D), or $m - dn = 0$, would have immediately given this same result, viz. $-dn = 0$, $-n = a$; so that these two equations lead to the same end. We now have

$$y = \frac{a(1+y'^2)^2}{2y'^3}, \; x = \int \frac{dy}{y'} = \frac{y}{y'} + \int \frac{y dy'}{y'^2};$$

Substituting for y its value, this integral is easily obtained; it then remains to eliminate y' between these values of x and y, and we shall obtain the equation of the curve required, containing two constants which we shall determine from the conditions given.

892. *What is the curve ABM [fig. 26], in which the area $BODM$, comprised between the arc BM, the radii of curvature BO, MD, at its extremities, and the arc OD of the evolute, is a minimum?* The element of the arc AM is $ds = dx\sqrt{(1+y'^2)}$; the radius of curvature MD is $\frac{(1+y'^2)^{\frac{3}{2}}}{y''}$ [N°. 733, p.]; and the product is the element of the area proposed; so that

$$Z = \frac{(1+y'^2)^2 dx}{y''} = \frac{(dx^2+dy^2)^2}{dx.d^2y}:$$

our object is to find the equation $y = fx$, which shall render $\int Z$ a *minimum*

Taking the variation δZ, and confining ourselves to the 2nd of the equations (D), which is sufficient for our purpose, we find

$$M = 0, \; N - dP = 4a,$$

$$N = \frac{dx^2+dy^2}{dx.d^2y}.4dy = \frac{1+y'^2}{y''}.4y', \; P = -\frac{(1+y'^2)^2}{y''^2.dx}:$$

and, putting $4a + dP$ for N,

$$d\left[\frac{(1+y'^2)^2}{y''}\right] = Ndy' + Pdy''dx = 4ady' + dP.dy' + Pdy''.dx.$$

But, $y''dx = dy'$ changes these two last terms into

$$(y''dP + Pdy'')\,dx = d(Py'').dx = -d\left[\frac{(1+y'^2)^2}{y''}\right];$$

so that, integrating,

$$\frac{(1+y'^2)^2}{2y''} = ay' + b,\ y'' = \frac{(1+y'^2)^2}{2(ay'+b)} = \frac{dy'}{dx},\ dx = \frac{2(ay'+b)\,dy'}{(1+y'^2)^2};$$

and

$$x = c + \frac{by' - a}{1 + y'^2} + b.\ \text{arc}\ (\tan = y'):$$

On the other hand, $y = \int y'dx = y'x - \int xdy'$, or

$$y = y'x - cy' - \int \frac{by' - a}{1 + y'^2}\,dy' - \int bdy'.\ \text{arc}\,(\tan = y\ );$$

this last term is integrable by parts, and we have

$$y = y'x - cy\ - (by' - a)\ \text{arc}\ (\tan = y') + f.$$

Eliminating the tangential arc between these values of x and y,

$$by = a\ (x - c) + \frac{(by' - a)^2}{1 + y'^2} + bf,$$

$$\sqrt{(by - ax + g)} = \frac{(by' - a)\,dx}{ds},\ ds = \frac{bdy - adx}{\sqrt{(by - ax + g)}};$$

and finally [IV, p. 336],

$$s = 2\sqrt{(by - ax + g)} + h.$$

This equation, compared with that of p. 369, shows that the curve required is a cycloid, the four constants in which will have to be determined from an equal number of given conditions.

893. Take, as the 3rd example, the function $Z = \frac{ds}{\sqrt{(z-h)}}$, s being an arc of a curve, or $ds^2 = dx^2 + dy^2 + dz^2$: it is required to render $\int Z$ a *minimum*. This problem is tantamount to finding the curve AC' [fig. 62] in which a falling body ought to descend, that the time of descent from C' to A may be the least possible [See my *Mec.* N°. 192]. Forming the variation δZ, we find

$$\mu = \frac{-ds}{2\sqrt{(z-h)^3}},\ n = \frac{dx}{ds\sqrt{(z-h)}},\ N = \frac{dy}{ds\sqrt{(z-h)}},\ \nu = \frac{dz}{ds\sqrt{(z-h)}};$$

$m = M = p... = 0$: and the equations D become

$$d\left(\frac{dx}{ds\sqrt{(z-h)}}\right) = 0,\ d\left(\frac{dy}{ds\sqrt{(z-h)}}\right) = 0...\ (1).$$

We here omit the 3rd equation, which may be shown to be included in the two others, a condition without which the problem proposed would be absurd. Integrating, and dividing one of the results by the

other, we obtain $dy = adx$; which proves that the projection of the curve on the plane xy is rectilinear, and that, consequently, this curve is situated in a plane perpendicular to xy. Take this plane as that of xz; the 1st then of the equations (1) will be sufficient, and we shall have $kdx = ds\sqrt{(z-h)}$; and since $ds^2 = dx^2 + dz^2$, we find $dx = \dfrac{dz\sqrt{(z-h)}}{\sqrt{(k^2+h-z)}}$. Assuming $z = k^2 + h - u$, we recognize this equation as that of a cycloid [p. 293], in which k^2 is the diameter of the generating circle.

When the limits are two fixed points A and C' [fig. 62], there is no other condition to be fulfilled, except that of making the cycloid pass through these two points, which determines the values of the constants k and h. If the 2nd limit be a fixed point C', and the 1st a curve AB, both of them situated in the vertical plane of xz; we have $\delta x_{\prime\prime} = dz_{\prime\prime} = 0$, and

$$L_{\prime} = \frac{dx_{\prime}}{ds_{\prime}\sqrt{(z_{\prime}-h)}}\delta x_{\prime} + \frac{dz_{\prime}}{ds_{\prime}\sqrt{(z_{\prime}-h)}}\delta z_{\prime}.$$

It will be sufficient if we render $L_{\prime} = 0$, having regard to the 1st limit which is a curve AB of which $x_{\prime} = fz_{\prime}$ is the given equation. We derive from it $\delta x_{\prime} - A\delta z_{\prime} = 0$; multiplying by λ, and adding to $L_{\prime}$, we find the two equations

$$\frac{dx_{\prime}}{ds_{\prime}\sqrt{(z_{\prime}-h)}} + \lambda = 0, \frac{dz_{\prime}}{ds_{\prime}\sqrt{(z_{\prime}-h)}} - A\lambda = 0;$$

and, λ being eliminated between these, we obtain $dz_{\prime} + Adx_{\prime} = 0$. The cycloid therefore must cut the given curve AB at right angles; the constant k will be determined by comparing the equation of the cycloid with the one preceding. We should arrive at the same consequences in the case of three dimensions, so that the curve of quickest descent, from any curve CD [fig. 63] to another AB, is a cycloid $A'C'$ which is a normal to the two latter curves. This would equally hold if the two limits were taken on two curve surfaces, as may easily be proved.

When the curve is to be traced on a surface given by its equation $u = 0$, (B) does not separate itself into three equations, until we have added $\lambda\delta u$, which gives, instead of the equations (1), three others, between which having eliminated λ, we shall have the equations of the curve required. If we had for limits two fixed points, the constants would be determined by the condition that the curve passed through these two points: when the limits are two curves, that of which we are in quest must cut them at right angles as above. Thus, the remainder of the problem is the same in both cases.

894. *What is the curve BM* [fig. 55] *in which, the length s of the arc being given, the area included between this arc, the terminal ordinates, BC, PM, and the axis Ax is the greatest possible?* $\int ydx$ must be a *maximum*, the arc s being constant: and we have therefore to combine the variation of $\int Z = \int ydx$ with that of $\int \surd(dx^2 + dy^2) - \text{const} = 0$, agreeably to what has been seen in N°. 888, so as to divide the equation (B) into two others. We find for the complete variation

$$\int \left(y.\delta dx + dx.\delta y + \frac{\lambda dx.\delta dx + \lambda dy.\delta dy}{ds}\right) = 0;$$

whence $\quad m = 0,\ n = y + \lambda \frac{dx}{ds},\ M = dx,\ N = \lambda \frac{dy}{ds},$

and $\quad y + \lambda \frac{dx}{ds} = c,\ x - \lambda \frac{dy}{ds} = c'.$

These equations are identical, since the integration of each of them leads to the same result; and we must not therefore attempt to eliminate λ between them.

The 1st gives, $\surd(dx^2 + dy^2)$ being put for ds,

$$\frac{dy}{dx} = \sqrt{\frac{[\lambda^2 - (y-c)^2]}{y-c}};\ \text{whence}\ (x - c')^2 + (y - c)^2 = \lambda^2.$$

The curve required is consequently a circle; the area being a *maximum* or a *minimum*, accordingly as this circle turns its concave or its convex side towards the axis of x. The constants c, c' and λ are to be determined from the conditions that the circle passes through the points B, M, and that the arc BM is of the specified length. This is the most simple of the *Isoperimetrical* problems.

895. *What is the curve BM* [fig. 55] *for which, the area BCMP being given, the arc BM is the least possible?* We here have $\int \surd(dx^2 + dy^2) = minimum$, with the condition $\int ydx - \text{constant} = 0$. Reasoning as above, we obtain

$$\frac{dx}{ds} + \lambda y = c,\ \lambda x - \frac{dy}{ds} = c',$$

equations which are obviously similar to those that have just been discussed; and the circle therefore is again the curve required.

896. *Required what must be the nature of the curve MK* [fig. 24], *so that it may be the least possible, the area MAK comprised between the two radii vectores AM, AK being given?*

We must have $s = \int \sqrt{(dx^2 + dy^2)} = minimum$, with the condition [N°. 729] $\int \frac{1}{2}(xdy - ydx) =$ constant: which gives

$$\frac{dx.\delta dx + dy.\delta dy}{ds} + \tfrac{1}{2}\lambda (dy.\delta x + x.\delta dy - dx.\delta y - y.\delta dx) = 0\,;$$

whence

$$\tfrac{1}{2}\lambda dy - d\left(\frac{dx}{ds} - \tfrac{1}{2}\lambda y\right) = 0,\ \tfrac{1}{2}\lambda dx + d\left(\frac{dy}{ds} + \tfrac{1}{2}\lambda x\right) = 0.$$

These equations evidently agree with each other, and the integration of the 1st will be sufficient, λ being taken as an arbitrary constant; the result is

$$\lambda y + c = \frac{dx}{ds}, \text{ or } (\lambda y + c)\, dy = dx \sqrt{[1 - (\lambda y + c)^2]}.$$

Making $\lambda y + c = z$, the integration of this will be easily effected [N°. 769, IV]; and we shall find $(\lambda x + b)^2 + (\lambda y + c)^2 = 1$, or, otherwise, $(x + b')^2 + (y + c')^2 = k^2$. The curve required therefore is a circle, subject to the condition of passing through the points M and K, and forming the area MAK of given size. So that every other curve, which passes through two points M and K of this circumference, and forms the same area, will have the arc intercepted within the angle MAK longer than the circular arc, wherever the points M and K be taken. It will in like manner be seen that the circle corresponds also to the inverse problem: *of all the curves, of equal length between two given points, which is that for which the area MAK is a maximum?*

897. *Among all the plane curves, terminated by two ordinates BC, PM* [fig. 51], *which generate in their revolutions bodies of which the surface is the same, required that which produces the greatest volume.*

We have $\int \pi y^2 dx = maximum$, and $\int 2\pi y \sqrt{(dx^2 + dy^2)} = constant$: whence it is easy to deduce

$$\frac{2\lambda y dx}{ds} + y^2 = c,\ ydx + \lambda ds = d\left(\frac{\lambda y dy}{ds}\right).$$

The equations accord with each other, and the 1st gives

$$dx = \frac{(c - y^2)\, dy}{\sqrt{[4\lambda^2 y^2 - (c - y^2)^2]}} \dots (1).$$

If the constant $c = 0$, we find $dx = \dfrac{-ydy}{\sqrt{(4\lambda^2 - y^2)}}$, whence ... $(x - b)^2 + y^2 = 4\lambda^2$, the equation of a circle which, its centre being in any point of the axis of x, must pass through the two given points. This circle however does not correspond to the problem unless the

surface generated by the revolution of the arc CM be of the extent required: the integral equation contains but two constants, which we shall determine from the condition that the line pass through the points C and M. The general solution of the problem is given by the equation (1).

898. *Of all the plane curves, of equal length between two given points, which is that which, in its revolution, generates a maximum volume or surface?*

In both cases, $\int \surd(dx^2 + dy^2) =$ constant: also, in the one $\int \pi y^2 dx$, and in the other $\int 2\pi y ds$ [N°. 752], must be a *maximum*. To commence with the 1st case, we have $Z = \pi y^2 dx$; and reasoning as above, we find

$$\pi y^2 + \frac{\lambda dx}{ds} = c; \text{ whence } dx = \frac{(c - \pi y^2)\,dy}{\surd\,[\lambda^2 - (c - \pi y^2)^2]}.$$

Thus, the curve now under consideration possesses the property of having its radius of curvature $R = \dfrac{\lambda}{2\pi y}$ [N°. 734, 6°]; for we have

$$y' = \surd\left[\left(\frac{\lambda}{c - \pi y^2}\right)^2 - 1\right],\ y'' = \frac{2\lambda^2 \pi y}{(c - \pi y^2)^3},\ s' = \frac{\lambda}{c - \pi y^2}:$$

and it consequently is the *Elastic curve*, the radius of curvature of which is in the inverse ratio of the ordinate. Besides c and λ, we have a third constant; the conditions that the curve pass through the two given points, and be of the requisite length, serve for determining these three quantities.

In the 2nd case, $Z = \int 2\pi y \surd(dx^2 + dy^2)$, whence

$$\frac{2\pi y dx + \lambda dx}{ds} = c,\ dx = \frac{cdy}{\surd[(2\pi y + \lambda)^2 - c^2]}.$$

The curve required is a *Catenary* [p. 345], the axis of which is horizontal: there is a *maximum* or a *minimum*, accordingly as it presents its concave or its convex side to the axis of x.

899. *Required the curve of given length s, between two fixed points, for which $\int yds$ is a maximum.* We shall easily find

$$(y + \lambda)\frac{dx}{ds} = c, \text{ whence } dx = \frac{cdy}{\surd[(y + \lambda)^2 - c^2]};$$

and we obtain the same curve as in the last instance. Since $\dfrac{\int yds}{s}$

is the vertical ordinate of the centre of gravity of a curvilinear arc of length s [see my *Mec.* N°. 64], it appears that the centre of gravity of any arc of the catenary is lower than that of an arc of any other curve terminated at the same points.

900. Reasoning in the same manner for $\int y^2 dx = minimum$, and $\int y dx =$ constant, we find $y^2 + \lambda y = c$, or rather $y = C$; and we have a straight line parallel to x. Since $\dfrac{\int y^2 dx}{2\int y dx}$ is the vertical ordinate of the centre of gravity of any plane area [see my *Mec.* N°. 68], that of a vertical rectangle, one side of which is horizontal, is the lowest possible. Thus, every body of water, the higher surface of which is horizontal, has its centre of gravity the most deeply situated.

Consult Euler's Work, entitled *Methodus inveniendi lineas curvas maximi minimique proprietate gaudentes.*

BOOK VIII

DIFFERENCES AND SERIES

DIRECT METHOD OF DIFFERENCES. INTERPOLATION.

901. A series $a, b, c, d\ldots$ being given, let each term be subtracted from the one succeeding: $a' = b - a$, $b' = c - b$, $c' = d - c\ldots$ will form the series $a', b', c', d'\ldots$ of the *first differences*.

We in like manner arrive at the series $a'', b'', c'', d''\ldots$ of the *second differences*, $a'' = b' - a'$, $b'' = c' - b'$, $c'' = d' - c'\ldots$; these give the *third differences* $a''' = b'' - a''$, $b''' = c'' - b''\ldots$; and so on. These differences are indicated by Δ, and we give to this characteristic an exponent which marks its order; Δ^n is a term of the series of the nth differences. Each difference also retains its proper sign, which is $-$, when the difference is derived from a decreasing series.

For example, the function $y = x^3 - 9x + 6$, making successively $x = 0, 1, 2, 3, 4\ldots$, gives a series of numbers, of which y is the general term, and whence we deduce the differences, as follows:

for $x =$	0,	1,	2,	3,	4,	5,	6,	7...
series $y =$	6,	$-$ 2,	$-$ 4,	6,	34,	86,	168,	286...
1st diff. $\Delta y =$	$-$8,	$-$ 2,	10,	28,	52,	82,	118,...	
2nd diff. $\Delta^2 y =$	6,	12,	18,	24,	30,	36, ...		
3rd diff. $\Delta^3 y =$	6,	6,	6,	6,	6,...			

902. It appears that in this instance the second differences form an equi-difference, and that the third are constant: constant differences are arrived at in all cases in which y is a rational and integral function of x, as we shall proceed to demonstrate.

In the monomial kx^m make $x=\alpha, \beta, \gamma\ldots \theta, \kappa, \lambda$ (these numbers having h for their constant difference); we have the series $k\alpha^m$, $k\beta^m$, $k\gamma^m\ldots k\kappa^m$, $k\lambda^m$. Since $\kappa = \lambda - h$, developing $k\kappa^m = k(\lambda - h)^m$, and denoting the coefficients of the binomial theorem by m, A', $A''\ldots$ we find

$$k(\lambda^m - \kappa^m) = kmh\lambda^{m-1} - kA'h^2\lambda^{m-2} + kA''h^3\lambda^{m-3}\ldots;$$

and this is the expression for the first difference between any two successive terms of the series $k\alpha^m$, $k\beta^m$, $k\gamma^m\ldots$ The difference between the two preceding terms, or $k(\kappa^m - \theta^m)$ is deduced by changing λ into κ, κ into θ; and since $\kappa = \lambda - h$, we must put $\lambda - h$ for λ in the 2nd side; which thus becomes

$$kmh(\lambda - h)^{m-1} - kA'h^2(\lambda - h)^{m-2}\ldots = kmh\lambda^{m-1} - [A' + m(m-1)]kh^2\lambda^{m-2}\ldots$$

These results being subtracted one from the other, the two first terms disappear, and there ensues, for the 2nd difference of an arbitrary rank,

$$km(m-1)h^2\lambda^{m-2} - kB'h^3\lambda^{m-3} + \ldots.$$

Similarly, changing λ into $\lambda - h$ in this last development, and subtracting, the two 1st terms again disappear, and we have, for the 3rd difference,

$$km(m-1)(m-2)h^3\lambda^{m-3} - kB''h^4\lambda^{m-4}\ldots;$$

and so on. Each of these differences has one term less in its development than the preceding one; the 1st has m terms, the 2nd has $m-1$, the 3rd $m-2,\ldots$ &c.; and from the form of the 1st term, which at length alone remains in the mth difference, we see that this difference is reduced to the constant quantity $1.2.3\ldots mkh^m$.

If, in the functions M and N, when for x we take two numbers, the results are m and n, that which arises for $M + N$ is $m + n$. Let m' and n' be the results similarly given for two other values of x; the 1st difference, arising from $M + N$, is obviously $(m - m') + (n - n')$. And the same may be said of the 3rd, 4th... differences: *the difference of the sum is the sum of the differences.*

Hence, if we make $x = \alpha, \beta, \gamma\ldots$ in $kx^m + px^{m-1} + \ldots$, the $(m-1)$th difference of px^{m-1} being constant, the mth will be nothing; so that, for our polynomial, the mth difference is the same as though we had but its first term kx^m. Consequently, *in a rational and integral function of the degree m, the mth difference is constant, when for x we substitute equi-different numbers.*

903. It appears, therefore, that, if circumstances require us to substitute equi-different numbers, as, for instance, in the solution of a nu-

merical equation [p. 67 & 138], it will be sufficient to investigate the first $(m + 1)$ results, and thence form the 1st, 2nd... differences; the mth will have but one term; and since we know that it is constant and $= 1.2.3 \ldots mkh^m$, we may continue this series as far as we think proper. We shall then, by means of successive additions, extend the series of the $(m - 1)$th differences beyond the two terms already known; and those of the $(m - 2)$th, $(m - 3)$th... differences being similarly continued, we shall at length have the series of the results arising from these substitutions also carried to any extent we think proper, by means of simple additions.

We have an instance of this in the following example: $x^3 - x^2 - 2x + 1$.

$x =$	0.	1.	2.	3.
gives.........	1.	−1.	1.	13.
1st diff.......	−2.	2.	12.	
2nd...........	4.	10.		
3rd...........	6.			

3rd Diff.	6.	6.	6.	6.	6.	6.	6...
2nd......	4.	10.	16.	22.	28.	34.	40...
1st....	−2.	2.	12.	28.	50.	78.	112...
Results	1.	−1.	1.	13.	41.	91.	169...
for $x =$	0.	1.	2.	3.	4.	5.	6...

These series are deduced from that which is throughout 6. 6. 6..., and the initial terms previously found for each: *any term is obtained by adding that on its left, to the one immediately above this last.* The series might also be continued in the contrary direction, for the purpose of obtaining the results corresponding to $x = -1, -2, -3 \ldots$

In this case, *any term is obtained by subtracting the number written above the one in question from that which is on the right of it.*

For the object proposed, that of solving an equation, it is not necessary to carry the series of results farther than the term beyond which we meet only with numbers of the same sign; and this takes place from the time that the terms of any column are all positive on the left side, or of alternate signs in the contrary direction, since the additions or subtractions, by means of which the series are extended, give these same signs to the results continually. In this way, therefore, we obtain limits of the roots, whether positive or negative.

904. For the future, we shall denote by y_x that function of x which is the general term of the series proposed, and generates all the terms by making $x = 0, 1, 2, 3 \ldots$; for example, y_5 will signify that we suppose $x = 5$, or that we have regard to the term which has five before it (the number 91, in the last example). Accordingly,

$$y_1 - y_0 = \Delta y_0, \quad y_2 - y_1 = \Delta y_1, \quad y_3 - y_2 = \Delta y_2 \ldots$$
$$\Delta y_1 - \Delta y_0 = \Delta^2 y_0, \quad \Delta y_2 - \Delta y_1 = \Delta^2 y_1, \quad \Delta y_3 - \Delta y_2 = \Delta^2 y_2 \ldots$$
$$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0, \quad \Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1, \quad \Delta^2 y_3 - \Delta^2 y_2 = \Delta^3 y_2 \ldots$$

And generally,

$$\begin{aligned} y_x - y_{x-1} &= \Delta y_{x-1}, \\ \Delta y_x - \Delta y_{x-1} &= \Delta^2 y_{x-1}, \\ \Delta^2 y_x - \Delta^2 y_{x-1} &= \Delta^3 y_{x-1}, \text{ \&c.} \end{aligned}$$

905. Suppose that we have formed the differences of any series

$$\begin{array}{r|l} a.\ b.\ c.\ d.\ e\ldots & b = a + a',\ c = b + b',\ d = c + c',\ldots, \\ \Delta^1\ldots a'.\ b'.\ c'.\ d'.\ e'\ldots & b' = a' + a'',\ c' = b' + b'',\ d' = c' + c'',\ldots, \\ \Delta^2\ldots a''.\ b''.\ c''.\ d''\ldots\ldots & b'' = a'' + a''',\ c'' = b'' + b''',\ d'' = c'' + c''',\ldots, \\ \Delta^3\ldots a'''.b'''.c'''.d'''\ldots\ldots & b''' = a''' + a^{\text{v}},\ c''' = b''' + b^{\text{v}},\ d''' = c''' + c^{\text{v}},\ldots, \end{array}$$

$b, b', c, c'\ldots$ being eliminated from the equations of the 1st line, they will contain in their 2nd sides only $a, a', a''\ldots$ And we can also obtain values for $a', a'', a'''\ldots$ which do not include any accented letter: we thus find

$$b = a + a',\ c = a + 2a' + a'',\ d = a + 3a' + 3a'' + a''',$$
$$e = a + 4a' + 6a'' + 4a''' + a^{\text{v}},\ f = a + 5a' + 10a'' + \text{\&c.}$$

and

$$a' = b - a,\ a'' = c - 2b + a,\ a''' = d - 3c + 3b - a,\ldots$$

But the initial letters $a', a'', a'''\ldots$ are $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0\ldots$, and $a, b, c, d\ldots$ are $y_0, y_1, y_2, y_3\ldots$: so that

$$\begin{array}{l|l} y_1 = y_0 + \Delta y_0, & \Delta y_0 = y_1 - y_0, \\ y_2 = y_0 + 2\Delta y_0 + \Delta^2 y_0 & \Delta^2 y_0 = y_2 - 2y_1 + y_0 \\ y_3 = y_0 + 3\Delta y_0 + 3\Delta^2 y_0 + \Delta^3 y_0 & \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0, \\ y_4 = y_0 + 4\Delta y_0 + 6\Delta^2 y_0 \text{ \&c.} & \Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - \text{\&c.} \end{array}$$

And, generally,

$$y_x = y_0 + x\Delta y_0 + x\,.\,\frac{x-1}{2}\,\Delta^2 y_0 + x\,.\,\frac{x-1}{2}\,.\,\frac{x-2}{3}\,\Delta^3 y_0\ldots\ (A),$$

$$\Delta^n y_0 = y_n - ny_{n-1} + n\,.\,\frac{n-1}{2}\,y_{n-2} - n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\,y_{n-3}\ldots\ (B).$$

906. These equations give, the one any term of rank x (the general term of the series), when we know the 1st term of all the orders of differences, the other the initial term of the series of the nth differences, knowing all the terms of the series, $y_0, y_1, y_2\ldots$ To apply the 1st series to the example of Nº. 903, we shall make

$$y_0 = 1,\ \Delta y_0 = -2,\ \Delta^2 y_0 = 4,\ \Delta^3 = 6,\ \Delta^4 = 0\ldots$$

whence

$$y_x = 1 - 2x + 2x(x-1) + x(x-1)(x-2) = x^3 - x^2 - 2x + 1.$$

The equations (A) and (B) will be the more strongly imprinted on the memory, by observing that

$$y_x = (1 + \Delta y_0)^x, \ \Delta^n y_0 = (y-1)^n,$$

with the proviso that, in the developments of these powers, we transform the powers of Δy_0 into exponents of Δ, so as to mark the order of the differences, and the powers of y into indices, putting y_0 in place of the 1st term 1.

907. Whatever be the proposed series $a, b, c, d\ldots$, we may always conceive it to have been derived from some other in which certain terms have been periodically omitted; in which, for instance, every 2nd, 3rd, 4th... term has been taken. The series $a, b, c\ldots$ being given, or rather its general term y_x (A), let it be proposed to regain this primitive series, which we shall denote by

$$a, a', a''\ldots a^{h-1}, \ b, b', b''\ldots b^{h-1}, \ c, c', c''\ldots c^{h-1}\ldots \ (C).$$

It is obvious that we here suppose $h-1$ terms to have been suppressed between a and b, between b and $c,\ldots$, which terms were subject to the same law of generation as the series $a, b, c\ldots$ which constitutes a part of the one preceding. The method of *interpolation* consists in inserting, between the terms of a proposed series, a specified number of terms subject to the same law: and in order, therefore, to *interpolate*, we must find these intermediate terms, or rather the general term of the equation (C). It will evidently be sufficient, for this purpose, to assume in (A), the general term of $a, b, c\ldots$, the condition $x = \frac{z}{h}$, z marking the rank of a term of the new series (C); for, making

$$z = 0, 1, 2, 3\ldots \ h, h+1, h+2,\ldots \ 2h\ldots \ \&c.$$

we have

$$x = 0,\ldots (h-1) \text{ terms}\ldots 1,\ldots (h-1) \text{ terms}\ldots 2\ldots \&c.;$$

and we thus obtain the same numbers $a, b, c\ldots$, as though we had made $x = 0, 1, 2\ldots$ in (A), and, likewise, $h-1$ intermediate terms. This substitution gives

$$y_z = y_0 + \frac{z\Delta'}{h} + \frac{z(z-h)\Delta^2}{2h^2} + \frac{z(z-h)(z-2h)\Delta^3}{2.3.h^3} \&c.\ldots (D);$$

an equation which will give y_x when $x = z$, z being integral or fractional. We deduce from the series $a, b, c\ldots$ proposed the differences of all the orders, and the initial terms of these series represent $\Delta^1, \Delta^2\ldots$ That this formula, however, may be applied practically, we must, in the course of the last mentioned operation, either arrive at differences actually constant, so that the series (D) may terminate, or we must at least have for $\Delta^1, \Delta^2\ldots$ decreasing values which shall render the series convergent: the development then gives the approximate value of a term corresponding to $x = z$; it being at the same time understood that the factors of Δ must not increase so far as to destroy this convergence, by which we are restricted from carrying z beyond a certain limit.

For example, we find [N°. 364, X] that

the arc of 60° has for its chord	1000·0		
65	1074·6	$\Delta^1 = 74{\cdot}6$	
70	1147·2	72·6	$\Delta^2 = -2{\cdot}0$
75	1217·5	70·3	$-2{\cdot}3$

and since the difference is very nearly constant, at least from 60° to 75°, we may, within this extent, employ the equation (D): making $h = 5$, there results, for the quantity to be added to $y_0 = 1000$,

$$\tfrac{1}{5} \times 74{\cdot}6 \times z - \tfrac{2}{50}\, z(z-5) = 15{\cdot}12 \times z - 0{\cdot}04 \times z^2.$$

Thus, taking $z = 1, 2, 3\ldots$, then adding 1000, we obtain the chords of 61°, 62°, 63°..., and if fractional values be assumed for z, we have the chord of any arc whatever between 60° and 75°. Differences, however, that have been thus obtained ought scarcely to be made use of, beyond the limits from which they have been derived. We give another example:

We have $\log 3100 = y_0$	$= 4913617$		
$\log 3110$	$= 4927604$	$\Delta^1 = 13987$	
$\log 3120$	$= 4941546$	13942	$\Delta^2 = -45$
$\log 3130$	$= 4955443$	13897	-45

the decimal part of the logarithm being here considered as an integer. Making $h = 10$, there results, for the part to be added to log 3100,

$$1400{\cdot}95 \times z - 0{\cdot}225 \times z^2.$$

To obtain the logarithms of 3101, 3102, 3103..., we make $z = 1, 2, 3\ldots$; and, if log 3107.58 be required, we shall take $z = 7{\cdot}58$;

whence there will result 10606 for the quantity to be added to the logarithm of 3100, viz. log 310758 = 5·4924223.

908. These methods are very useful for shortening the calculations of tables of logarithms, sines, chords, &c. Results are investigated by the direct processes from one point to another, and the interval is then filled up by *Interpolation.*

Most frequently, the proposed series $a, b, c \ldots$, or the table of numbers between which we wish to interpolate, corresponds to the order 1, 2, 3... ; in which case $h = 1$, and we have to investigate some term intermediate to y_0 and y_1, answering to the rank z: the equation (D) then becomes

$$y_z = y_0 + z\Delta^1 + z.\frac{z-1}{2}\Delta^2 + z.\frac{z-1}{2}.\frac{z-2}{3}\Delta^3 + \&c.\ldots (E).$$

1°. When Δ^2 is nothing, or very small, the series is reduced to $y_0 + z\Delta^1$; whence we infer that the difference $z\Delta^1$ increases proportionally to z, *i. e. there must be added to y_0 a part of Δ^1 proportional to z.* Frequent use has previously been made of this remark [N^{os}. 91, 111, and 586].

2°. When Δ^2 is constant, or Δ^3 very small, which is most frequently the case,

$$y_z = y_0 + z[\Delta^1 + \tfrac{1}{2}(z-1)\Delta^2].$$

Thus, *form $\frac{1}{2}(z-1)\Delta^2$, and add this quantity as a correction to Δ^1; then consider the series as having this result for the 1st difference, and that the 2nd difference is nothing:* the investigation will thus be reduced to that of the case preceding.

For example,

log 310 = 2·4913617 = y_0			
log 311 = 27604	$\Delta^1 = 13987$		
log 312 = 41546	13942	$\Delta^2 = -45$	
log 313 = 55443	13897	-45	

Since Δ^2 is constant, to obtain log 310·758, we make $z = 0·758$; whence $\frac{1}{2}z(z-1)\Delta^2 = 0·121 \times 45 = 5·445$; this being added to Δ^1, we have 13992·445 which is to be multiplied by z; the product is 10606·27; and consequently log 310·758 = 2·4924223.

3°. When Δ^3 is constant, or Δ^4 so small that it may be neglected, the series (E) has only four terms; Δ^2 must now be corrected by the addi-

tion of $\frac{1}{3}(z-2)\,\Delta^3$, and the result considered as a constant 2nd difference; and so on.

Applications of this theory may be seen in p. 101 of the logarithmic tables of Callet, where these logarithms are calculated to 20 places of decimals.

4°. Conversely, if the terms y_z and y_0 be given, and the rank z of the 1st be required, the 2nd difference being constant, we have

$$z = \frac{y_z - y_0}{\Delta^1 + \frac{1}{2}(z-1)\Delta^2} \dots (F).$$

We first go through the calculation, neglecting the 2nd term of the denominator, which gives an approximate value of z; and this is then substituted for z in the formula (F), no part being neglected.

From the result of the calculation, in the preceding example, we find the numerator $y_z - y_0 = 10606$; which, divided by $\Delta^1 = 13987$, gives a first approximation, $z = 0{\cdot}758$: this value, substituted for z in F, gives $z = \frac{10606}{13992} = 0{\cdot}758$. This inverse problem may be solved in a similar manner, when Δ^3 is constant, &c. [See *Conn. des Temps* 1819, p. 303].

909. The following is a convenient mode of conducting the calculation when Δ^2 is constant, and we wish to find n successive numbers between y_0 and y_1. Changing z into $z+1$ in D and subtracting, we have the general value of the 1st difference of the new series interpolated; and by repeating the operation on this value, we arrive at the 2nd difference: viz.

$$\text{1st Diff. } \delta^1 = \frac{\Delta^1}{h} + \frac{2z-h+1}{2h^2}\Delta^2, \text{ 2nd Diff. } \delta^2 = \frac{\Delta^2}{h^2}.$$

But n terms are to be inserted between y_0 and y_1; so that we must take $h = n+1$; and then, making $z=0$, we have the initial terms of the differences

$$\delta^2 = \frac{\Delta^2}{(n+1)^2}, \quad \delta^1 = \frac{\Delta^1}{n+1} - \frac{1}{2}n\delta^2:$$

we hence calculate δ^2, then δ^1; this initial term δ^1 will serve for composing the series of the 1st differences of the series interpolated (δ^2 is its constant 2nd difference); and, finally, we have the series itself by means of simple additions.

Suppose that, in the example of p. 466, we wish to calculate the

logarithms of 3101, 3102, 3103...; 9 numbers are to be interpolated between those which are given; whence $n = 9$, $\delta^2 = -0{\cdot}45$, $\delta^1 = 1400{\cdot}725$. We first form the equidifference which has δ^1 for its initial term and $-0{\cdot}45$ for the constant difference; and the first differences are

$$1400{\cdot}725,\ 1400{\cdot}275,\ 1399{\cdot}825,\ 1399{\cdot}375,\ 1398{\cdot}925,\ldots$$

Successive additions, commencing from log 3100, will give the consecutive logarithms required.

Suppose that a physical phenomenon has been observed every 12 hours, and that the measured results have given

at	$0^h\ldots$	$y_0 = 78$		
	$12\ldots$	300	$\Delta' = 222$	
	$24\ldots$	666	366	$\Delta^2 = 144$
	$36\ldots$	1176	510	$144.$

To find the circumstances of the phenomenon corresponding to 4^h, 8^h, $12^h\ldots$, two terms must be interpolated, whence $n = 2$, $\delta^2 = 16$, $\delta' = 58$; composing the equi-difference which commences from 58, and the ratio of which is 16, we shall have the 1st differences of the new series, and thence the series itself:

1st diff. $\delta'\ldots$ 58, 74, 90, 106, 122, 138...
Series ... 78, 136, 210, 300, 406, 528, 666...
0^h, 4^h, 8^h, 12^h, 16^h, 20^h, $24^h\ldots$

The supposition of the 2nd differences being constant holds in almost every case, when suitable intervals can be taken. This method is repeatedly made use of in Astronomy; and even when the observation, or the calculation, gives results, the 2nd differences of which are by no means regular, this irregularity is imputed to errors, which we correct by establishing a uniform course.

910. The astronomical, geodesical, and other such tables are formed on these principles. We first calculate by the direct processes a number of different terms, taken sufficiently near to each other that the 1st or 2nd differences shall be constant; and the intermediate quantities are then obtained by interpolation.

Thus, when a convergent series gives the value of y, by means of that of a variable x; if the formula be one of frequent use, instead of calculating y each time that x is known, we determine the results y for certain gradually increasing values of x, so that the values of y shall be but slightly different from each other; and we then insert, in the form of a table, each value of y near to the corresponding one of x, which is

called the *Argument* of this table. For intermediate numbers x, simple proportions will give y, as we have seen in respect to logarithms [1st Vol. p. 100], and the results required are obtained simply by inspection.

When the series has two variables, or arguments, x and z, the values of y are disposed in a table of *double entry*, as that of Pythagoras [N°. 14]; x and z are taken as co-ordinates, and the result is contained in the square thus determined. For example, having taken $z = 1$, we must arrange in the 1st line the several values of y corresponding to $x = 1, 2, 3 \ldots$; in the 2nd line must be placed those given by $z = 2$; in the 3rd, those for $z = 3 \ldots$ To obtain the result corresponding to $x = 3$ and $z = 5$, we shall stop at the square which, in the 3rd column, occupies the 5th rank. The intermediate values are obtained in a manner analogous to the one just detailed.

911. We have hitherto supposed x to increase in the manner of an equi-difference. If this be not the case, and we are acquainted with the results $y = a, b, c, d \ldots$ arising from certain suppositions $x = \alpha, \beta, \gamma, \delta \ldots$, recourse may be had to the theory explained in N°. 465, when it was required to make a parabolic curve pass through a series of given points; this problem is in fact no other than an interpolation. We may also proceed in the manner following:

By means of the known corresponding values, $a, \alpha, b, \beta \ldots$, form the consecutive fractions:

$$A = \frac{b-a}{\beta-\alpha},\ A_{\prime} = \frac{c-b}{\gamma-\beta},\ A_{2} = \frac{d-c}{\delta-\gamma},\ A_{3} = \frac{e-d}{\varepsilon-\delta} \ldots,$$

$$B = \frac{A_{\prime}-A}{\gamma-\alpha},\ B_{\prime} = \frac{A_{2}-A_{\prime}}{\delta-\beta},\ B_{2} = \frac{A_{3}-A_{2}}{\varepsilon-\gamma} \ldots,$$

$$C = \frac{B_{\prime}-B}{\delta-\alpha},\ C_{\prime} = \frac{B_{2}-B_{\prime}}{\varepsilon-\beta} \ldots,\ D = \frac{C_{\prime}-C}{\varepsilon-\alpha} \ldots$$

Eliminating between these equations, we find successively

$$b = a + A(\beta-\alpha),$$
$$c = a + A(\gamma-\alpha) + B(\gamma-\alpha)(\gamma-\beta),$$
$$d = a + A(\delta-\alpha) + B(\delta-\alpha)(\delta-\beta) + C(\delta-\alpha)(\delta-\beta)(\delta-\gamma),$$

and, generally,

$$y_x = a + A(x-\alpha) + B(x-\alpha)(x-\beta) + C(x-\alpha)(x-\beta)(x-\gamma) \ldots$$

We must therefore investigate the 1st differences between the results $a, b, c \ldots$, and divide by the differences between the suppositions $\alpha, \beta, \gamma \ldots$ which will give $A, A_{\prime}, A_{2} \ldots$; these numbers being treated in a

similar manner; we shall from them deduce B, $B_{,}$, B_{2}...; which will give C, $C_{,}$...; and, lastly, substituting, we shall have the general term required.

The multiplications being carried into effect, the expression takes the form $a + a'x + a''x^2$..., that of any rational and integral polynomial; this arises from our having neglected the higher differences [N°. 902].

Chord of 60° $= 1000$		$A = 15$		$B = -0.035$
	35		-0.18	$B_{,} = -0.031$
$62{\cdot}20' = 1035$		$A_{,} = 14{\cdot}82$		
	42		-0.21	
$65{\cdot}10 = 1077$		$A_{2} = 14{\cdot}61$		
	56			
$69.0 = 1133$				

$$\alpha = 0,\ \beta = 2\tfrac{1}{3},\ \gamma = 5\tfrac{1}{6},\ \delta = 9:$$

we may neglect the 3rd differences, and assume

$$y_x = 1000 + 15{\cdot}082 \times x - 0{\cdot}035x^2.$$

912. Considering every function y_x of x as being the general term of the series given by $x = m,\ m + h,\ m + 2h$..., if we take the differences between these results, in order to obtain a new series, the general term will be what is called the *first Difference* of the proposed function y_x, and which is represented by Δy_x. Thus, this difference is obtained by changing x into $x + h$ in y_x, and subtracting y_x from the result; the remainder will generate the series of 1st differences, by making $x = m,\ m + h,\ m + 2h$, &c.

Thus

$$y_x = \frac{x^2}{a + x} \text{ gives } \Delta y_x = \frac{(x + h)^2}{a + x + h} - \frac{x^2}{a + x}.$$

It will remain to reduce this expression, or to develop it according to the powers of h...

Generally, it follows from Taylor's theorem, that

$$\Delta y_x = y'h + \tfrac{1}{2}y''h^2 + \tfrac{1}{6}y'''h^3 + \ldots$$

To obtain the 2nd difference, we must repeat, on Δy_x, the same operation that has been performed on the function proposed; and so on for the 3rd, 4th... differences.

INTEGRATION OF DIFFERENCES. SUMMATION OF SERIES.

913. The object of the integration here is to trace back a given difference in x to the function whence it has sprung; *i. e.* to regain the general term y_x of a series y_m, y_{m+h}, y_{m+2h}..., knowing that of the

series of a difference of some given order. This operation is indicated by the sign Σ.

For example, supposing that $h = 1$, $\Sigma\,(3x^2 + x - 2)$ must lead to this idea: a function y_x generates a certain series, by assuming in it $x = 0, 1, 2, 3\ldots$; and the 1st differences which ensue from it form another series, of which $3x^2 + x - 2$ is the general term (this series is $-2, 2, 12, 28\ldots$). The object, therefore, which we propose to ourselves in integrating, is to find the function y_x which, if we put $x + 1$ for x and subtract y_x from the result, will give the remainder $3x^2 + x - 2$.

It is easily seen that 1° the signs Σ and Δ destroy each other (in the same manner as $\int$ and d): thus, $\Sigma\Delta fx = fx$.

2°. $\Delta\,(ay) = a\,\Delta y$; and therefore $\Sigma ay = a\Sigma y$.

3°. As $\Delta\,(At - Bu) = A\Delta t - B\Delta u$, we similarly have

$$\Sigma\,(At - Bu) = A\Sigma t - B\Sigma u,$$

t and u being functions of x.

914. The problem of determining y_x from its 1st difference does not include the given quantities necessary for its complete solution; thus, to recompose the series corresponding to y_x, from the one $-2, 2, 12, 28$, if we make the 1st term $y_0 = a$, we find, by successive additions, $a, a - 2, a, a + 12\ldots$, and a remains arbitrary.

We may consider every integral to be included in the equation (A) p. 464; for taking $x = 0, 1, 2, 3\ldots$ in the 1st difference, which is given in x, we shall have the series of the 1st differences; and subtracting these consecutively, we shall thus form the 2nd differences, the 3rd, 4th.... The initial terms of these series will be $\Delta^1 y_0$, $\Delta^2 y_0\ldots$, and these values substituted in (A) give y_x. Thus, in the example above (which is the same with that of N°. 903, when $a = 1$), we have [N°. 906.]

$$\Delta' y_0 = -2,\ \Delta^2 y_0 = 4,\ \Delta^3 y_0 = 6,\ \Delta^4 y_0 = 0\ldots;$$

whence $$y_x = y_0 - 2x - x^2 + x^3.$$

In general, the 1st term y_0 of the equation (A) is the arbitrary constant, which is to be added to the integral. If the given function be a 2nd difference, we must, by a first integration, ascend to the 1st difference, and from this to y_x; thus, we shall have two arbitrary constants; and, in fact, the equation (A) still gives us y_x, Δ^2, $\Delta^3\ldots$ being found, only y_0 and $\Delta' y_0$ remain indeterminate. And so on for the higher orders.

915. Let it be proposed to find Σx^m, the exponent m being integral and positive. We shall represent this development by

$$\Sigma x^m = px^a + qx^b + rx^c \ldots,$$

$a, b, c\ldots$ being decreasing exponents now to be determined, as also the coefficients $p, q\ldots$. Let the 1st difference be taken, by suppressing the symbol Σ on the 1st side, then changing x into $x + h$ on the second, and subtracting. Confining ourselves to the two 1st terms, we have from this

$$x^m = pahx^{a-1} + \tfrac{1}{2} pa(a-1)h^2x^{a-2}\ldots + qbhx^{b-1}\ldots$$

That the identity therefore may be established, the exponents must give the equations $a - 1 = m$, $a - 2 = b - 1$; whence $a = m + 1$, $b = m$; also, the coefficients give

$$1 = pah, \ -\tfrac{1}{2} pa(a-1)h = qb; \text{ whence } p = \frac{1}{h(m+1)}, \ q = -\tfrac{1}{2}.$$

As to the other terms, it is evident that the exponents are all integral and positive; and, as would be found by continuing the process, they decrease successively by 2.

Assuming therefore that

$$\Sigma x^m = px^{m+1} - \tfrac{1}{2}x^m + \alpha x^{m-1} + \beta x^{m-3} + \gamma x^{m-5}\ldots,$$

let us now determine $\alpha, \beta, \gamma\ldots$ Take, as above, the 1st difference, by putting $x + h$ for x, and subtracting: having first transposed $px^{m+1} - \frac{1}{2}x^m$, we find that the 1st side, considering that $ph(m+1) = 1$, reduces itself to

$$A'.\frac{h^2}{2.3}x^{m-2} + A''.\frac{m-3}{4}.\frac{3h^4}{2.5}x^{m-4} + A^{\text{IV}}.\frac{m-5}{6}.\frac{5h^6}{2.7}x^{m-6}\ldots$$

For conciseness, we here omit the alternate terms of the development, which would be proved by the calculation to destroy each other; and we also denote by $1, m, A', A''\ldots$ the coefficients of the binomial. Proceeding now to the 2nd side, and performing the same operation on $\alpha x^{m-1} + \beta x^{m-3}\ldots$, we shall have, with the same respective powers of x and h,

$$(m-1)\alpha + (m-1)\frac{m-2}{2}.\frac{m-3}{3}\alpha + (m-1)\frac{m-2}{2}\ldots\frac{m-4}{5}\alpha + \ldots$$
$$+ (m-3)\beta + (m-3)\frac{m-4}{2}.\frac{m-5}{3}\beta + \ldots$$
$$+ (m-5)\gamma + \ldots$$

Comparing the homologous terms, we easily deduce

$$\alpha = \frac{mh}{3.4},\ \beta = \frac{-A''h^3}{2.3.4.5},\ \gamma = \frac{A^{IV}h^5}{6.6.7}\ldots;$$

and hence we finally obtain

$$\Sigma x^m = \frac{x^{m+1}}{(m+1)h} - \frac{x^m}{2} + mahx^{m-1} + A''bh^3x^{m-3}$$
$$+ A^{IV}ch^5x^{m-5} + A^{VI}dh^7x^{m-7} + \ldots \ (D).$$

The development has for coefficients those of the alternate terms of the binomial, multiplied by certain numerical factors $a, b, c\ldots$, which are called *Bernouillian* numbers, from Jacques Bernouilli having been the first to determine them. These factors are of frequent use in the theory of series; we shall give a more ready mode of arriving at their values in N°. 917: in the meantime these values are

$$a = \frac{1}{12},\ b = -\frac{1}{120},\ c = \frac{1}{252},\ d = -\frac{1}{240},\ e = \frac{1}{132}.$$
$$f = -\frac{691}{32760},\ g = \frac{1}{12},\ h = -\frac{3617}{8160},\ i = \frac{43867}{14364}\ldots$$

916. Hence we conclude that, to obtain Σx^m, m being a given positive integer, we must take, besides the two first terms $\frac{x^{m+1}}{(m+1)h} - \frac{x^m}{2}$, the development of $(x+h)^m$, rejecting the terms of the odd ranks, the 1st, 3rd, 5th..., and multiply the terms retained respectively by $a, b, c\ldots$ *When m is odd, x and h have no other than even exponents, and the converse;* so that the last term h^m must also be rejected, when it occurs in one of the ranks neglected: the number of the terms is $\frac{1}{2}m + 2$ when m is even; and $\frac{1}{2}(m+3)$ when m is odd, *i. e.* the same for an even number and the succeeding odd one.

Is Σx^{10} required? besides $\frac{x^{11}}{11h} - \frac{1}{2}x^{10}$, we must develop $(x+h)^{10}$, and retain the 2nd, 4th, 6th... terms; we shall have

$$10x^9ah + 120x^7bh^3 + 252\ldots;$$

and therefore

$$\Sigma x^{10} = \frac{x^{11}}{11h} - \frac{1}{2}x^{10} + \frac{5}{6}x^9h - x^7h^3 + x^5h^5 - \frac{1}{2}x^3h^7 + \frac{5}{66}xh^9.$$

In this manner we obtain

$$\Sigma x^0 = \frac{x}{h},\ \Sigma x^1 = \frac{x^2}{2h} - \frac{x}{2},$$

$$\Sigma x^2 = \frac{x^3}{3h} - \frac{x^2}{2} + \frac{hx}{6},$$

$$\Sigma x^3 = \frac{x^4}{4h} - \frac{x^3}{2} + \frac{hx^2}{4},$$

$$\Sigma x^4 = \frac{x^5}{5h} - \frac{x^4}{2} + \frac{hx^3}{3} - \frac{h^3x}{30},$$

$$\Sigma x^5 = \frac{x^6}{6h} - \frac{x^5}{2} + \frac{5hx^4}{12} - \frac{h^3x^2}{12},$$

$$\Sigma x^6 = \frac{x^7}{7h} - \frac{x^6}{2} + \frac{hx^5}{2} - \frac{h^3x^3}{6} + \frac{h^5x}{42},$$

$$\Sigma x^7 = \frac{x^8}{8h} - \frac{x^7}{2} + \frac{7hx^6}{12} - \frac{7h^3x^4}{24} + \frac{h^5x^2}{12},$$

$$\Sigma x^8 = \frac{x^9}{9h} - \frac{x^8}{2} + \frac{2hx}{3} - \frac{7h^3x^5}{15} + \frac{2h^5x^3}{9} - \frac{h^7x}{30},$$

$$\Sigma x^9 = \frac{x^{10}}{10h} - \frac{x^9}{2} + \frac{3hx^8}{4} - \frac{7h^3x^6}{10} + \frac{h^5x^4}{2} - \frac{3h^7x^2}{20},$$

$\Sigma x^{10} =$ &c. (See above).

917. We shall now give an easy mode of continuing the Bernouillian numbers $a, b, c\ldots$ to any extent whatever. Let $x = h = 1$ in the equation (D): Σx^m is the general term of the series which has x^m for its 1st difference; we are here considering $\Sigma 1$, and the series is that of the natural numbers 0, 1, 2, 3... Take zero therefore for the 1st side, and transpose $\frac{1}{m+1} - \frac{1}{2} = \frac{1-m}{2(1+m)}$; then

$$\frac{m-1}{2(m+1)} = am + bA'' + cA^{\text{IV}} + dA^{\text{VI}} \ldots + km.$$

If now we make $m = 2$, the 2nd side of this is reduced to am, whence we deduce $a = \frac{1}{12}$; $m = 4$ gives $am + bA''$, or $4a + 4b$, for the 2nd side; we similarly find $am + bA'' + 6c$, for $m = 6, \ldots$; and following this up for all the even numbers $m = 2, 4, 6, 8\ldots$, we obtain each time an equation which has one term more than the preceding one; and these equations serve to determine their successive last terms $2a, 4b, 6c \ldots mk$.

918. Form the difference of the product

$$y_x = (x-h)\,x\,(x+h)(x+2h)\ldots(x+ih),$$

by putting $x+h$ for x, and subtracting: there results

$$\Delta y_x = x(x+h((x+2h)\ldots(x+ih) \times (i+2)h;$$

dividing by this last constant factor, integrating, and replacing for y_x its value, we find

$$\Sigma x(x+h)(x+2h)\ldots(x+ih)$$

$$= \frac{x-h}{(i+2)h} \times x(x+h)(x+2h)\ldots(x+ih).$$

This equation gives the integral of the product of factors which form an equi-difference.

919. Take the difference of the 2nd side, and we shall see verified the equation

$$\Sigma \frac{1}{x(x+h)(x+2h)\ldots(x+ih)} = \frac{-1}{ihx(x+h)\ldots[x+(i-1)h]}.$$

920. Let $y_x = a^x$; the difference is

$$\Delta y_x = a^x(a^h - 1); \text{ whence } y_x = \Sigma a^x(a^h-1) = a^x;$$

and, therefore,

$$\Sigma a^x = \frac{a^x}{a^h-1} + \text{const.}$$

921. The note, p. 309, 1st vol. contains the following equation

$$\cos B - \cos A = 2\sin\tfrac{1}{2}(A+B).\sin\tfrac{1}{2}(A-B);$$

but

$$\Delta\cos x = \cos(x+h) - \cos x = -2\sin(x+\tfrac{1}{2}h)\sin\tfrac{1}{2}h;$$

integrating and changing $x + \frac{1}{2}h$ into z, we have

$$\Sigma\sin z = -\frac{\cos(z-\frac{1}{2}h)}{2\sin\frac{1}{2}h} + \text{const.};$$

and we should in like manner find

$$\Sigma\cos z = \frac{\sin(z-\frac{1}{2}h)}{2\sin\frac{1}{2}h} + \text{const.}$$

When powers of sines and cosines are proposed for integration, we transform them into sines and cosines of multiple arcs [p. 171], and we

have terms of the form $A \sin qx$, $A \cos qx$; then making $qx = z$, the integration is given by the preceding equations.

922. Let the integral of a product uz be represented by

$$\Sigma uz = u\Sigma z + t,$$

u, z and t denoting functions of x, the last unknown, u and z given. Changing x into $x + h$ in $u\Sigma z + t$, u becomes $u + \Delta u$, z becomes $z + \Delta z$, and we have

$$u\Sigma z + uz + \Delta u. \Sigma(z + \Delta z) + t + \Delta t:$$

our 2nd side $u\Sigma z + t$ being subtracted, we obtain its difference, or that of uz; and hence results the equation

$$0 = \Delta u. \Sigma(z + \Delta z) + \Delta t; \text{ whence } t = - \Sigma[\Delta u. \Sigma(z + \Delta z)]$$

Consequently, $\quad \Sigma(uz) = u\Sigma z - \Sigma[\Delta u. \Sigma(z + \Delta z)]$,

a formula which corresponds to that of integration by parts in the case of differential functions [p. 336].

923. There is but a small number of functions of which the finite integral can be found; when the integration cannot be accomplished in an exact form, recourse is had to series.

That of Taylor, $\Delta y_x = y'h + \ldots$ [N°. 912] gives

$$y_x = h\Sigma y' + \tfrac{1}{2}h^2 \Sigma y'' + \ldots,$$

where $y', y'' \ldots$ are the successive derivatives of y_x.

Let y' be considered as a given function z of x; then we have $y' = z$ $y'' = z', y''' = z'' \ldots$, and $y_x = \int y'dx = \int zdx$; whence

$$\int zdx = h\Sigma z + \tfrac{1}{2}h^2\Sigma z' + \ldots;$$

and consequently

$$\Sigma z = h^{-1} \int zdx - \tfrac{1}{2}h\Sigma z' - \tfrac{1}{6}h^2\Sigma z'' \ldots,$$

an equation which gives Σz, when $\Sigma z'$, $\Sigma z'' \ldots$ can be found. Take the derivatives of its two sides; that of the 1st will be $\Sigma z'$, as we may easily assure ourselves. Hence we shall derive $\Sigma z''$, then $\Sigma z''' \ldots$; and, even without going through these calculations, it is easy to see that the result from the substitution of these values will be of the form

$$\Sigma z = h^{-1} \int zdx + Az + Bhz' + Ch^2z'' \ldots;$$

where it remains to determine the factors $A, B, C \ldots$

If, now, $z = x^m$, we thence deduce $\int zdx$, z', z''...; and, substituting, there results a series which must be identical with (D); and which, consequently, must be devoid of the powers $m-2$, $m-4$,...; so that we shall assume

$$\Sigma z = \frac{\int zdx}{h} - \frac{z}{2} + \frac{ahz'}{1} + \frac{bh^3z'''}{1.2} + \frac{ch^5z^{v}}{2.3.4} + \frac{dh^7z^{vii}}{2...6} \&c.$$

a, b, c... being the Bernouillian numbers.

For example, if $z = lx$, $h = 1$, $\int lxdx = xlx - x$, $z' = x^{-1}$, $z'' = \&c.$; and consequently

$$\Sigma lx = C + xlx - x - \tfrac{1}{2}lx + ax^{-1} + bx^{-3} + cx^{-5} + \&c.$$

924. The series $a, b, c, d...k, l$, having for its first differences a', b', c'..., it has been seen [N°. 905] that

$$b = a + a',\ c = b + b',\ d = c + c' \ldots l = k + k';$$

equations the sum of which gives

$$l = a + a' + b' + c' \ldots + k'.$$

If the numbers a', b', c'... be known, we may look upon them as being the 1st differences of some other series a, b, c..., since it is easy to compose the latter of these series from the former one and the initial term a. But, from the definition of N°. 913, it appears that any term l', taken in the given series a', b', c'... is no other than Δl, since $l' = m - l$; and integrating $l' = \Delta l$, we have $\Sigma l' = l$, or

$$\Sigma l' = a' + b' + c' + \ldots + k',$$

supposing the initial a to be included in the constant of the sign Σ. Hence, *by taking the integral of any term of a series, we obtain the sum of all the terms which precede it*:

$$\Sigma y_x = y_0 + y_1 + y_2 \ldots + y_{x-1}.$$

It follows, therefore, that to obtain the sum of the series, the general term y_x included, y_x must be added to the integral, or x changed in it into $x+1$; or, lastly, x be changed into $x+1$ in y_x before the integration is carried into effect. To conclude, the constant is determined by making the sum $= y_0$, when $x = 1$.

925. *We can therefore find the term of summation of every series of which we know the general term, in a rational and integral function of* x. Let $y_x = Ax^m - Bx^n + C$, m and n being integral and positive; we have $A\Sigma x^m - B\Sigma x^n + C\Sigma x^0$ for the sum of the terms as far as y_x;

this term itself excluded. This integral being found by the equation D, we must change x into $x+1$, and determine the constant according to the circumstances of the question.

Take, for example, $y_x = x(2x-1)$; changing x into $x+1$, and integrating the result, we find

$$2\Sigma x^2 + 3\Sigma x + \Sigma x^0 = \frac{4x^3+3x^2-x}{2.3} = x.\frac{x+1}{2}.\frac{4x-1}{3};$$

we add no constant, because $x=0$ must reduce the sum to nothing [See p. 21].

The series $1^m, 2^m, 3^m \ldots$ of the mth powers of the natural numbers is found by taking Σx^m [Equ. D]: but we must then add the xth term, which is x^m, *i. e.* we must change $-\frac{1}{2}x^m$, the 2nd term of D, into $+\frac{1}{2}x^m$: it will then remain to determine the constant, according to the term at which we wish the series to commence. For example, for the series of the squares, we take Σx^2, p. 475, changing the sign of the 2nd term, and we have

$$\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x = x.\frac{2x+1}{2}.\frac{x+1}{3};$$

the constant is $=0$, because the sum is nothing when $x=0$.

Had it been required to take the sum from n^2 to x^2, this sum would have been nothing when $x=n-1$; and we should have had the constant $= -n.\dfrac{n-1}{2}.\dfrac{2n-1}{3}$.

This theory applies to the summation of the figurate numbers. For example, to find the sum of the x first pyramidal numbers, 1, 4, 10, 20... [p. 19], we must integrate the general term $\frac{1}{6}x(x+1)(x+2)$; we find [N°. 918] $\frac{1}{24}(x-1)x(x+1)(x+2)$: lastly, x must be changed into $x+1$, and we have, for the sum required, $\frac{1}{24}x(x+1)(x+2)(x+3)$. The constant is nothing.

926. *The inverse figurate numbers are fractions*, having 1 for their numerator, and a figurate series for their denominator. The xth term of the order p is [p. 19] $\dfrac{1.2.3\ldots(p-1)}{x(x+1)\ldots(x+p-2)}$, and the integral therefore is [N°. 919]

$$C - \frac{1.2.3\ldots(p-1)}{(p-2)\,x(x+1)\ldots(x+p-3)}.$$

Changing x into $x+1$, and determining the constant by making the

sum $= 0$ when $x = 0$, so that $C = \frac{p-1}{p-2}$, we have for the sum of the x first terms

$$\frac{p-1}{p-2} - \frac{1.2.3\ldots(p.-1)}{(p-2)(x+1)(x+2)\ldots(x+p-3)}.$$

Make $p = 3, 4, 5\ldots$ successively, and we shall have

$$\tfrac{1}{1} + \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{10}\ldots \frac{1.2}{x(x+1)} \qquad = \tfrac{2}{1} - \frac{2}{x+1},$$

$$\tfrac{1}{1} + \tfrac{1}{4} + \tfrac{1}{10} + \tfrac{1}{20}\ldots \frac{1.2.3}{x(x+1)(x+2)} = \tfrac{3}{2} - \frac{3}{(x+1)(x+2)},$$

$$\tfrac{1}{1} + \tfrac{1}{5} + \tfrac{1}{15} + \tfrac{1}{35}\ldots \frac{1.2.3.4}{x\ldots(x+3)} \qquad = \tfrac{4}{3} - \frac{2.4}{(x+1)\ldots(x+3)},$$

$$\tfrac{1}{1} + \tfrac{1}{6} + \tfrac{1}{21} + \tfrac{1}{56}\ldots \frac{1.2.3.4.5}{x\ldots(x+4} \qquad = \tfrac{5}{4} - \frac{2.3.5}{(x+1)\ldots(x+4)},$$

and so on. To obtain the total sum of our series, we must render x infinite, which gives $\frac{p-1}{p-2}$ for the limit to which they continually approach.

For the series $\sin a$, $\sin(a+h)$, $\sin(a+2h)$,..., we have [N° 921]

$$\Sigma \sin(a+hx) = C - \frac{\cos(a+hx-\frac{1}{2}h)}{2\sin\frac{1}{2}h};$$

and changing x into $x+1$, and determining C from the condition that $x = -1$ reduces the sum to nothing, we find, for the term of summation,

$$\frac{\cos(a-\frac{1}{2}h) - \cos(a+hx+\frac{1}{2}h)}{2\sin\frac{1}{2}h},$$

or
$$\frac{\sin(a+\frac{1}{2}hx).\sin[\frac{1}{2}h(x+1)]}{\sin\frac{1}{2}h},$$

by virtue of the equation in the note, p. 309, 1st vol. The series $\cos a$, $\cos(a+h)$, $\cos(a+2h)$... similarly gives, for the term of summation,

$$\frac{\cos(a+\frac{1}{2}hx).\sin[\frac{1}{2}h(x+1)]}{\sin\frac{1}{2}h}.$$

FINIS.

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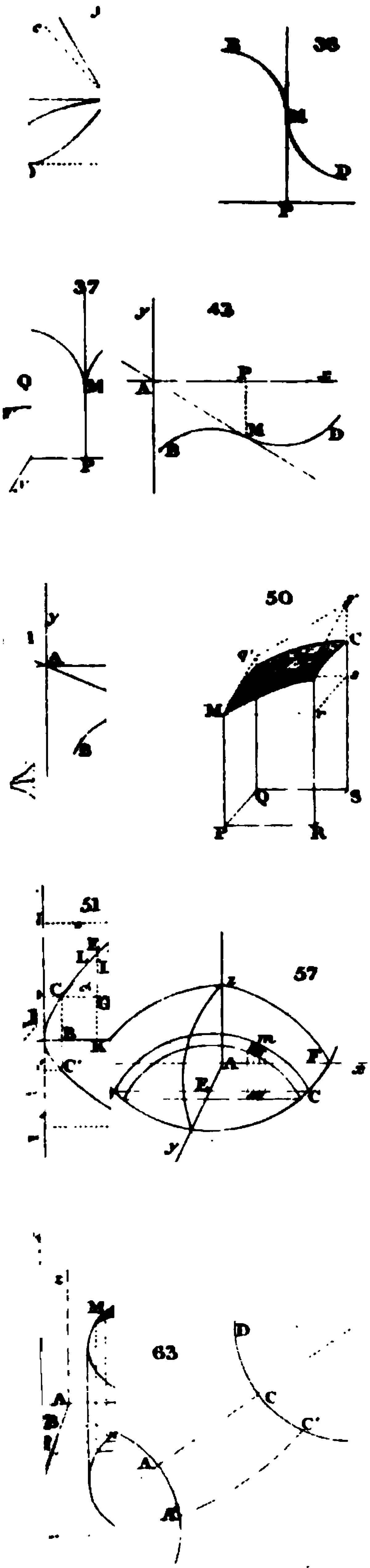
Pure Mathematics V. I. Plate 2.



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